CSC165

September 16, 2014

Announcements

➔ Tutorials start this week, exercise posted on course web page, work on them **before** tutorial.

Tutorial section and time	TA, tutorials 1–5	TA, tutorials 5–9	Room	Surnames
L0101, Tuesday 9:10–10:30	Jason	Jason	BA3012	A–F
	Eleni	Eleni	BA3116	G–L
	Madina	Madina	BA2185	M–T
	Siamak	Siamak	BA2175	V–Z
L0201, Monday 7:10–8:30	Ekaterina	Ekaterina	BA2159	A–D
	Gal	Gal	BA3008	E–Li
	Yana	Adam	BA3012	Liang–S
	Christina	Nadira	BA3116	T–Z
L5101, Thursday 7:10–8:30	Christine	Christine	BA3116	A–F
	Elias	Elias	BA2135	G–Li
	Yiyan	Yiyan	BA2159	Lin–U
	Natalie	Natalie	BA3008	V–Z

➔ Submit **slogURL.txt** to MarkUs by Friday, include a paragraph on topics from Week 1~2.

Lecture 2.1 Quantifiers (cont.)

Course Notes: Chapter 2

Today's topics

- → Quantifiers, verify / falsify
- → Sentences, statements
- → Predicates
- → Implications, equivalence

Learning the language of math

Before today, you talk like ...

"A course is worthy only when it is a prerequisite of some course."

After today, you talk like ...

$$\forall x \in C, W(x) \Rightarrow \exists y \in C, P(x, y)$$

Last week

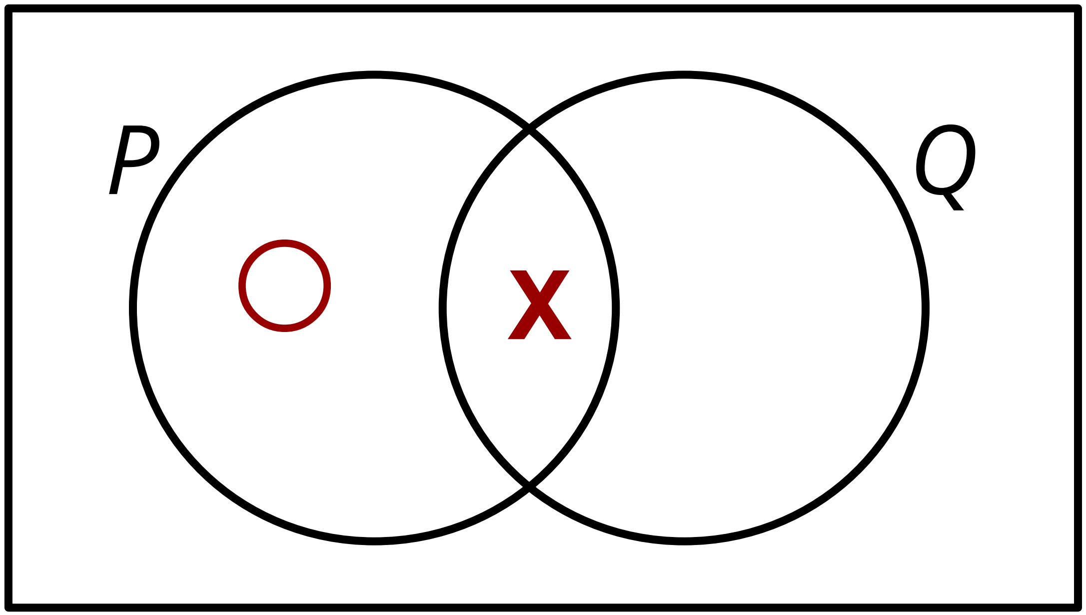
- ➔ Universal quantifier: $\forall$, “for all”, “every”
- ➔ Existential quantifier: $\exists$, “there is”, “some”

Fill with
“no”, “one”, “example”, “counter-example”

	Universal	Existential
Verify (prove)	no counter-example	one example
Falsify (disprove)	one counter-example	no example

