Announcements

→ Tutorials start this week, exercise posted on course web page, work on them **before** tutorial.

<table>
<thead>
<tr>
<th>Tutorial section and time</th>
<th>TA, tutorials 1–5</th>
<th>TA, tutorials 5–9</th>
<th>Room</th>
<th>Surnames</th>
</tr>
</thead>
<tbody>
<tr>
<td>L5101, Thursday 7:10–8:30</td>
<td>Christine Elias Yiyang Natalie</td>
<td>Christine Elias Yiyang Natalie</td>
<td>BA3116 BA2135 BA2159 BA3008</td>
<td>A–F G–Li Lin–U V–Z</td>
</tr>
</tbody>
</table>

→ Submit **slogURL.txt** to MarkUs by Friday, include a paragraph on topics from Week 1~2.
Lecture 2.1  Quantifiers (cont.)

Course Notes: Chapter 2
Today’s topics

➔ Quantifiers, verify / falsify
➔ Sentences, statements
➔ Predicates
➔ Implications, equivalence

Learning the language of math
Before today, you talk like ...

“A course is worthy only when it is a prerequisite of some course.”

After today, you talk like ...

\[ \forall x \in C, W(x) \Rightarrow \exists y \in C, P(x, y) \]
Last week

→ Universal quantifier: $\forall$, “for all”, “every”

→ Existential quantifier: $\exists$, “there is”, “some”
Fill with “no”, “one”, “example”, “counter-example”

<table>
<thead>
<tr>
<th></th>
<th>Universal</th>
<th>Existential</th>
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<tbody>
<tr>
<td><strong>Verify</strong> (prove)</td>
<td><em>no</em> counter-example</td>
<td><em>one</em> example</td>
</tr>
<tr>
<td><strong>Falsify</strong> (disprove)</td>
<td><em>one</em> counter-example</td>
<td><em>no</em> example</td>
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“Duality”, “anti-symmetric”
Visualization with Venn Diagram

X: this part must be **empty**, i.e., with no element

O: this part must be **occupied**, i.e., there must be some element in here
Verify $\forall x \in P, x \in Q$
Falsify $\forall x \in P, x \in Q$
Verify $\exists x \in P, x \in Q$
Falsify $\exists x \in P, x \in Q$
Quantifiers as claims about sets
Quantifiers as claims about **sets**

Some symbols about **sets**.
(prerequisites in Chapter 1.5):

\[ A \subseteq B \quad A \not\subseteq B \]
\[ A \cap B \quad A \cup B \]
\[ \overline{A} \quad \emptyset \]
### Quantifiers as claims about sets

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- $E$: set of all employees
- $M$: set of male employees
- $F$: set of female employees
- $O$: set of employees who earn over $42,000$

"All employees earn over $42,000."

$$\forall x \in E, x \in O$$

$E \subseteq O$
Quantifiers as claims about sets

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\( E: \) set of all employees

\( M: \) set of male employees

\( F: \) set of female employees

\( O: \) set of employees who earn over $42,000

“No male employee earns over $42,000.”

\[ \forall x \in M, x \in \overline{O} \]

\[ M \subseteq \overline{O} \]

\[ M \cap O = \emptyset \]
Quantifiers as claims about sets

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\[ E: \text{ set of all employees} \]
\[ M: \text{ set of male employees} \]
\[ F: \text{ set of female employees} \]
\[ O: \text{ set of employees who earn over$42,000} \]

"Some female employee earns over $42,000."

\[ \exists x \in F, \ x \in O \]

\[ F \cap O \neq \emptyset \quad F \not\subseteq \overline{O} \]
Evaluating quantified claims
(using Python functions)
Evaluating quantified claims

\[ \text{def q2(S1, S2):} \]
\[ \quad \text{return any\{x in S2 for x in S1\}} \]

\[ \text{def q3(S1, S2):} \]
\[ \quad \text{return all\{x in S2 for x in S1\}} \]

\( E \): set of all employees

\( M \): set of male employees

\( F \): set of female employees

\( O \): set of employees who earn over $42,000

“All employees earn over $42,000.”

\[ \text{q3}(E, O) \]
Evaluating quantified claims

def q2(S1, S2):
    return any({x in S2 for x in S1})

def q3(S1, S2):
    return all({x in S2 for x in S1})

E: set of all employees
M: set of male employees
F: set of female employees
O: set of employees who earn over $42,000

"Some female employee earns over $42,000."

q2(F, O)
Evaluating quantified claims

```
def q1(S1, S2):
    return not all({x in S2 for x in S1})
```

```
def q4(S1, S2):
    return not any({x in S2 for x in S1})
```

\[ E: \text{ set of all employees} \]
\[ M: \text{ set of male employees} \]
\[ F: \text{ set of female employees} \]
\[ O: \text{ set of employees who earn over $42,000} \]

“Some male employee does not earn over $42,000.”

\[ q1(M, O) \]
Evaluating quantified claims

\[
\begin{align*}
def q1(S1, S2):
    & \text{return not all(\{x in S2 for x in S1\})} \\

def q4(S1, S2):
    & \text{return not any(\{x in S2 for x in S1\})}
\end{align*}
\]

\(E\): set of all employees  
\(M\): set of male employees  
\(F\): set of female employees  
\(O\): set of employees who earn over $42,000

“No male employee earns over $42,000.”

