Linear programming (LP)

The diet problem (cost minimization)

m nutrients $N_1, N_2, \ldots, N_m$
n foods $F_1, F_2, \ldots, F_n$

A unit of food F_j

- costs c_j
- provides a_{ij} units of nutrient N_i

Each day, a person must get $\geq b_i$ units of nutrient N_i, for every $i \in 1..m$

Question: What is a cheapest diet (combination of foods) that satisfies the daily requirement in nutrients?

The diet problem (continued)

LP formulation of the problem:

Variables: $x_1, x_2, \ldots, x_n$
(x_j is the amount of food F_j in the diet)

Objective function: minimize $c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$
(cost of the diet)

Constraints:

- $\underbrace{a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n}_{\text{amount of } N_1 \text{ in diet}} \geq b_1$
- ...
- $a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \geq b_m$
- $x_1, x_2, \ldots, x_n \geq 0$

Input:
- c_j, for $j \in 1..n$
- b_i, for $i \in 1..m$
- a_{ij}, for $i \in 1..m$ and $j \in 1..n$

Output:
Values for $x_1, \ldots, x_n$ that minimize $\sum_{j=1}^{n} c_j x_j$ and satisfy all constraints

Product selection (profit maximization)

m resources $R_1, R_2, \ldots, R_m$
n products $P_1, P_2, \ldots, P_n$

A unit of product P_j

- yields profit c_j
- requires a_{ij} units of resource R_i

The manufacturer of the products has $\leq b_i$ units of resource R_i, for every $i \in 1..m$

Question: What is the most profitable combination of products to manufacture with the available resources?

The product selection problem (continued)

LP formulation of the problem:

Variables: $x_1, x_2, \ldots, x_n$
(x_j is the amount of product P_j to manufacture)

Objective function: maximize $c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$
(profit from product lineup)

Constraints:

- $\underbrace{a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n}_{\text{amount of } R_1 \text{ needed for product lineup}} \leq b_1$
- ...
- $a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m$
- $x_1, x_2, \ldots, x_n \geq 0$

Input:

- c_j, for $j \in 1..n$
- b_i, for $i \in 1..m$
- a_{ij}, for $i \in 1..m$ and $j \in 1..n$

Output:

Values for $x_1, \ldots, x_n$ that maximize $\sum_{j=1}^{n} c_j x_j$ and satisfy all constraints

Concrete instance of product selection

A farmer has 3 resources:

- labour: $\leq$ 40 hours
- seed: $\leq$ 100 kg
- fertilizer: $\leq$ 10 bags

and can grow 2 types of corn:

- C1 (for human consumption)
- C2 (for animal feed)

Concrete instance of product selection

Corn of type C1:

- yields profit $3 per hectare
- requires (per hectare)
  - 2 hours of labour
  - 1 kg seed
  - 1 bag fertilizer

Corn of type C2:

- yields profit $2 per hectare
- requires (per hectare)
  - 1 hour of labour
  - 3 kg seed
  - no fertilizer

Question: How many hectares of C1 and how many of C2 should the farmer grow to maximize her profit?

Concrete instance of product selection

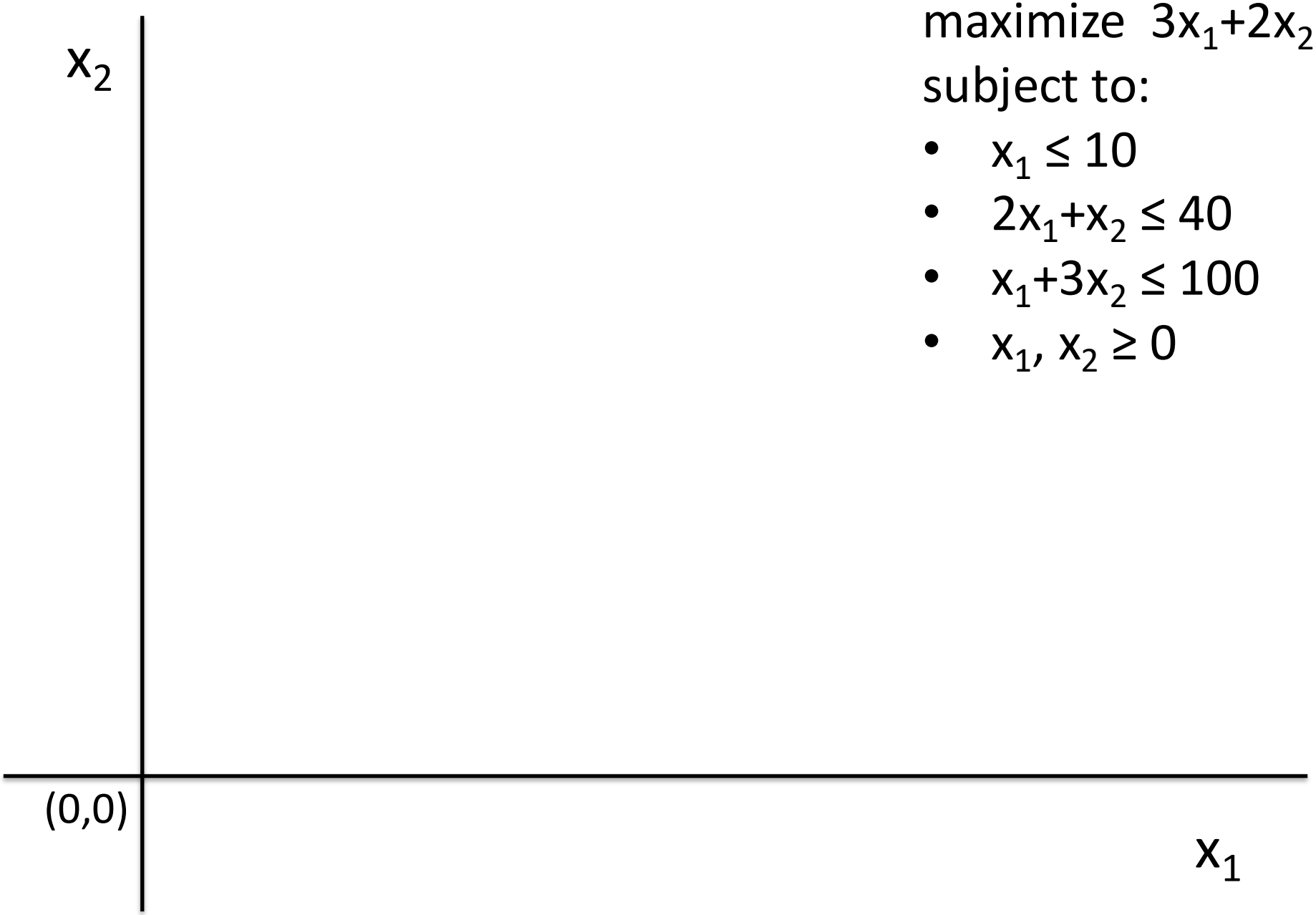
LP formulation of the problem:

Variables: x_1, x_2
(x_i is the number of hectares on which to grow corn type Ci)

Objective function: maximize $3x_1 + 2x_2$
(profit from crop mix)

Constraints:

- $2x_1 + x_2 \leq 40$ (labour constraint)
- $x_1 + 3x_2 \leq 100$ (seed constraint)
- $x_1 \leq 10$ (fertilizer constraint)
- $x_1, x_2 \geq 0$