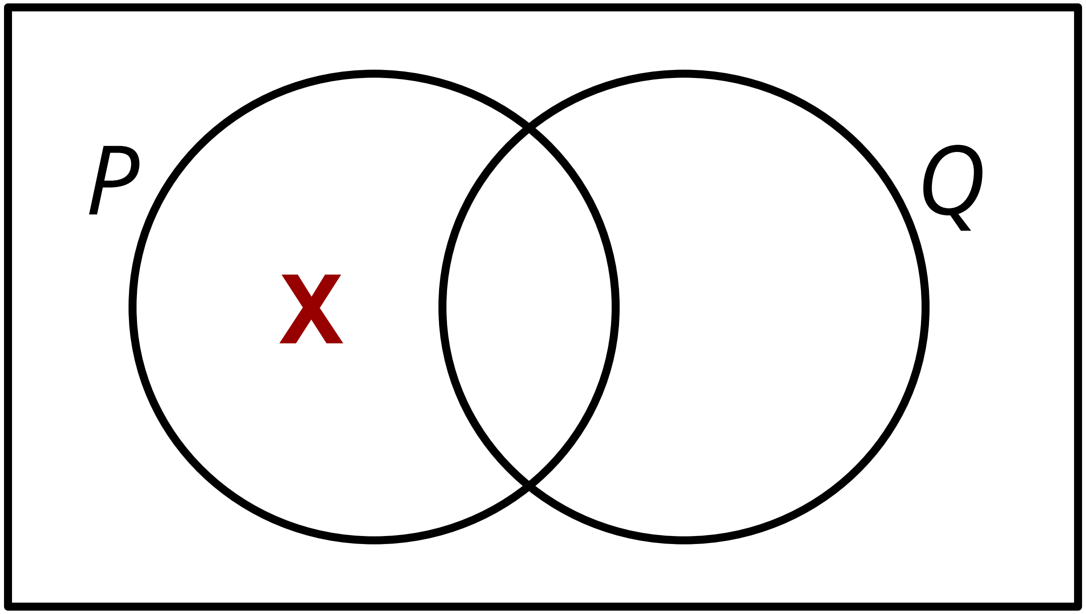
“Duality”, “anti-symmetric”