\(q4(M, O)\)
Summary

Be able to **understand** and **express** quantifications in different ways

→ using quantifiers (for all, ∀, ∃, etc.)

→ using Venn Diagrams

→ using set relations (⊆, ∈, ∩, ∅, etc.)

→ using quantifying functions (q1, q2, etc.)
The employee earns less than $55,000.

Every employee earns less than $55,000.

This is a sentence.

This is a statement.
Sentence vs Statement

- A **statement** is always a **sentence**.
- A **sentence** is NOT always a **statement**.
- A **statement** is a **sentence** that is NOT “open”.
- The object in an **open sentence** is unspecified (unquantified), thus the sentence cannot be evaluated.
- The object in a **statement** is **quantified**, thus a **statement** can be evaluated **true** or **false**.
Exercise: Is it a statement?

- Roses are red.
  - YES
- Someone ate my sandwich.
  - YES
- The sandwich tastes delicious.
  - NO
- At least one student is hungry.
  - YES
- The exercise is fun!
  - NO
Predicate
**Predicates**

$L$: the employees who earn less than $55,000.

To say: employee $x$ earns less than $55,000.

We can write: $x \in L$

or we can write: $L(x)$

$L(x)$ is a boolean function returning True or False (whether $x$ earns less than $55,000)

$L(x)$ is called a predicate.
Predicates

$L(x) : x$ earns less than $55,000$

$L(\text{Carlos})$ Carlos earns less than $55,000$.

$\neg L(\text{Carlos})$ Carlos earns no less than $55,000$

“$\neg$” means “not”: negation
Predicates

\[ F(x) : x \text{ is a female employee} \]

\[ F(\text{Ellen}) \]

Ellen is a female employee.

\[ \neg F(\text{Carlos}) \]

Carlos is not a female employee.
**Predicates**

$L(x)$ without specifying $x$ is an **open sentence**.

Turn it into a **statement** by **universally quantifying** it.

*For each* employees, the employee earns less than $55,000.

∀ employee $x$, $x$ earns less than $55,000$.

∀ $x \in E$, $L(x)$
Predicates

Similarly, turn it into a statement by existentially quantifying it.

There exists an employee, the employee earns less than $55,000.

∃ employee x, x earns less than $55,000.

∃x ∈ E, L(x)
Predicate or Set, which to use?

If \( L \) is more like a **property**, use **predicate**.

e.g., “\( x \) is a prerequisite of \( y \)”

\( P(x, y) \) (simple and good)

**How to do this using set?**

\( (x, y) \in P \), where \( P \) is the set of pairs \( (x, y) \) such that \( x \) is a prerequisite of \( y \).
Implications
Implication

If an employee is male, then he makes less than $55,000.

If $P$, then $Q$.

Antecedent (Assumption)

Consequent (Conclusion)
$P \implies Q$

“P implies Q”
Verify / falsify implication

If an employee is male, then he makes less than $55,000.

\[ \forall x \in M, L(x) \]

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Same as verifying / falsifying a universal quantification.
Be careful with “if”

Mom: If you eat your vegetables, then you can have dessert.

eat vegetables => can have dessert

don’t eat vegetables => cannot have dessert

Our “=>” logic only means this!
Be careful with “if”

If it rains yesterday, then the sun rose today.

A true implication
(Try falsify it if you don’t buy it)

$P \Rightarrow Q$ does not mean $P$ causes $Q$. 
Be careful with “if”

*If you wear big shirts, then you wear big shoes.*

\[ P \implies Q \text{ does not mean } P \text{ causes } Q. \]
Converse of implication

$E$: set of all employees
$F(x)$: $x$ is female
$L(x)$: $x$ earns less than $55,000$

Original: $\forall x \in E, F(x) \Rightarrow L(x)$

Converse: $\forall x \in E, L(x) \Rightarrow F(x)$

What’s the relation between the two?
Contrapositive

\( E \): set of all employees
\( F(x) \): \( x \) is female
\( L(x) \): \( x \) earns less than $55,000

Original: \( \forall x \in E, F(x) \Rightarrow L(x) \)

Contrapositive: \( \forall x \in E, \neg L(x) \Rightarrow \neg F(x) \)

What’s the relation between these two?
Summary

➔ Sentence vs Statements
➔ Predicates
➔ Implications
  ◆ Converse
  ◆ Contrapositive
  ◆ more ...
Lecture 2.3 Implications cont.

