Question 1. (10 marks) Let $F = (G, s, t, c)$ be a flow network and $(S, T)$, $(S', T')$ be minimum cuts of $F$. Prove that $(S \cap S', T \cup T')$ and $(S \cup S', T \cap T')$ are also minimum cuts of $F$.

**Hint:** Prove both facts together. In thinking about this problem you may find the following diagram useful:
**Question 2.** (10 marks) Given a flow network in which all edges have capacity one and a positive integer $k$, we want to remove $k$ edges from the flow network so as to reduce as much as possible the maximum flow in the resulting flow network.

Give a polynomial-time algorithm that does this. That is, your algorithm takes as input (a) a flow graph $(G, s, t, c)$, where $G = (V, E)$ and $c(e) = 1$ for each $e \in E$, and (b) a positive integer $k \leq |E|$, and produces as output a set $E' \subseteq E$ such that (i) $|E'| = k$, and (ii) the maximum flow in the flow graph $(G', s, t, c')$ is as small as possible, where $G' = (V, E - E')$ and $c'(e) = 1$ for each $e \in E - E'$.

Prove that your algorithm is correct and analyze its running time.

**Question 3.** (15 marks) Let $F = (G, s, t, c)$ be a flow graph with integer capacities and $f$ be a flow in $F$. A **cycle of flow** $f$ is a cycle of $G$ every edge of which carries some flow; that is, a path $u_1, u_2, \ldots, u_k$ of $G$ such that $u_1 = u_k$ and $f(u_i, u_{i+1}) > 0$, for each $i$ such that $1 \leq i < k$. A flow $f$ is called **acyclic** if it has no cycle.

a. Suppose we use the version of the Ford-Fulkerson algorithm that always chooses an augmenting path with the fewest edges. Give an example flow network in which this algorithm **necessarily** produces a cyclic flow. Your example should be as simple as you can. You should identify the augmenting paths that the algorithm chooses in your example flow network and the cycle of the maximum flow that it creates.

b. Prove that for any flow network $F$ and any flow $f$ in $F$, there is an acyclic flow of $F$ with the same value as $f$. You may **not** assume that $f$ is an integral flow.