Visualization with Venn Diagram

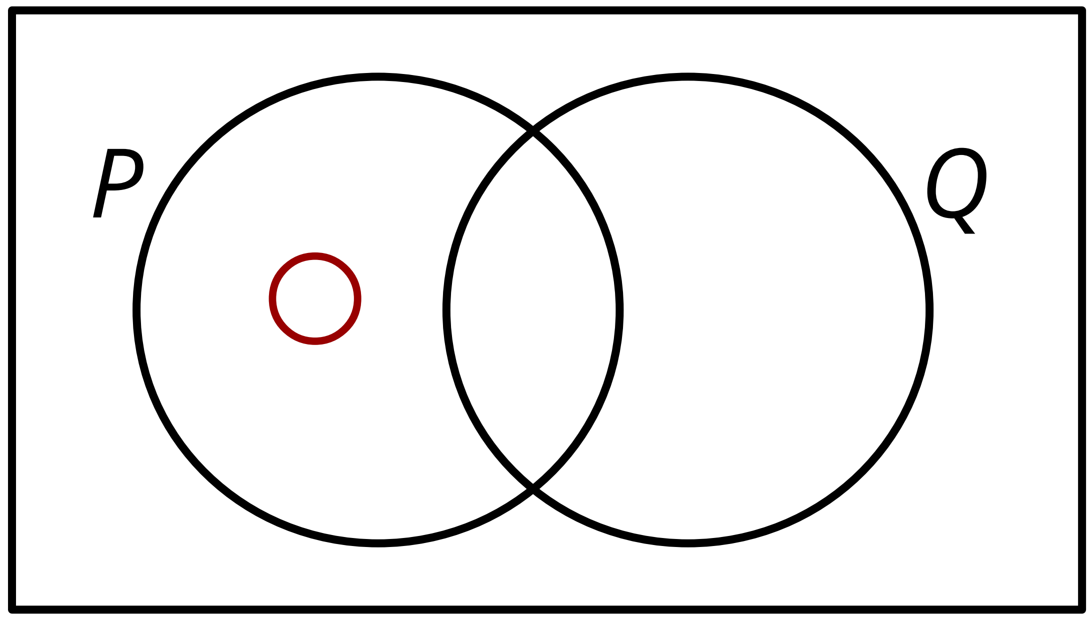


X: this part must be **empty**, i.e., with no element

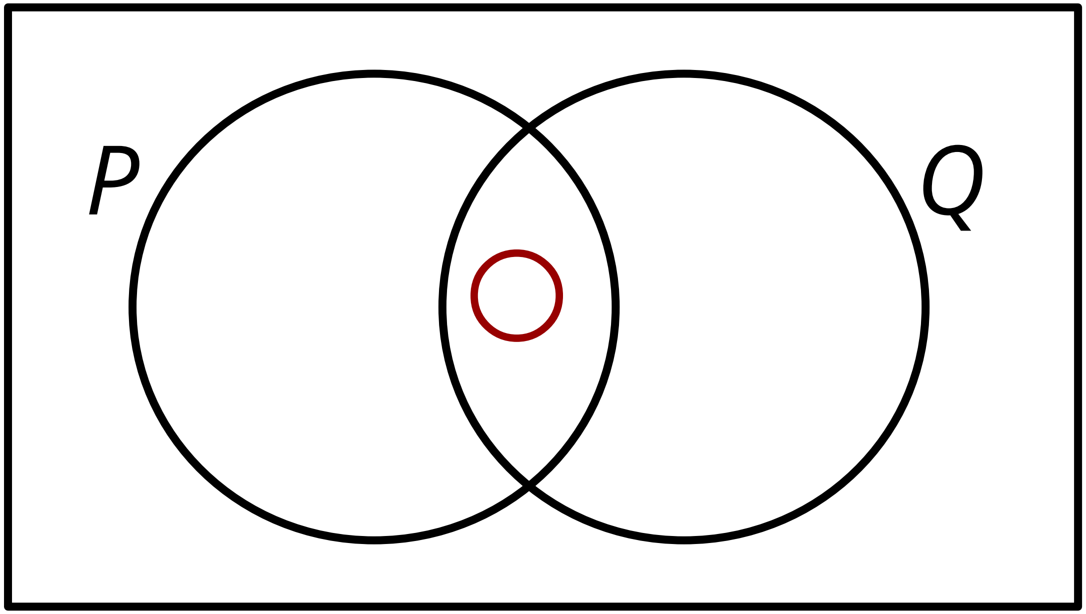
○: this part must be **occupied**, i.e., there must be some element in here



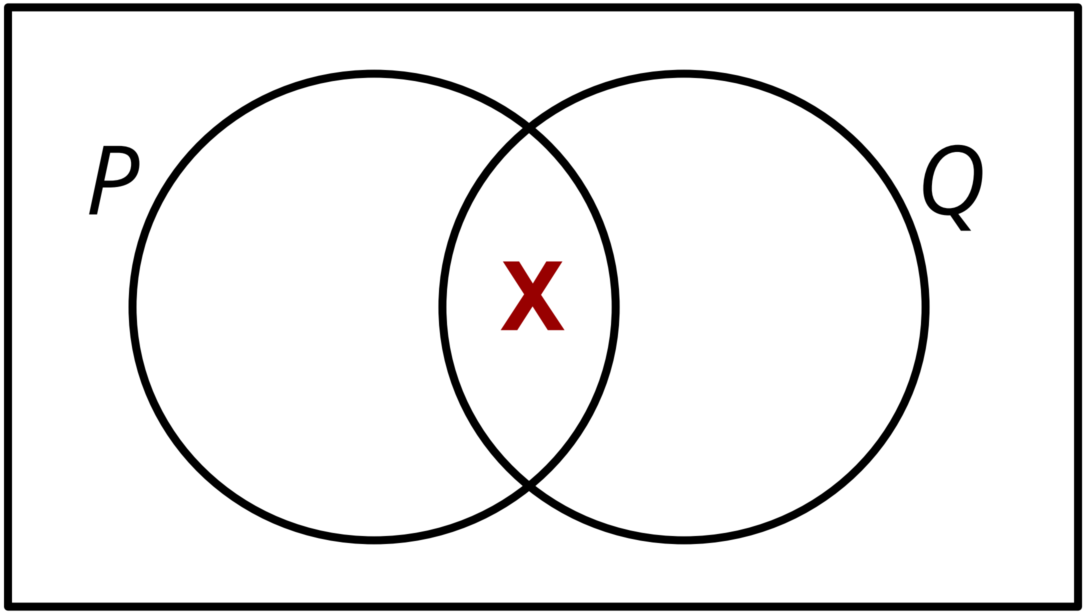
Verify $\forall x \in P, x \in Q$



Falsify $\forall x \in P, x \in Q$



Verify $\exists x \in P, x \in Q$



Falsify $\exists x \in P, x \in Q$

Quantifiers as claims about sets

Quantifiers as claims about **sets**

Some symbols about **sets**.