Course Notes: Chapter 2
Numerical example

Define $P(n)$: $n$ is a multiple of 4, and $Q(n)$: $n^2$ is a multiple of 4. We know that

$$\forall n \in \mathbb{N}, P(n) \Rightarrow Q(n)$$

What do the following tell you?

→ $k$ is a multiple of 4  \[\text{k}^2 \text{ must be a multiple of 4} \]
→ $k$ is not a multiple of 4
→ $k^2$ is a multiple of 4  \[\text{k must NOT be a multiple of 4} \]
→ $k^2$ is not a multiple of 4
Everyday language for $P \implies Q$

If nominated, I will not stand.

If you think I’m lying, then you’re a liar!

If $P$, $[\text{then}]$ $Q$. 
Everyday language for $P \implies Q$

Whenever I hear that song, I think about ice cream.

When[ever] $P$, [then] $Q$. 

I get heartburn whenever I eat supper too late.

When[ever] $P$, [then] $Q$. 
Everyday language for $P \implies Q$

Differentiability is sufficient for continuity.

Matching fingerprints and a motive are enough for guilt.

$P$ is sufficient/enough for $Q$. 
Everyday language for \( P \implies Q \)

There are no rights without responsibilities.

You can’t stay enrolled in CSC165 without a pulse.

Can’t have \( P \) without \( Q \).
Everyday language for $P \implies Q$

Successful programming requires skill.

Login is required to view content.

$P$ requires $Q$. 

\[ P \implies Q \]
Everyday language for $P \implies Q$

A student need to get 40% on the final to pass CSC165.

To buy this phone, I must sell my kidney.

For $P$ to be true, $Q$ must / need to be true / is necessary.
Everyday language for $P \Rightarrow Q$

I will go only if you insist.

\[ \text{P} \quad \text{Q} \]

I dance only when I’m drunk.

\[ \text{P} \quad \text{Q} \]

$P$ only if / only when $Q$. 
Everyday language for \( P \implies Q \)

Don’t knock on it unless you have tried it.

\[
P \implies Q
\]

It is not beautiful if it is not round.

\[
P \implies Q
\]

Not \( P \) unless / if not \( Q \).
Everyday language for $P \implies Q$

Replying “come in” kills a knock-knock joke.

$P$: successful knock-knock joke  
$Q$: no replying “come in”.

$P$: replying “come in”  
$Q$: failed knock-knock joke

Contrapositive!
Takeaway

Though all these ways in everyday language, in the **language of math**, it is just...

\[ P \implies Q \]
something weird...
Vacuous truth

To falsify “P(x) => Q(x)”
➔ find an \( x \) such that P(\( x \)) is true but Q(\( x \)) is false.

\[
\forall x \in \mathbb{R}, \; x^2 < 0 \implies x > x + 5
\]

➔ All employees earning over $80 trillion are female.
➔ All employees earning over $80 trillion are male.
➔ All employees earning over $80 trillion have pink toenails.
Equivalence
Equivalence

If P then Q, and if Q then P

P if and only only if Q

P iff Q

“=>”: P only if Q

“<=”: P if Q
Equivalence

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Every male employee earns between $25,000 and $45,000.

Every employee earning between $25,000 and $45,000 is male.
Equivalence

“The employee is male” is equivalent to “the employee earns between $25,000 and $45,000.”

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Other sayings of equivalence

→ **P implies Q**, and **conversely**.

→ **P is true exactly when Q is true**.

→ **P is necessary and sufficient** for Q

\[
\text{necessary: } P \leq Q \\
\text{sufficient: } P \Rightarrow Q
\]
something weird...
Equivalence (weird)

\[ \forall x \in \mathbb{R}, x^2 < 0 \iff x > x + 5 \]
some language conventions
Idiom

Sometimes there are more than one way, even in math. Some ways are more commonly used than others.

**Every D that is a P is also a Q.**

common: \( \forall x \in D, P(x) \implies Q(x) \)

less common: \( \forall x \in D \cap P, Q(x) \)
Idiom

Sometimes there are more than one way, even in math. Some ways are more commonly used than others.

Some \( D \) that is a \( P \) is also a \( Q \).

common: \( \exists x \in D, P(x) \land Q(x) \)

less common: \( \exists x \in D \cap P, Q(x) \)
as promised ...
Before today, you talk like ...

“A course is worthy only when it is a prerequisite of some course.”

After today, you talk like ...

$$\forall x \in C, W(x) \Rightarrow \exists y \in C, P(x, y)$$
Summary

We learned elements of the **math language**
- quantifiers
- sentences
- symbols
- statements
- predicates
- implications
- equivalence

*there is more ...*
Next week

➔ Conjunction
➔ Disjunction
➔ Negation
➔ Boolean algebra
➔ ...