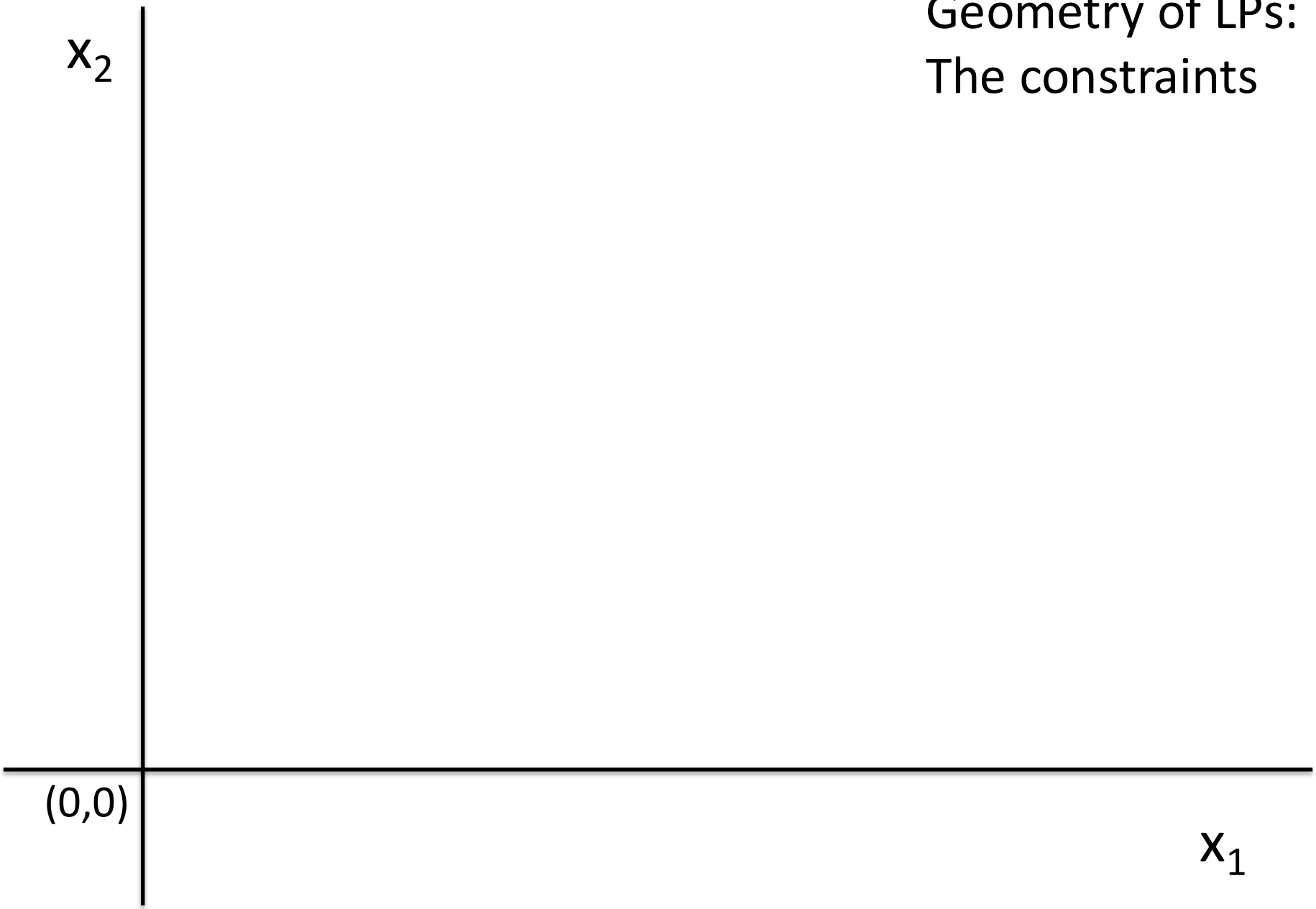


maximize $3x_1+2x_2$

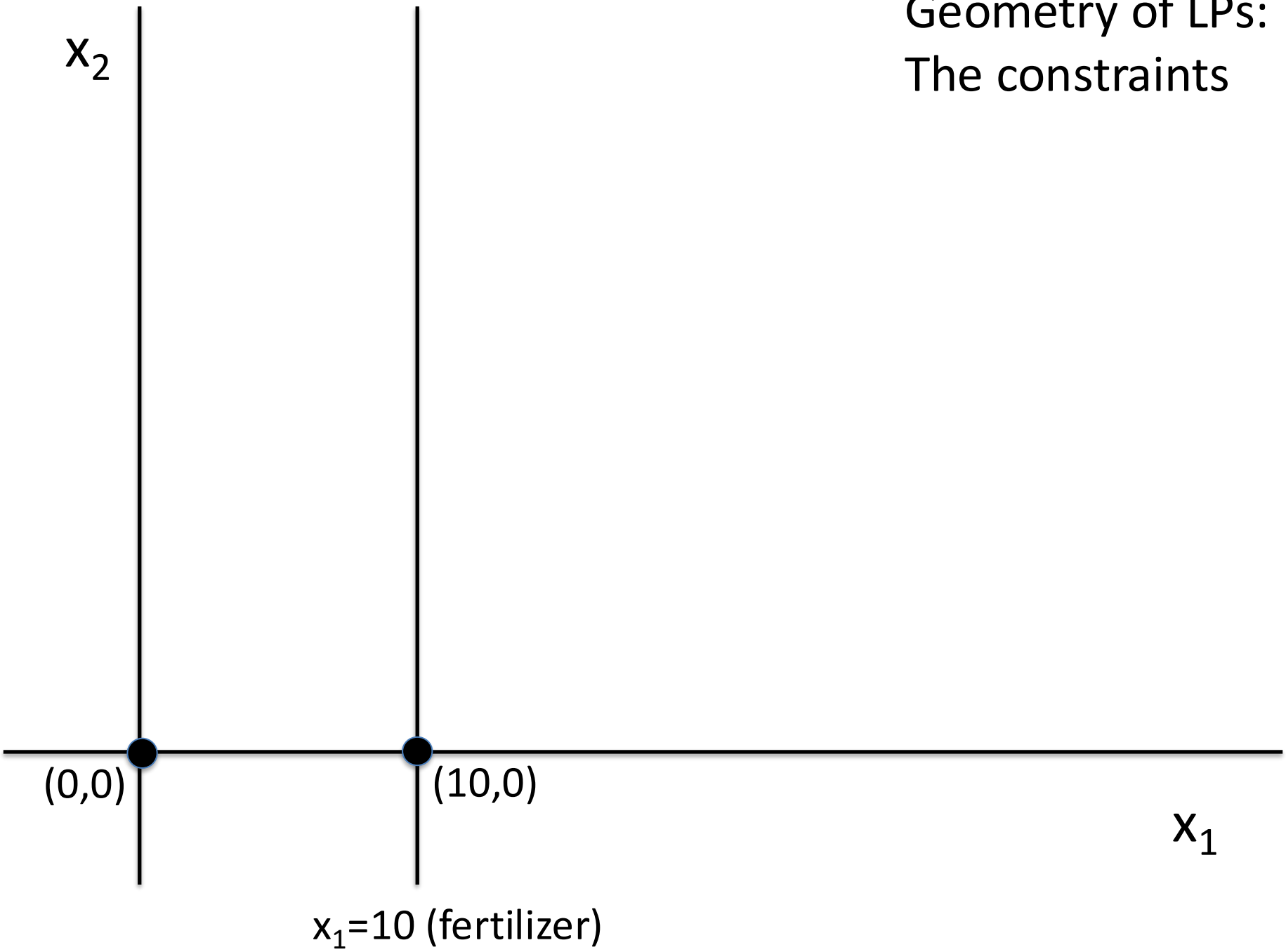
subject to:

- $x_1 \leq 10$
- $2x_1+x_2 \leq 40$
- $x_1+3x_2 \leq 100$
- $x_1, x_2 \geq 0$

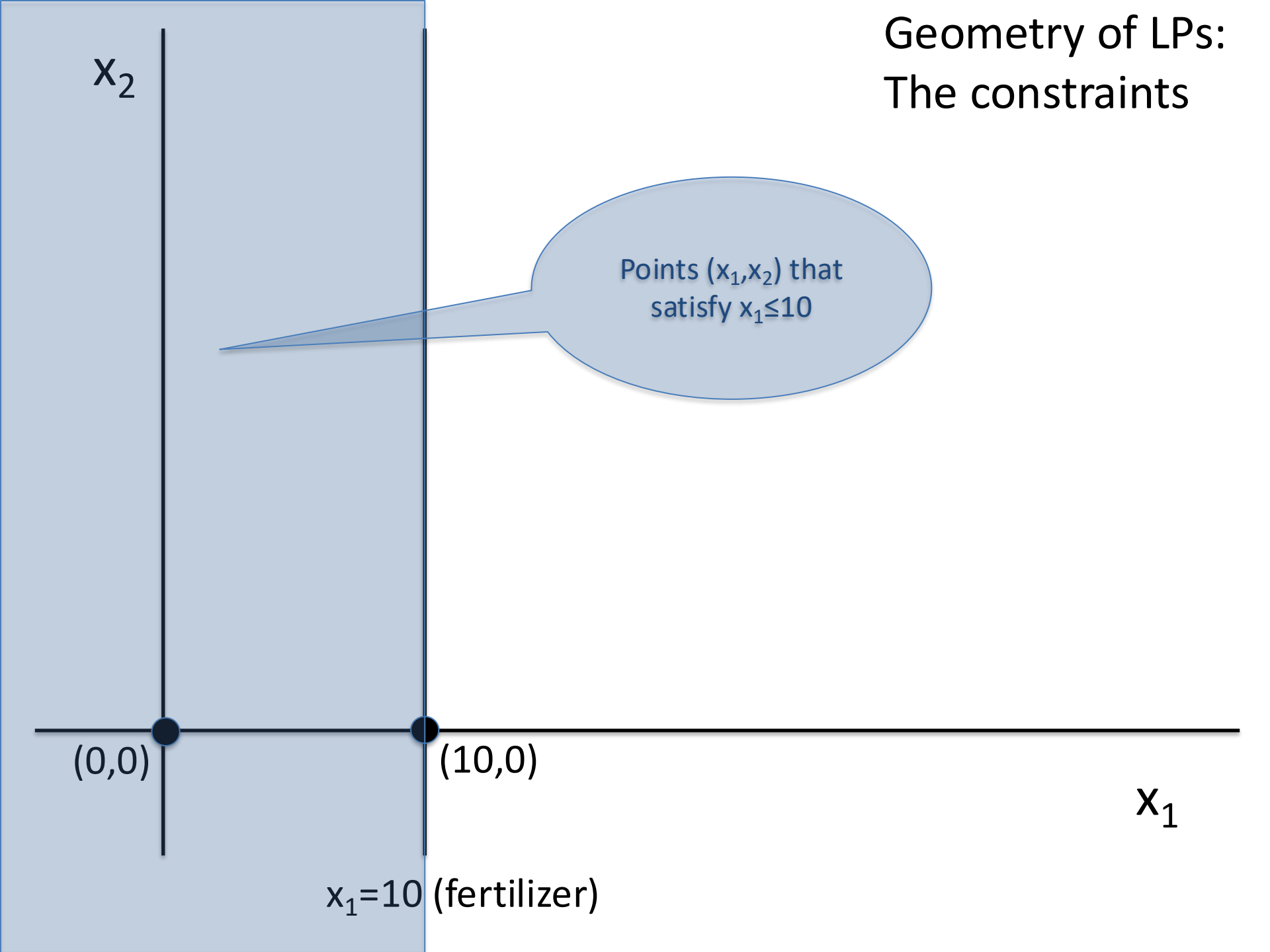
Geometry of LPs: The constraints



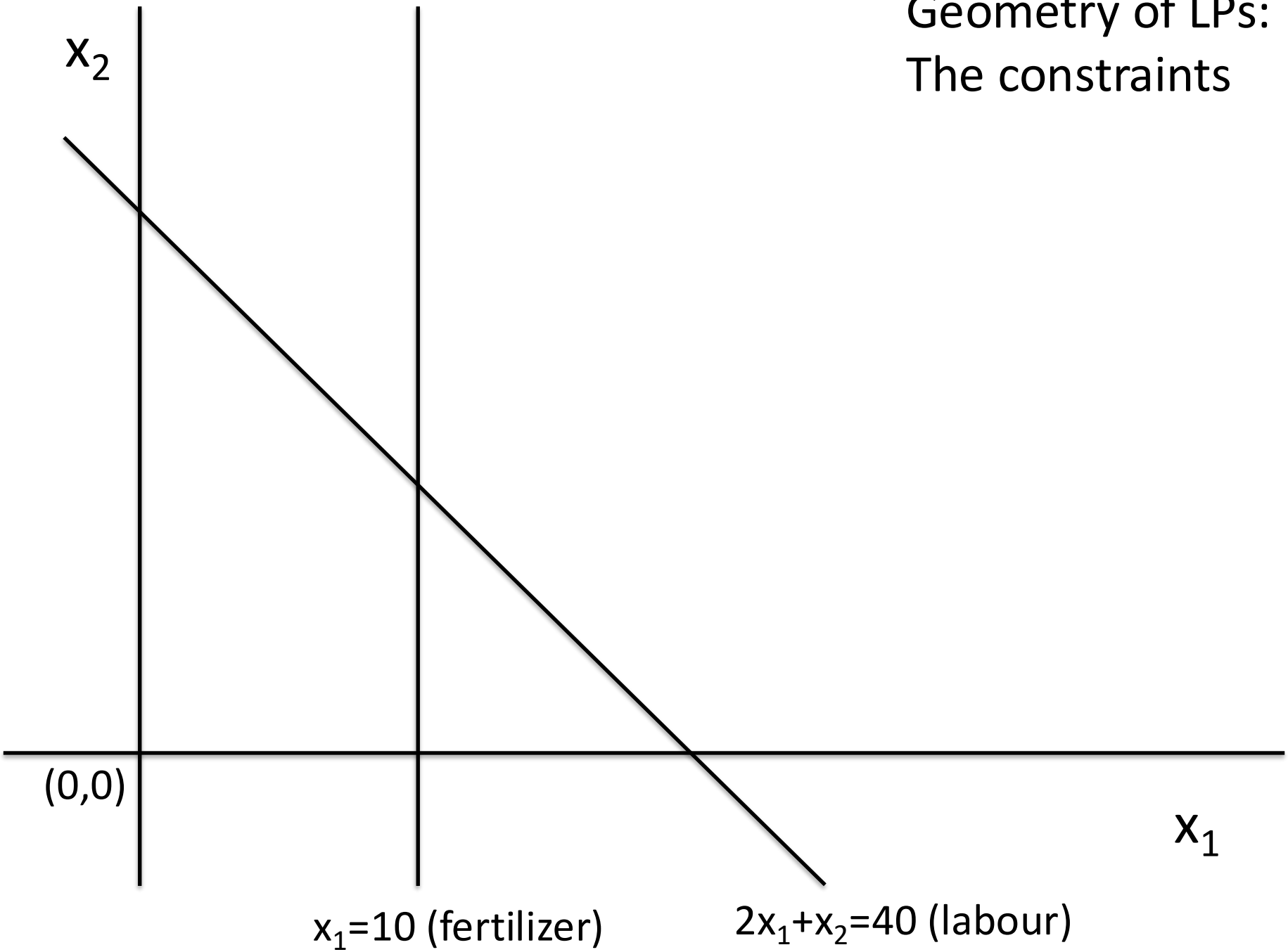
Geometry of LPs: The constraints



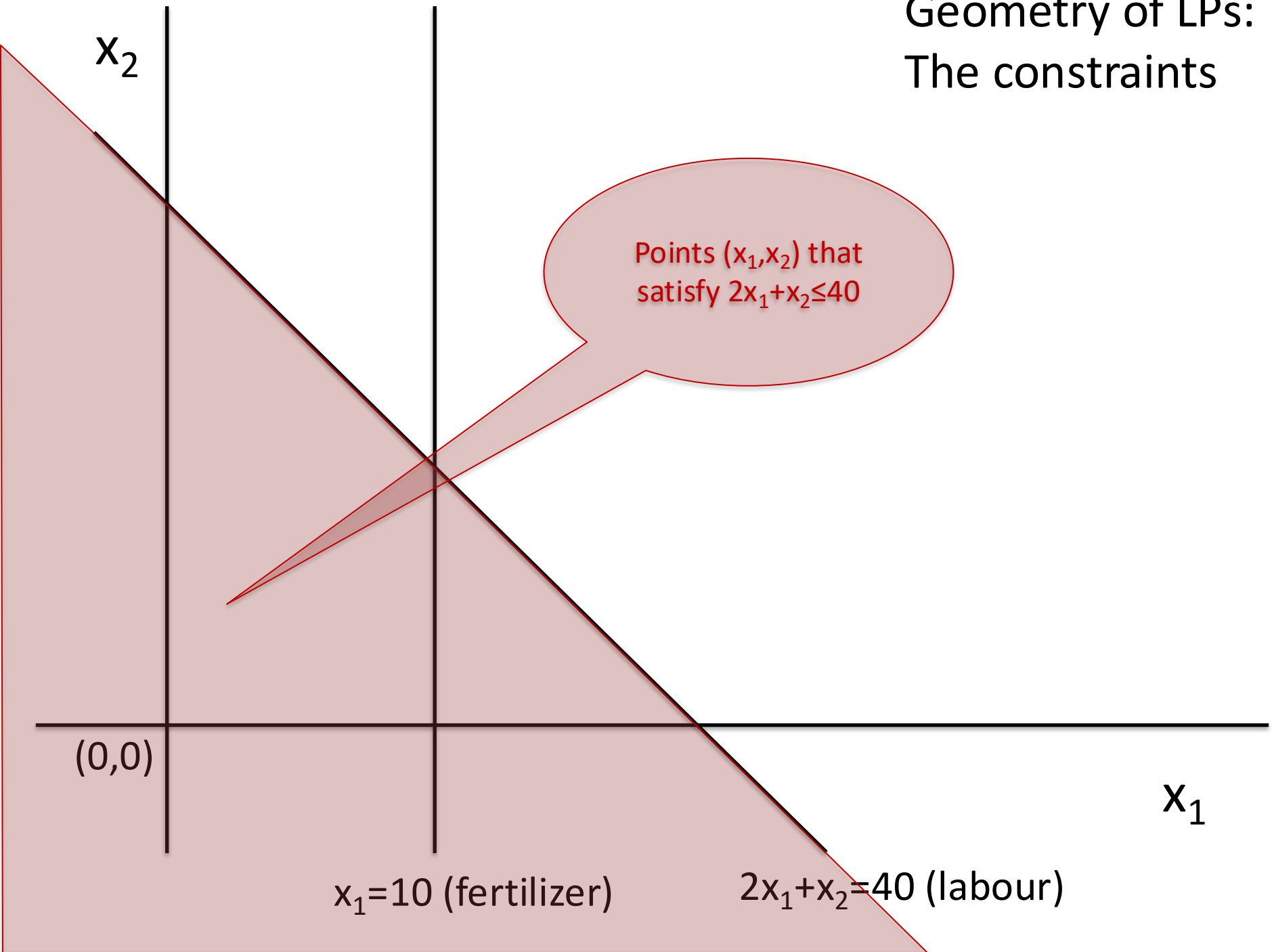