(prerequisites in Chapter 1.5):

$$A \subseteq B \qquad A \not\subseteq B$$

$$A \cap B \qquad A \cup B$$

$$\overline{A} \qquad \varnothing$$

Quantifiers as claims about **sets**

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000

E: set of all employees

M: set of male employees

F: set of female employees

O: set of employees who earn over $42,000

“All employees earn over $42,000.”

$$\forall x \in E, x \in O$$

$$E \subseteq O$$

Quantifiers as claims about **sets**

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000

E: set of all employees

M: set of male employees

F: set of female employees

O: set of employees who earn over $42,000

"No male employee earns over $42,000."

$$\forall x \in M, x \in \overline{O}$$

$$M \subseteq \overline{O} \qquad M \cap O = \varnothing$$

Quantifiers as claims about **sets**

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000

E: set of all employees

M: set of male employees

F: set of female employees

O: set of employees who earn over $42,000

"Some female employee earns over $42,000."

$$\exists x \in F, x \in O$$

$$F \cap O \neq \varnothing \qquad F \not\subseteq \overline{O}$$

Evaluating quantified claims

(using Python functions)

Evaluating quantified claims

```
def q2(S1, S2):

    return any({x in S2 for x in S1})
```

```
def q3(S1, S2):

    return all({x in S2 for x in S1})
```

E: set of all employees

M: set of male employees

F: set of female employees

O: set of employees who earn over $42,000

"All employees earn over $42,000."

$$\texttt{q3}(E,\ O)$$

Evaluating quantified claims

```
def q2(S1, S2):

    return any({x in S2 for x in S1})
```

```
def q3(S1, S2):

    return all({x in S2 for x in S1})
```

E: set of all employees

M: set of male employees

F: set of female employees

O: set of employees who earn over $42,000

“Some female employee earns over $42,000.”

q2(*F*, *O*)

Evaluating quantified claims

```
def q1(S1, S2):

    return not all({x in S2 for x in S1})
```

```
def q4(S1, S2):

    return not any({x in S2 for x in S1})
```

E: set of all employees

M: set of male employees

F: set of female employees

O: set of employees who earn over $42,000

"Some male employee does not earn over $42,000."

`q1(M, O)`

Evaluating quantified claims

```
def q1(S1, S2):
    return not all({x in S2 for x in S1})
```

```
def q4(S1, S2):
    return not any({x in S2 for x in S1})
```

E: set of all employees

M: set of male employees

F: set of female employees

O: set of employees who earn over $42,000

“No male employee earns over $42,000.”

q4(*M*, *O*)

Summary

Be able to **understand** and **express** quantifications in different ways

- ➔ using quantifiers (for all, $\forall$, $\exists$, etc.)
- ➔ using Venn Diagrams
- ➔ using set relations ($\subseteq$, $\nsubseteq$, $\cap$, $\varnothing$, etc.)
- ➔ using quantifying functions (q1, q2, etc.)

Lecture 2.2: Sentences, Statements and Implications

Course Notes: Chapter 2

Sentence and Statement

The employee earns less than $55,000.

This is a sentence.

Every employee earns less than $55,000.

This is a statement.

Sentence vs Statement

- A **statement** is always a **sentence.**
- A **sentence** is **NOT** always a **statement**.
- A **statement** is a **sentence** that is NOT “**open**”.
- The object in an **open sentence** is **unspecified (unquantified)**, thus the sentence cannot be evaluated.
- The object in a **statement** is **quantified**, thus a **statement** can be evaluated **true** or **false**.

Exercise: Is it a statement?