Geometry of LPs: The constraints



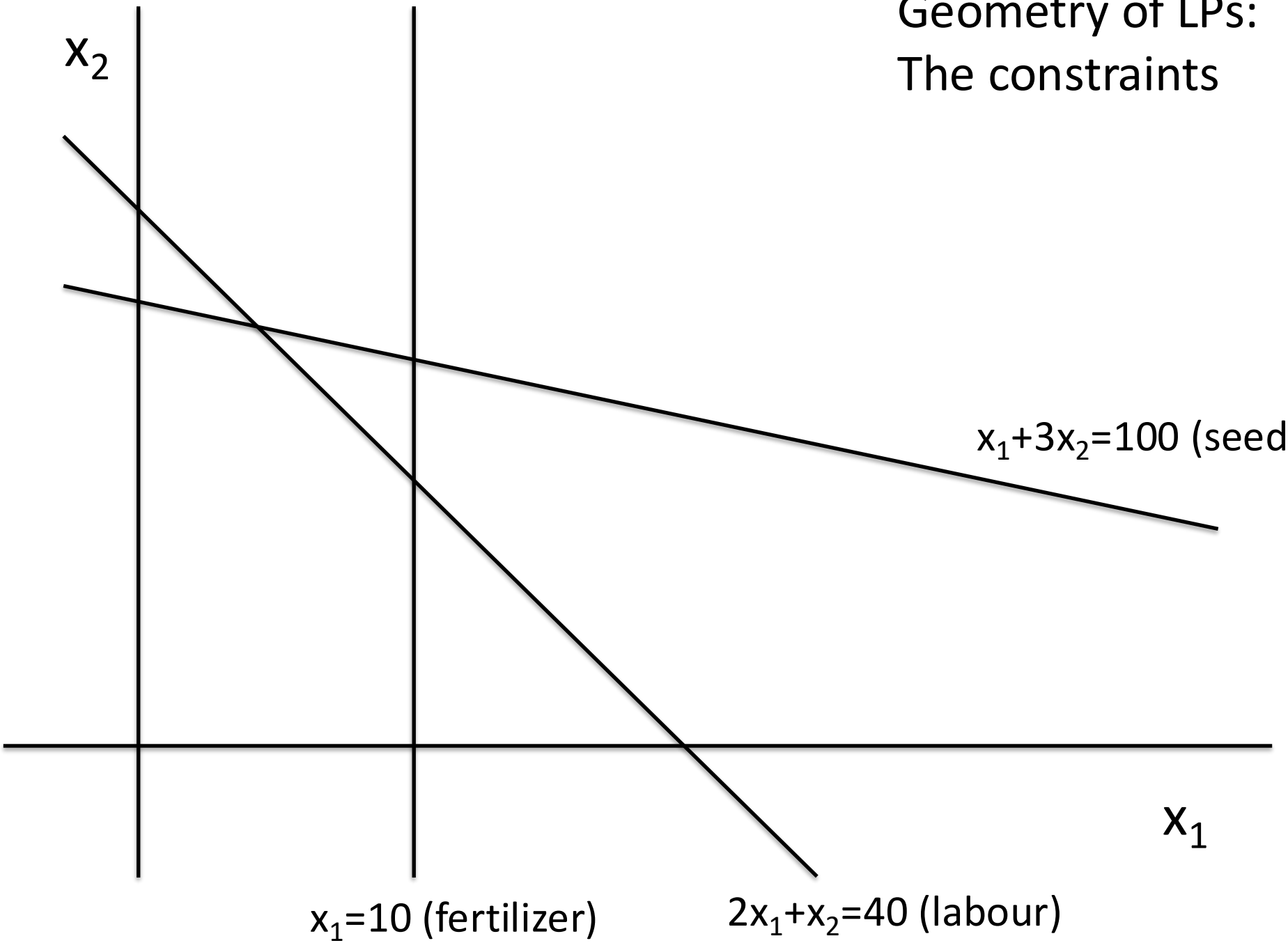
Geometry of LPs: The constraints



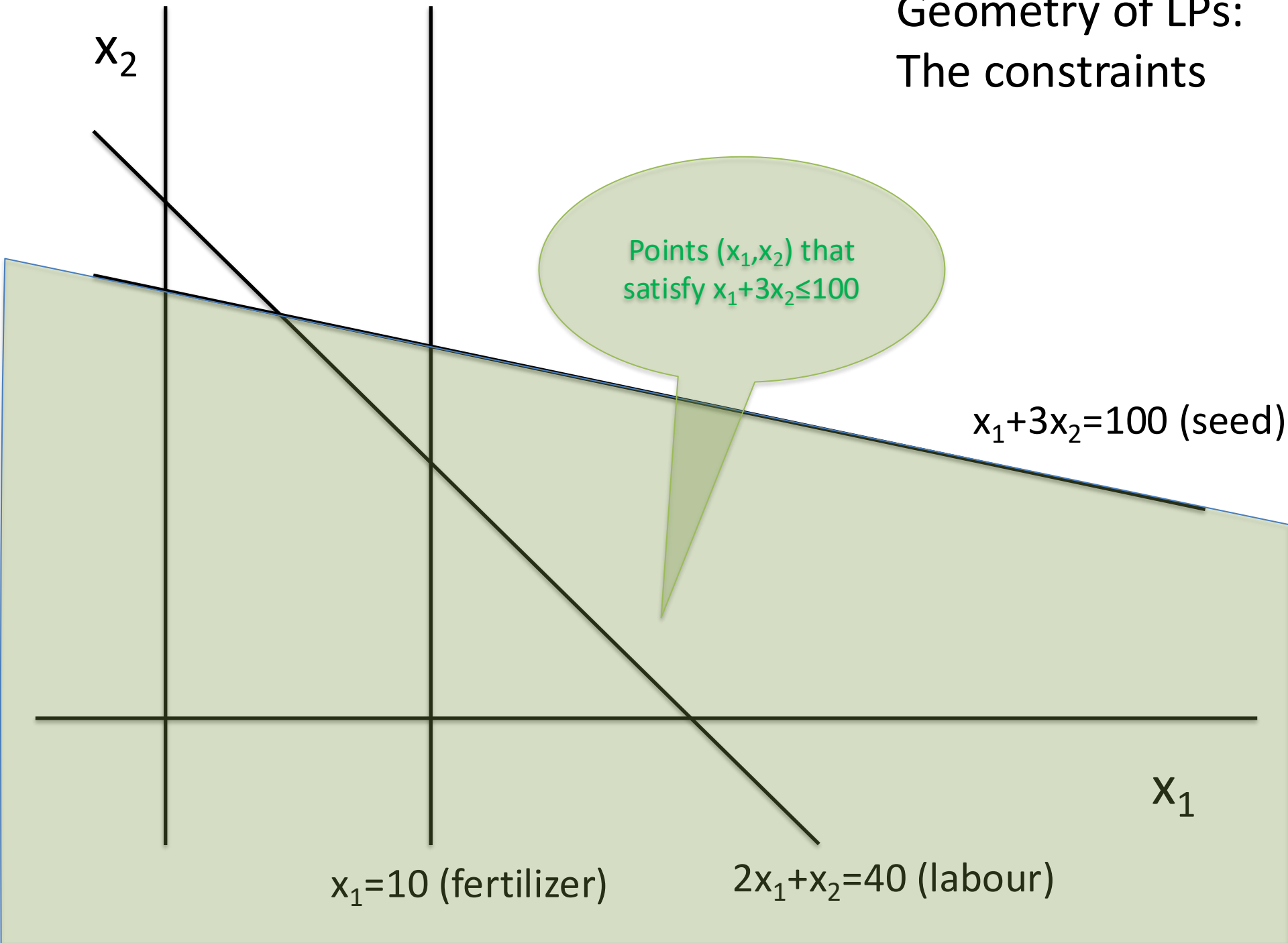
Geometry of LPs: The constraints

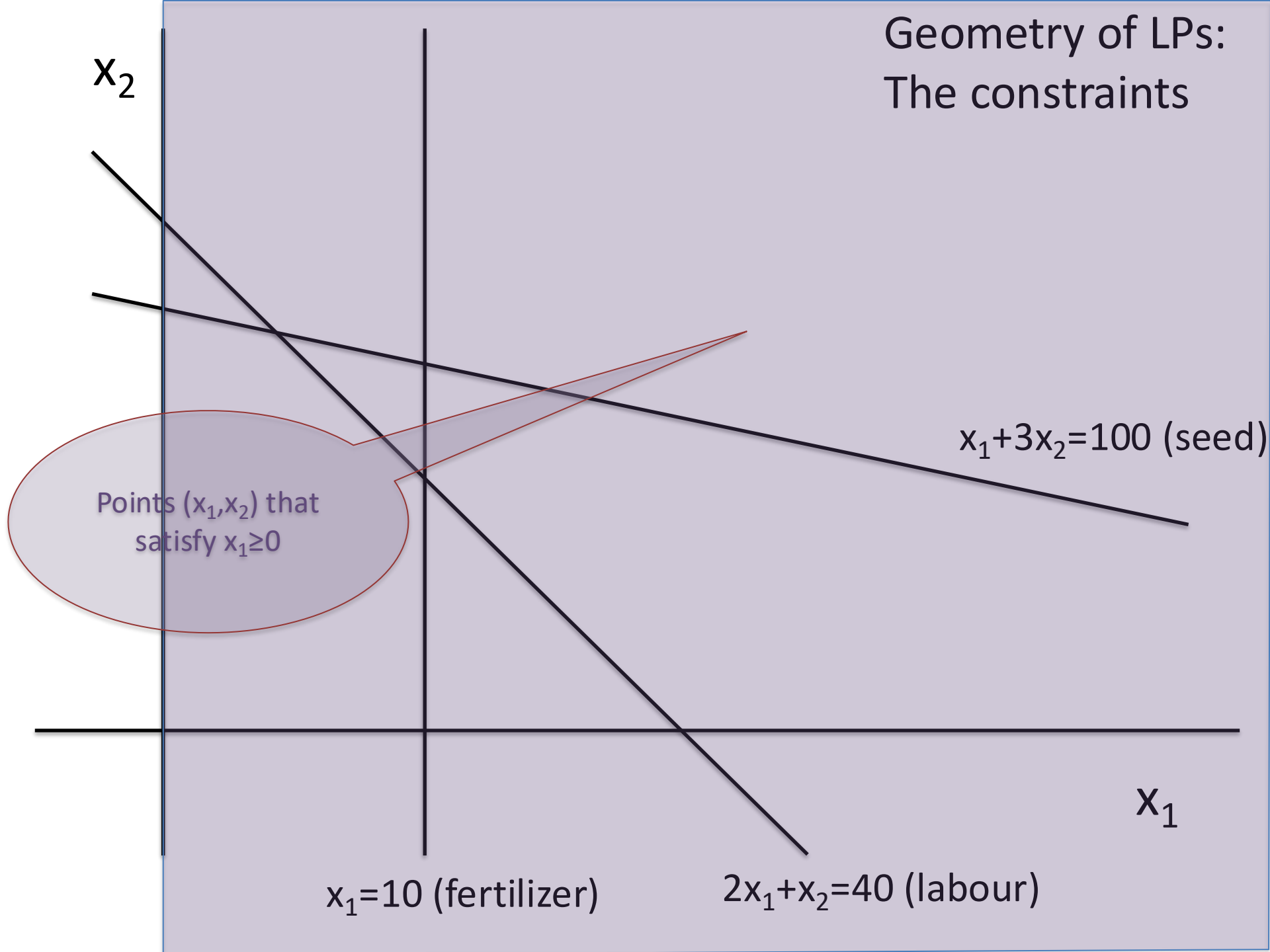


Geometry of LPs: The constraints

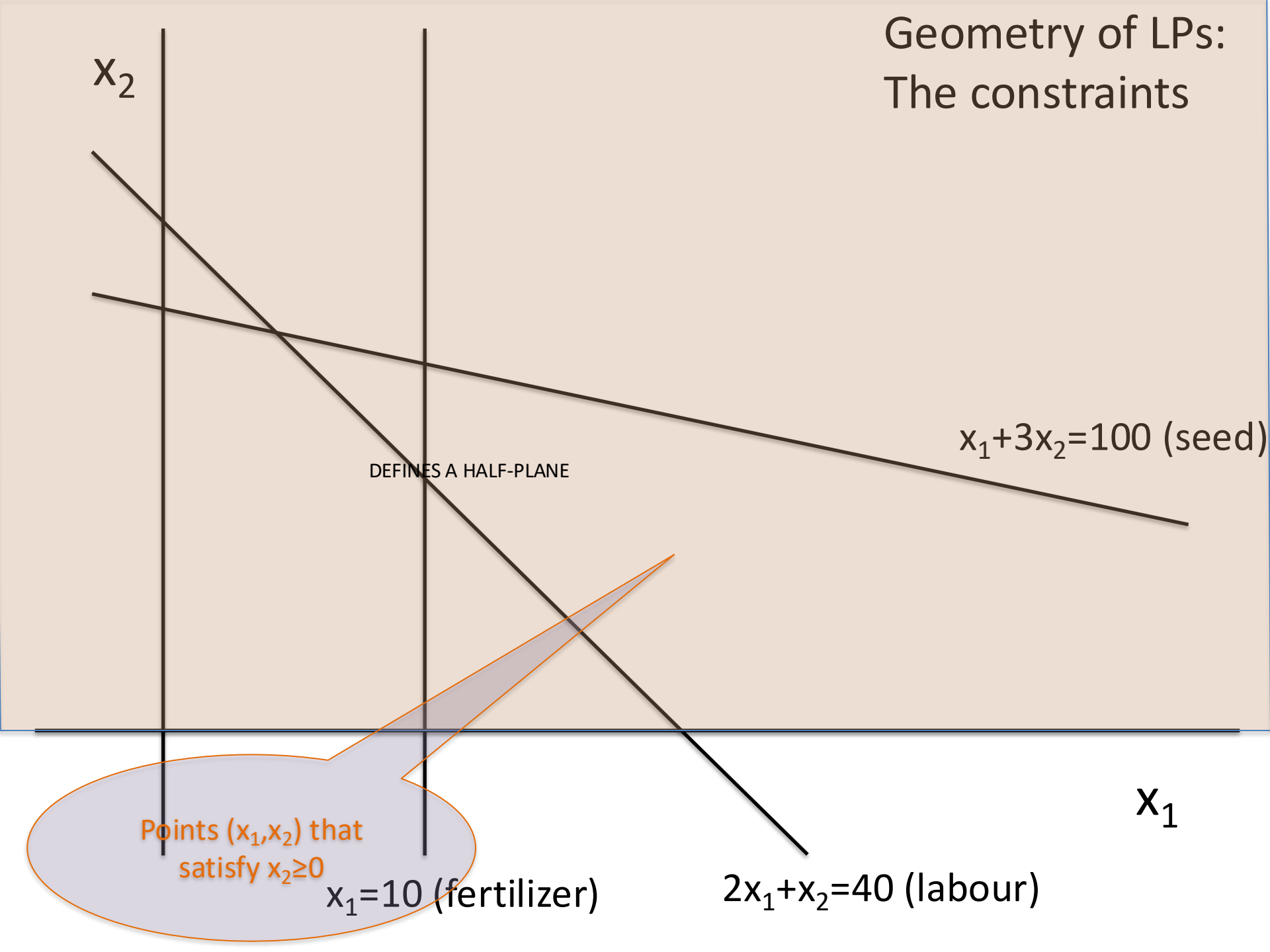


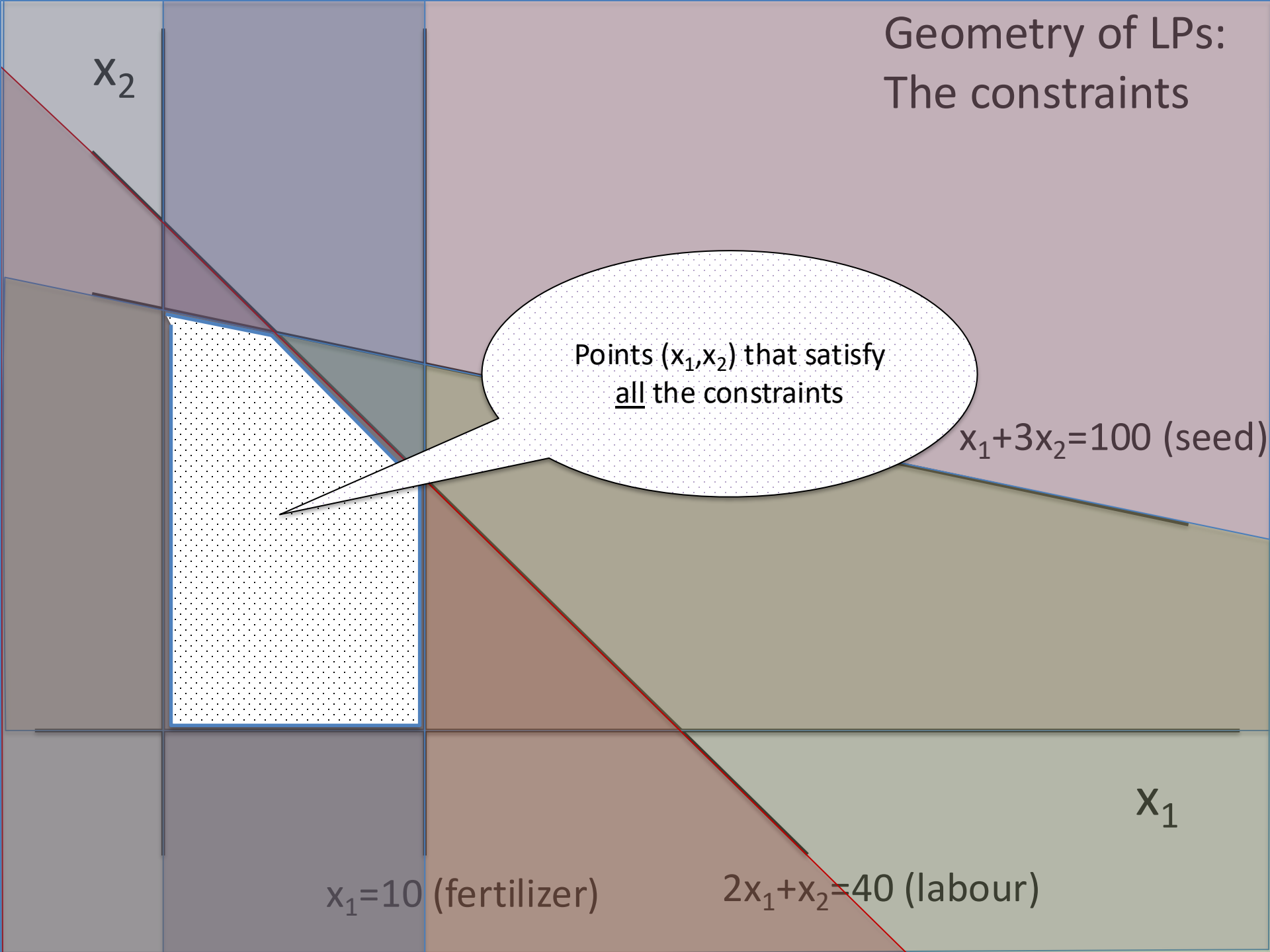
Geometry of LPs: The constraints



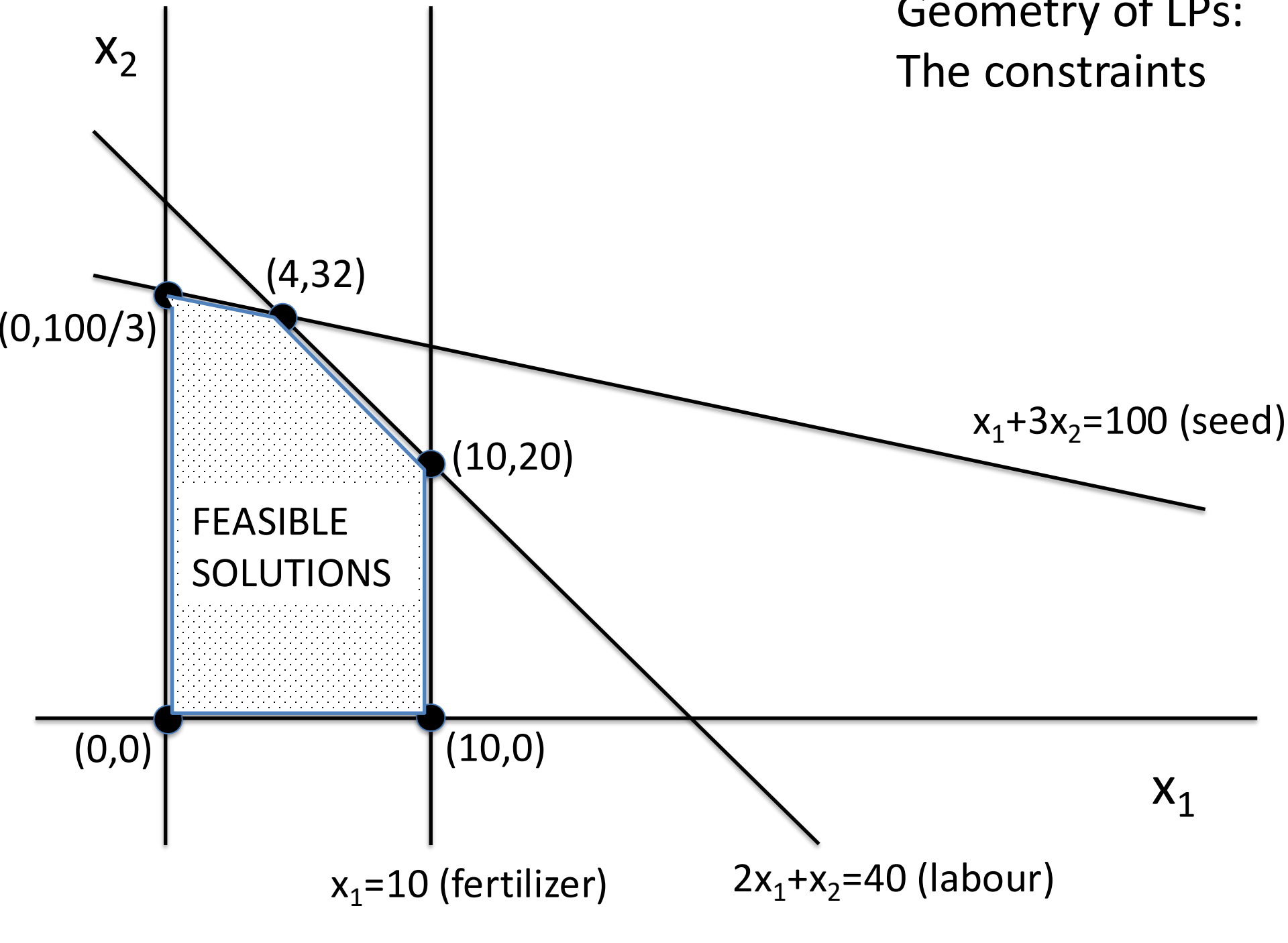

Geometry of LPs:
The constraints
x_2
x_1
$x_1+3x_2=100$ (seed)
$2x_1+x_2=40$ (labour)
$x_1=10$ (fertilizer)
Points (x_1,x_2) that satisfy $x_1 \geq 0$