- Roses are red.
  - YES
- Someone ate my sandwich.
  - YES
- The sandwich tastes delicious.
  - NO
- At least one student is hungry.
  - YES
- The exercise is fun!
  - NO

Predicate

Predicates

L: the employees who earn less than $55,000.

To say: employee ***x*** earns less than $55,000.

We can write: ***x* ∈ *L***

x* is in the set *L

or we can write: *L(x)*

x* has property *L

***L(x)* is a boolean function returning True or False (whether *x* earns less than $55,000)**

***L(x)* is called a predicate.**

Predicates

$$L(x) : x \text{ earns less than } \$55{,}000$$

$$L(\text{Carlos})$$ Carlos earns less than $55,000.

$$\neg L(\text{Carlos})$$

Carlos earns no less than $55,000

“$\neg$” means “not”: negation

Predicates

$$F(x) : x \text{ is a female employee}$$

$$F(\text{Ellen})$$ Ellen is a female employee.

$$\neg F(\text{Carlos})$$

Carlos is not a female employee

Predicates

L(x) without specifying ***x*** is an **open sentence.**

Turn it into a **statement** by **universally** quantifying it.

For each employees, the employee earns less than $55,000.

∀ employee ***x***, ***x*** earns less than $55,000.

$$\forall x \in E, L(x)$$

Predicates

Similarly, turn it into a **statement** by **existentially** quantifying it.

There exists an employee, the employee earns less than $55,000.

∃ employee ***x***, ***x*** earns less than $55,000.

$$\exists x \in E, L(x)$$

$$L(x) \text{ or } x \in L?$$

Predicate or Set, which to use?

If ***L*** is more like a **property**, use **predicate**.

e.g., **"x** is a prerequisite of **y"**

P(x, y) (simple and good)

How to do this using set?

(x, y) ∈ **P**, where P is the set of pairs **(x, y)** such that **x** is a prerequisite of **y**.

Implications

Implication

If an employee is male, **then** he makes less than $55,000.

If P, **then** Q.

Antecedent (Assumption)

Consequent (Conclusion)

$$P \Rightarrow Q$$

“P implies Q”

Verify / falsify implication

If an employee is male, **then** he makes less than $55,000.

Employee	Gender	Salary
Al	male	60,000
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000

$$\forall x \in M, L(x)$$

Same as verifying / falsifying a universal quantification.

Be careful with “if”

Mom: If you eat your vegetables, then you can have dessert.

eat vegetables => can have dessert

don’t eat vegetables => cannot have dessert

Our “=>” logic only means this!

Be careful with "if"

If it rains yesterday, then the sun rose today.

A true implication
(Try falsify it if you don't buy it)

P => Q does **not** mean **P causes Q**.

Be careful with "if"

*If you **wear big shirts**, then you **wear big shoes**.*

P => Q does **not** mean **P causes Q**.

Converse of implication

E: set of all employees

F(x): ***x*** is female

L(x): ***x*** earns less than $55,000

$$\text{Original: } \forall x \in E, F(x) \Rightarrow L(x)$$

$$\text{Converse: } \forall x \in E, L(x) \Rightarrow F(x)$$

What's the relation between the two?

Contrapositive

E: set of all employees

F(x): ***x*** is female

L(x): ***x*** earns less than $55,000

$$\text{Original: } \forall x \in E, F(x) \Rightarrow L(x)$$

$$\text{Contrapositive: } \forall x \in E, \neg L(x) \Rightarrow \neg F(x)$$

What's the relation between these two?

Summary

- Sentence vs Statements
- Predicates
- Implications
  - Converse
  - Contrapositive
  - more …

Lecture 2.3 Implications cont.

Course Notes: Chapter 2

Numerical example

Define P(n): n is a multiple of 4, and Q(n): n^2 is a multiple of 4. We know that

$$\forall n \in \mathbb{N}, P(n) \Rightarrow Q(n)$$

What do the following tell you?

- ➔ k is a multiple of 4 — **k^2 must be a multiple of 4**
- ➔ k is not a multiple of 4
- ➔ k^2 is a multiple of 4
- ➔ k^2 is not a multiple of 4 — **k must NOT be a multiple of 4**

Everyday language for P => Q

If $\underbrace{\text{nominated}}_{P}$, $\underbrace{\text{I will not stand}}_{Q}$.