Geometry of LPs: The constraints

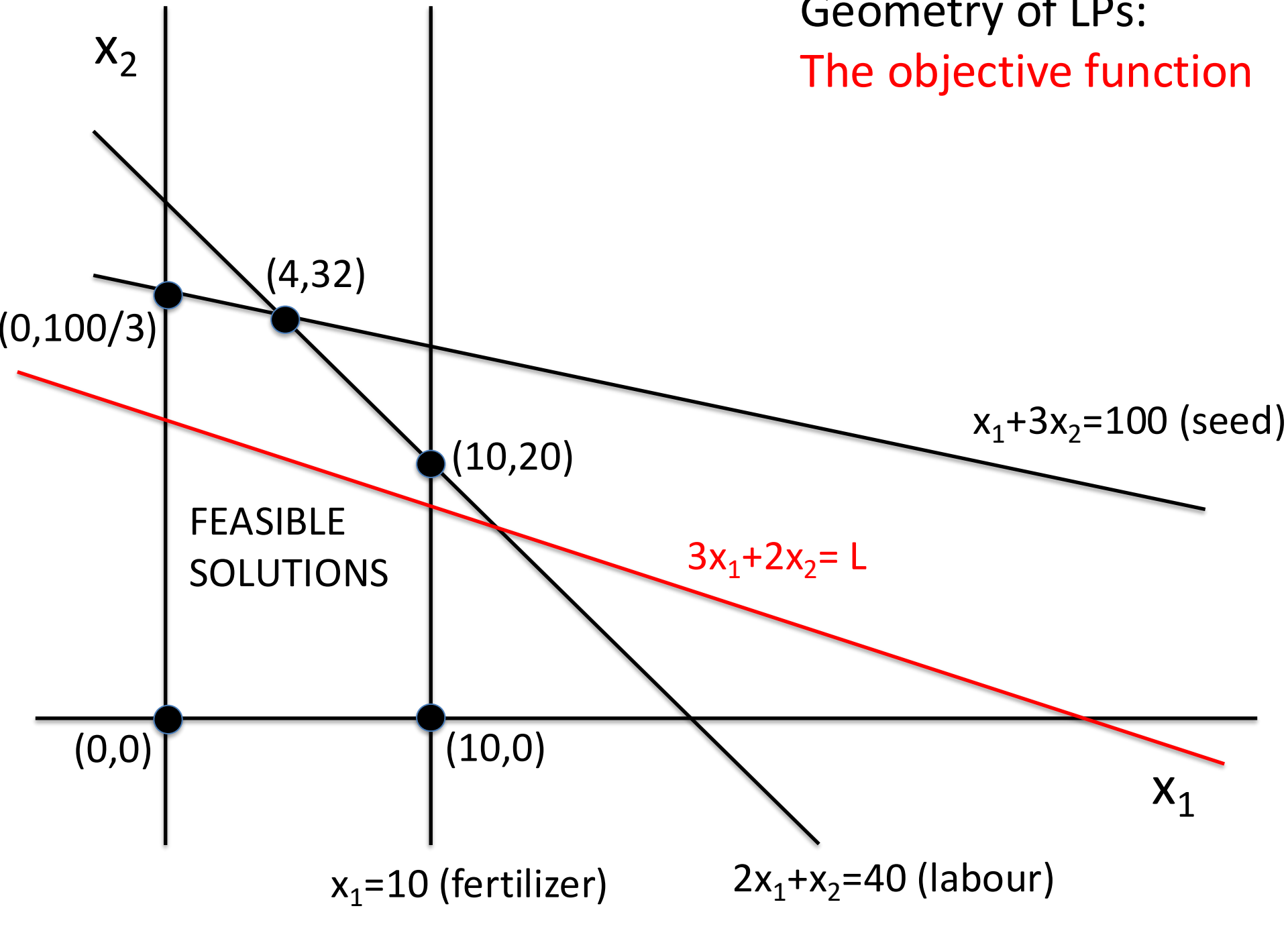


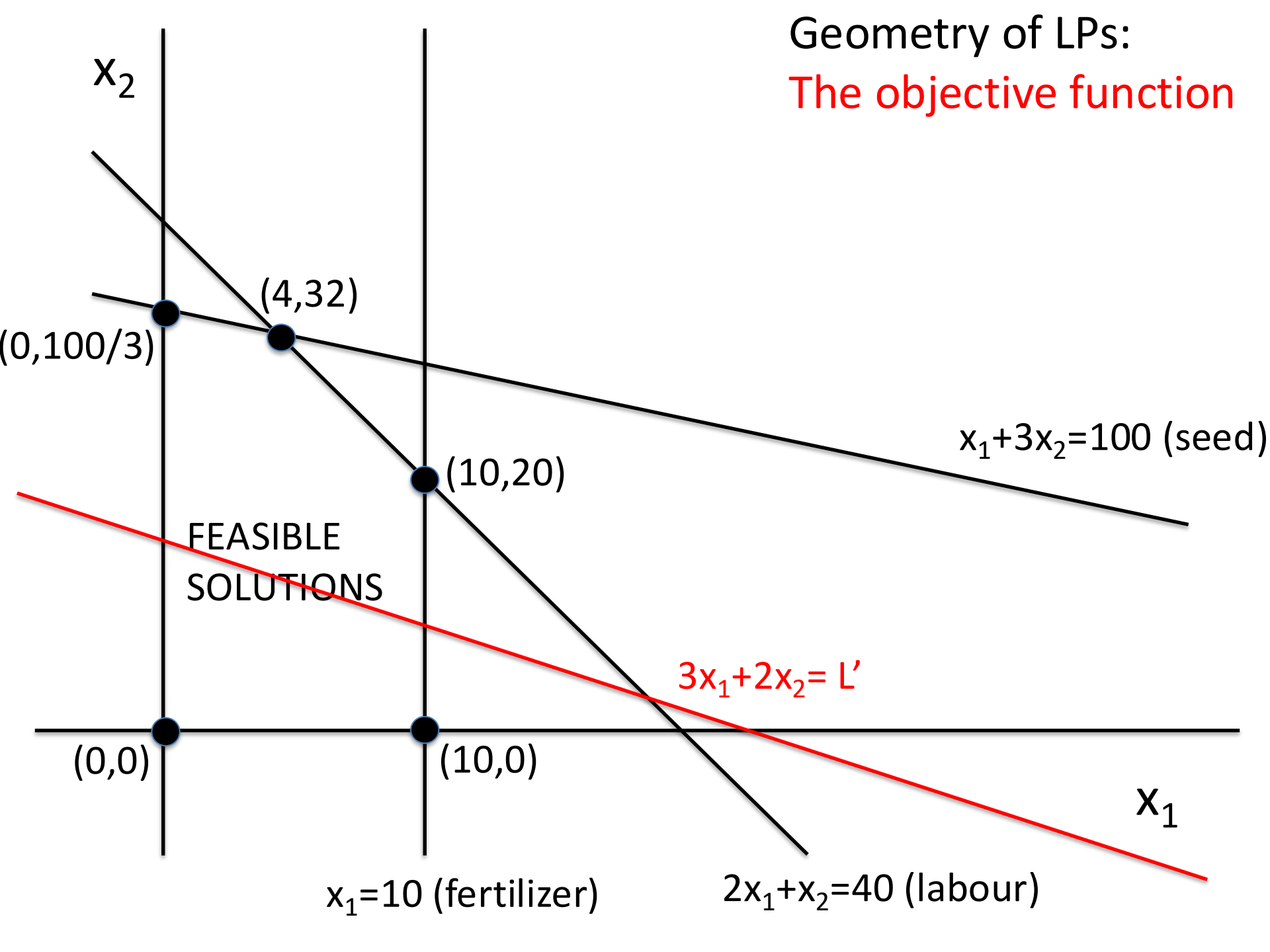

Geometry of LPs:
The constraints
x_2
Points (x_1,x_2) that satisfy
all the constraints
$x_1+3x_2=100$ (seed)
x_1
$x_1=10$ (fertilizer)
$2x_1+x_2=40$ (labour)

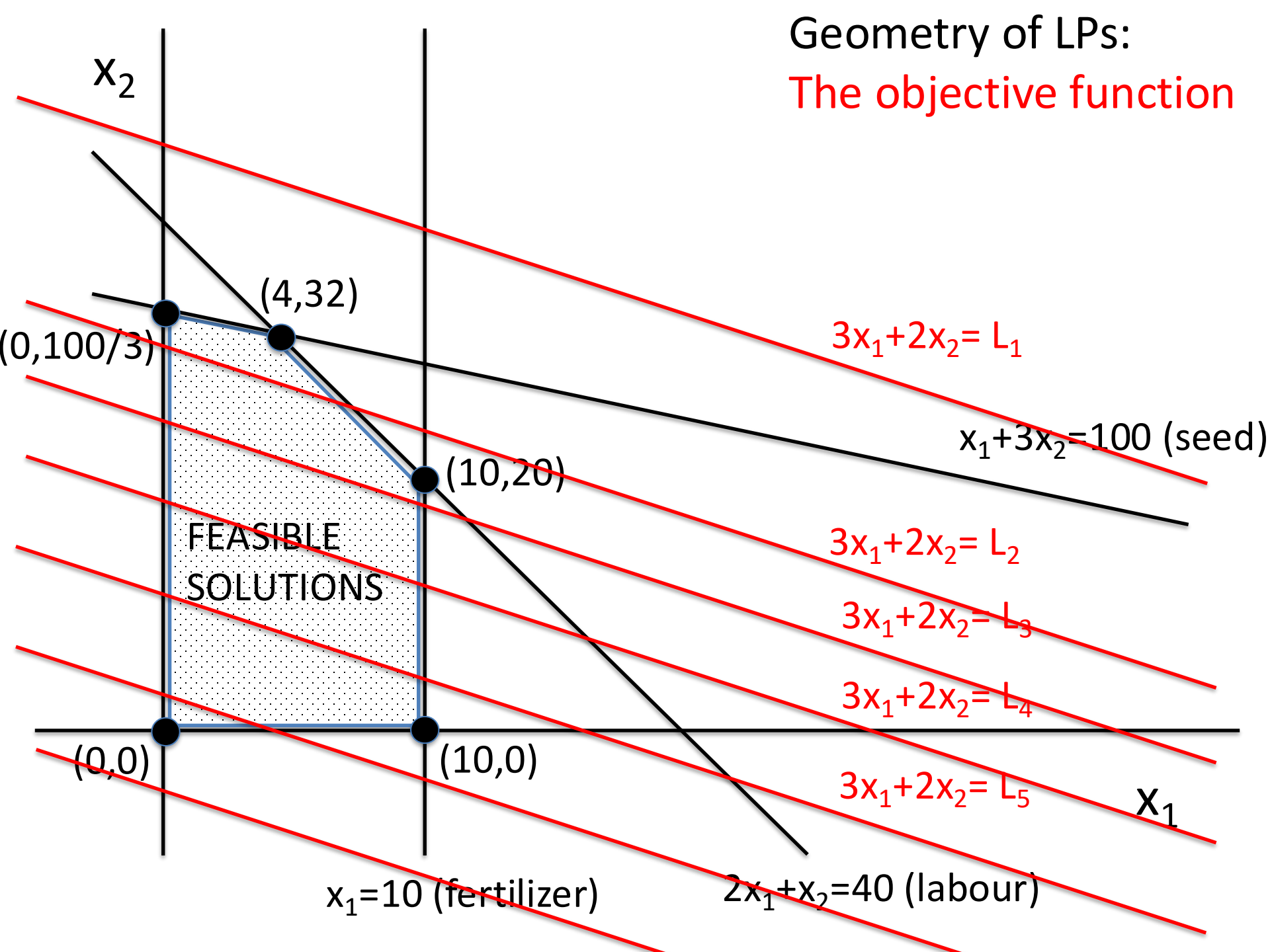
Geometry of LPs: The constraints

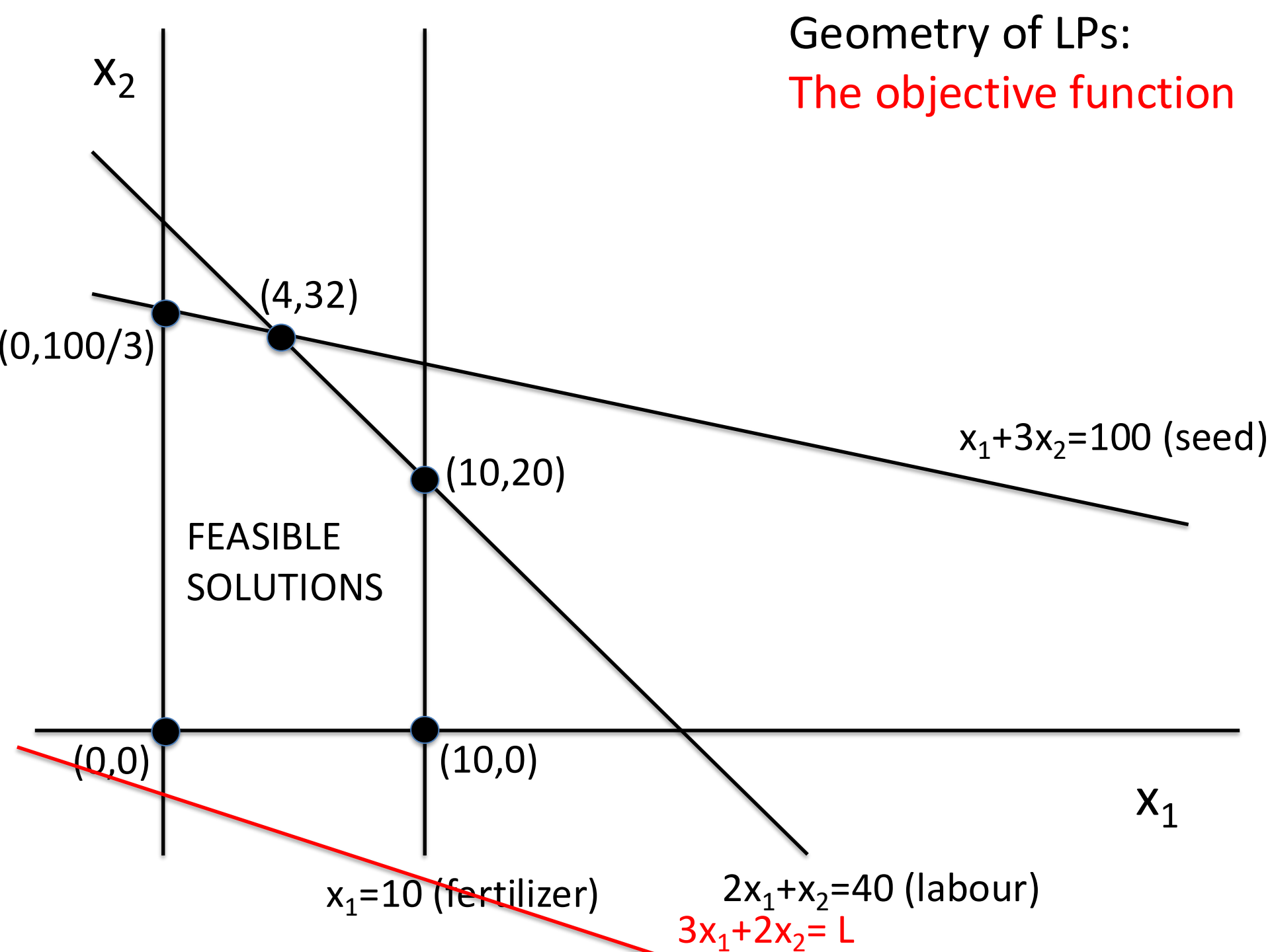


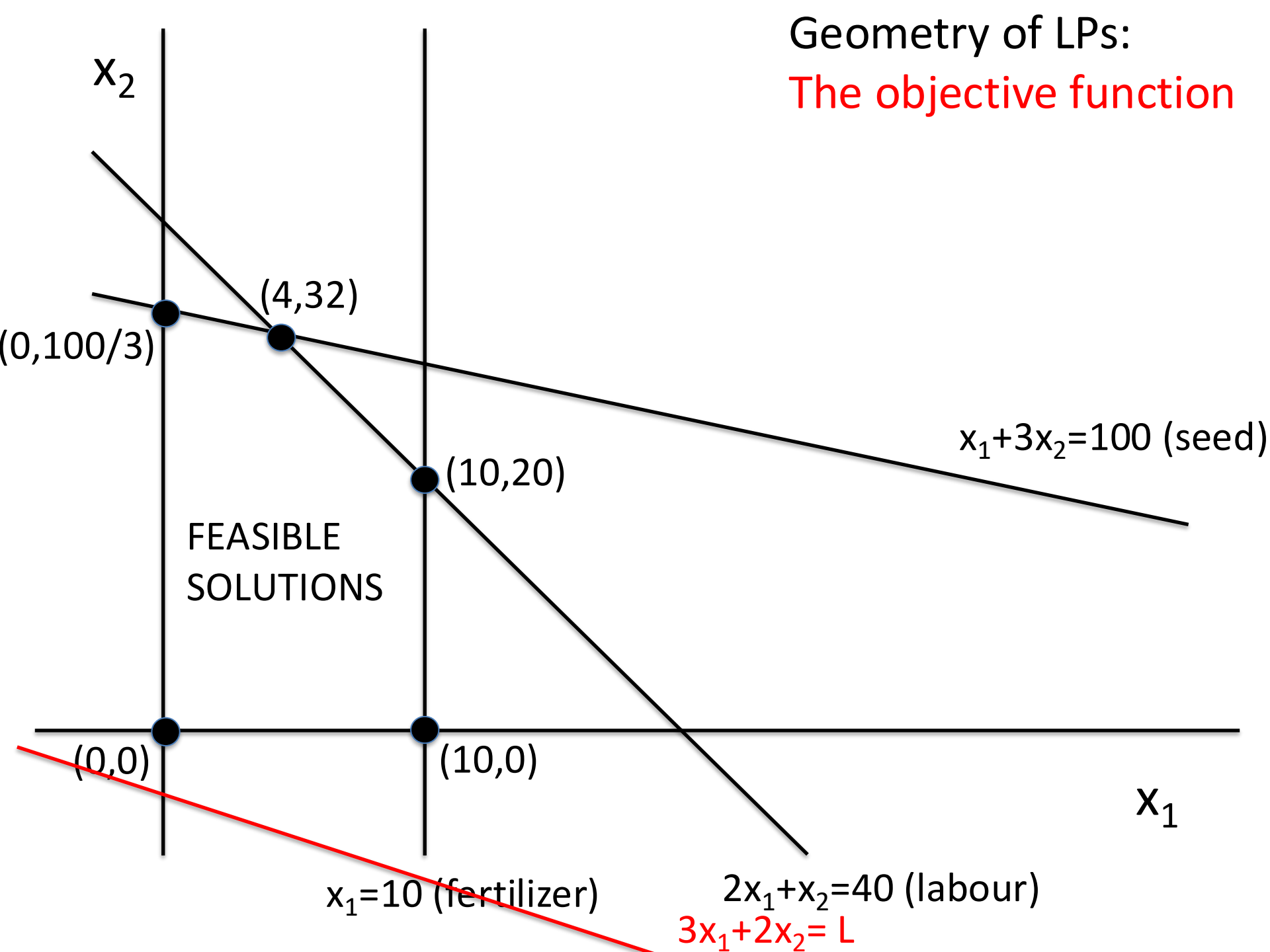
Geometry of LPs:
The objective function

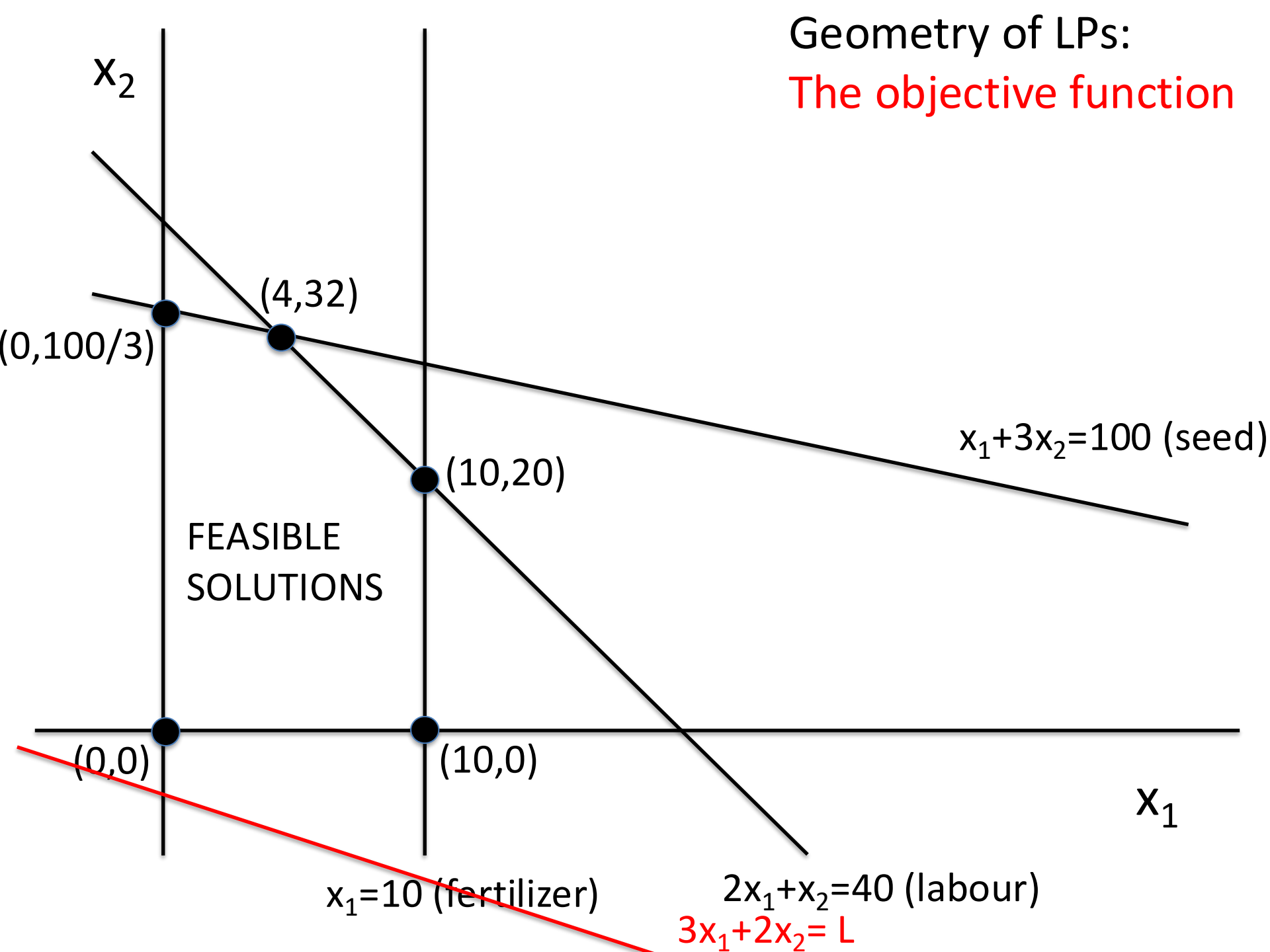


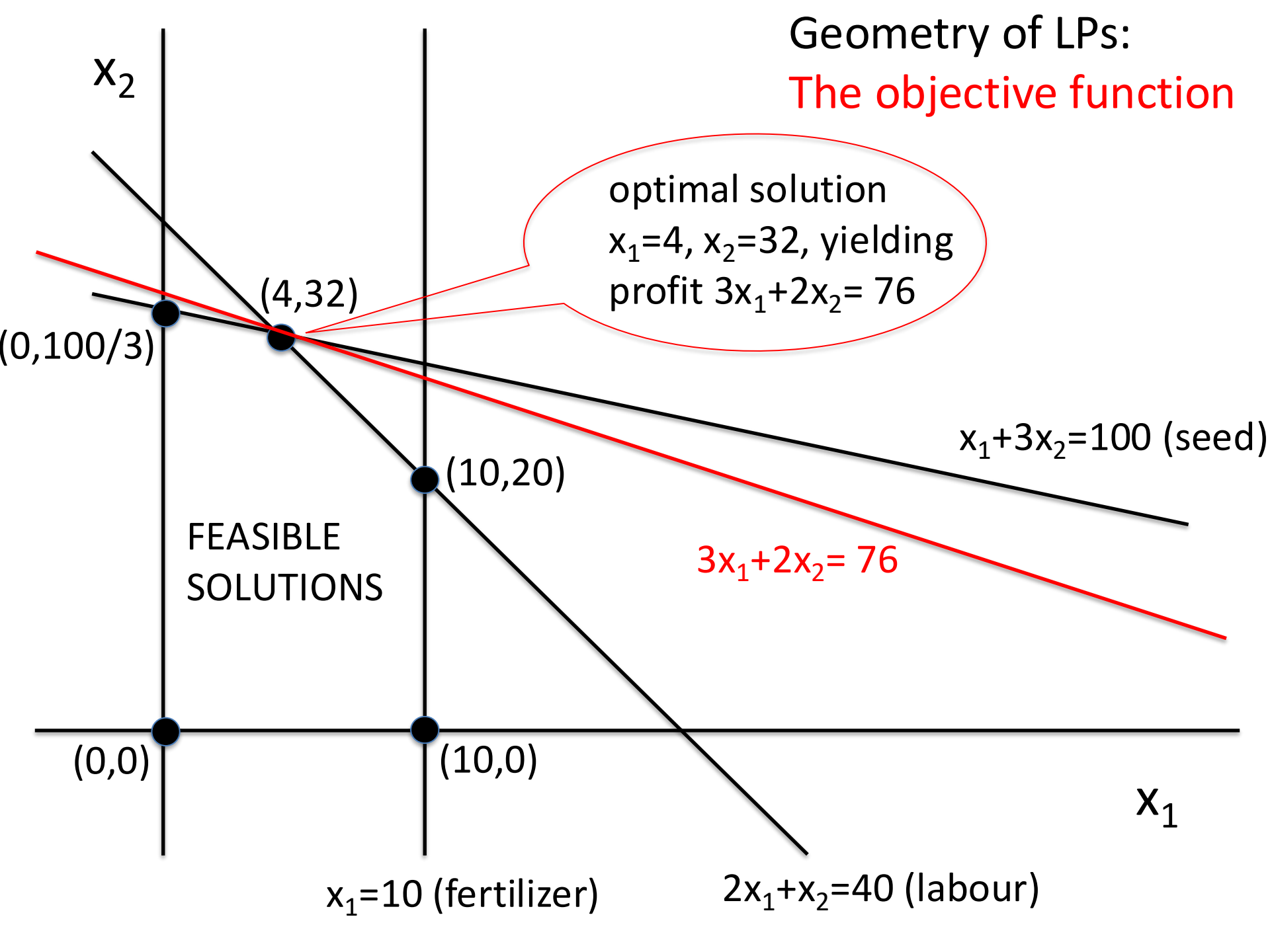

Geometry of LPs:
The objective function
x_2
(4,32)
(0,100/3)
$x_1+3x_2=100$ (seed)
(10,20)
FEASIBLE
SOLUTIONS
$3x_1+2x_2= L'$
(0,0)
(10,0)
x_1
$x_1=10$ (fertilizer)
$2x_1+x_2=40$ (labour)


Geometry of LPs:
The objective function
x_2
x_1
(4,32)