If $\underbrace{\text{you think I'm lying}}_{P}$, then $\underbrace{\text{you're a liar}}_{Q}$!

If P, [then] Q.

Everyday language for P => Q

Whenever I hear that song, I think about ice cream.

P Q

I get heartburn whenever I eat supper too late.

Q P

When[ever] P, [then] Q.

Everyday language for P => Q

$\underbrace{\text{Differentiability}}_{P}$ is sufficient for $\underbrace{\text{continuity}}_{Q}$.

$\underbrace{\text{Matching fingerprints and a motive}}_{P}$ are enough for $\underbrace{\text{guilt}}_{Q}$.

P is sufficient/enough for Q.

Everyday language for P => Q

There are no rights without responsibilities.

P Q

You can't stay enrolled in CSC165 without a pulse.

P Q

Can't have P without Q.

Everyday language for P => Q

Successful programming requires skill.

P Q

Login is required to view content.

Q P

P requires Q.

Everyday language for P => Q

A student need to get 40% on the final
to pass CSC165.

Q

P

To buy this phone, I must sell my kidney.

P

Q

For P to be true, Q must / need to be true / is necessary.

Everyday language for P => Q

I will go only if you insist.

P **Q**

I dance only when I'm drunk.

P **Q**

P only if / only when Q.

Everyday language for P => Q

Don't knock on it unless you have tried it.

P Q

It is not beautiful if it is not round.

P Q

Not P unless / if not Q.

Everyday language for P => Q

Replying “come in” kills a knock-knock joke.

P: successful knock-knock joke
Q: no replying “come in”.

P: replying “come in”
Q: failed knock-knock joke

Contrapositive!

Takeaway

Though all these ways in everyday language, in the **language of math**, it is just...

$$P \Rightarrow Q$$

something weird...

Vacuous truth

To falsify “P(x) => Q(x)”

➔ find an **x** such that P(**x**) is true but Q(**x**) is false.

$$\forall x \in \mathbb{R}, x^2 < 0 \Rightarrow x > x + 5$$

➔ All employees earning over $80 trillion are female.

➔ All employees earning over $80 trillion are male.

➔ All employees earning over $80 trillion have pink toenails.

Equivalence

Equivalence

If P then Q, and if Q then P

P if and only if Q

P iff Q

$$P \Leftrightarrow Q$$

"=>": P only if Q

"<=": P if Q

Equivalence

Employee	Gender	Salary
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000
Gwen	female	95,000

Every male employee earns between $25,000 and $45,000.

Every employee earning between $25,000 and $45,000 is male.

Equivalence

Employee	Gender	Salary
Betty	female	500
Carlos	male	40,000
Doug	male	30,000
Ellen	female	50,000
Flo	female	20,000
Gwen	female	95,000

“The employee is male”

is equivalent to

“the employee earns between $25,000 and $45,000.”

Other sayings of equivalence

→ P **implies** Q, and **conversely**.

→ P is true **exactly when** Q is true.

→ P is **necessary** and **sufficient** for Q

necessary: P <= Q

sufficient: P => Q

something weird...

Equivalence (weird)

$$\forall x \in \mathbb{R}, x^2 < 0 \Leftrightarrow x > x + 5$$

TRUE

some language conventions

Idiom

Sometimes there are more than one way, even in math.
Some ways are more commonly used than others.

Every D that is a P is also a Q.

$$\text{common: } \forall x \in D, P(x) \Rightarrow Q(x)$$

$$\text{less common: } \forall x \in D \cap P, Q(x)$$

Idiom

Sometimes there are more than one way, even in math.
Some ways are more commonly used than others.

Some D that is a P is also a Q.

$$\text{common: } \exists x \in D, P(x) \wedge Q(x)$$

$$\text{less common: } \exists x \in D \cap P, Q(x)$$

as promised …

Before today, you talk like ...

“A course is worthy only when it is a prerequisite of some course.”

After today, you talk like ...

$$\forall x \in C, W(x) \Rightarrow \exists y \in C, P(x, y)$$

Summary

→ We learned elements of the **math language**

- quantifiers
- sentences
- symbols
- statements
- predicates
- implications
- equivalence
- **there is more ...**

Next week

- Conjunction
- Disjunction
- Negation
- Boolean algebra
- …