(0,100/3)
(10,20)
(0,0)
(10,0)
FEASIBLE SOLUTIONS
$3x_1+2x_2= L_1$
$3x_1+2x_2= L_2$
$3x_1+2x_2= L_3$
$3x_1+2x_2= L_4$
$3x_1+2x_2= L_5$
$x_1+3x_2=100$ (seed)
$x_1=10$ (fertilizer)
$2x_1+x_2=40$ (labour)

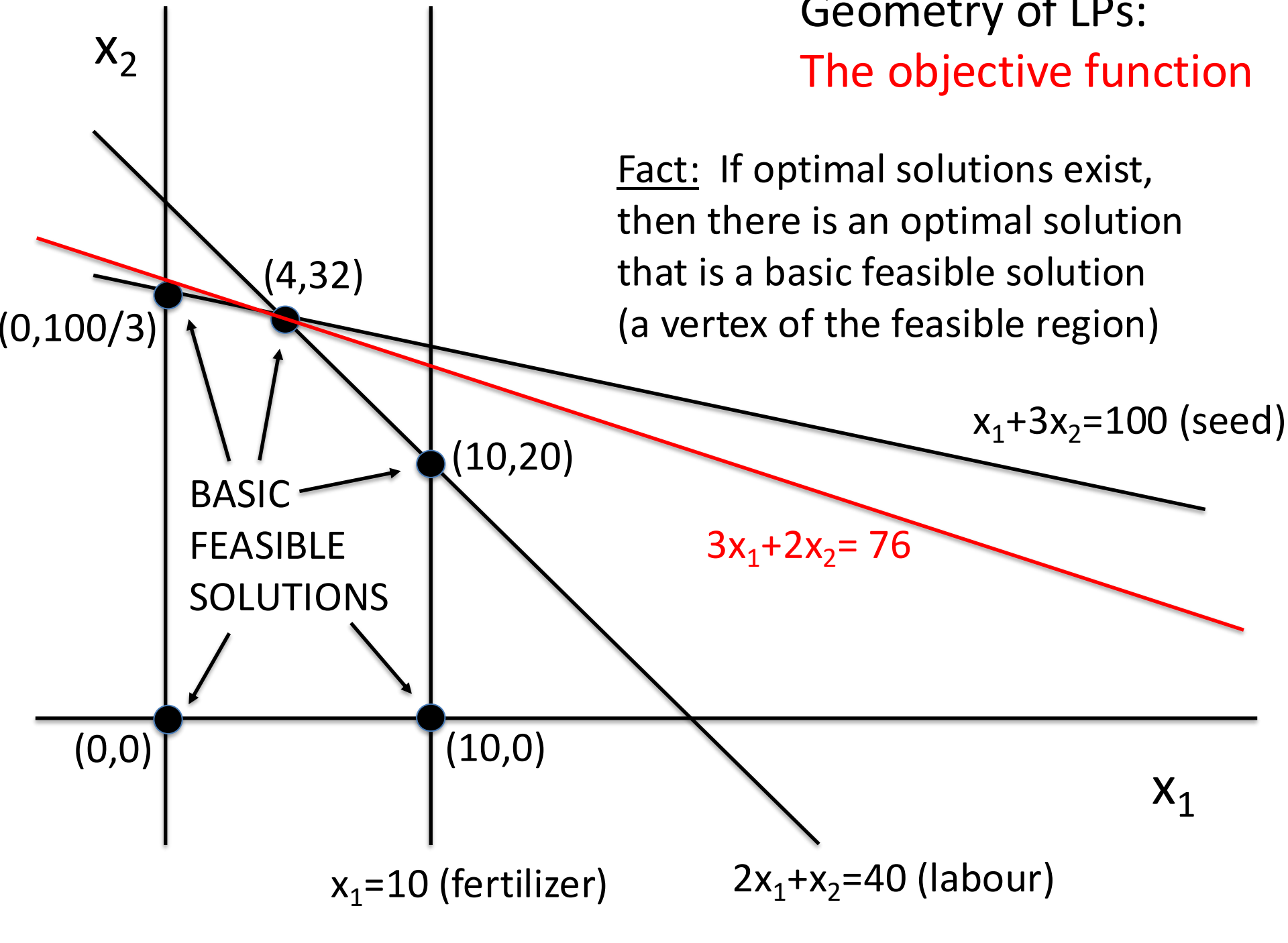

Geometry of LPs:
The objective function
x_2
(4,32)
(0,100/3)
$x_1+3x_2=100$ (seed)
(10,20)
FEASIBLE
SOLUTIONS
(0,0)
(10,0)
x_1
$x_1=10$ (fertilizer)
$2x_1+x_2=40$ (labour)
$3x_1+2x_2= L$


Geometry of LPs:
The objective function
x_2
(4,32)
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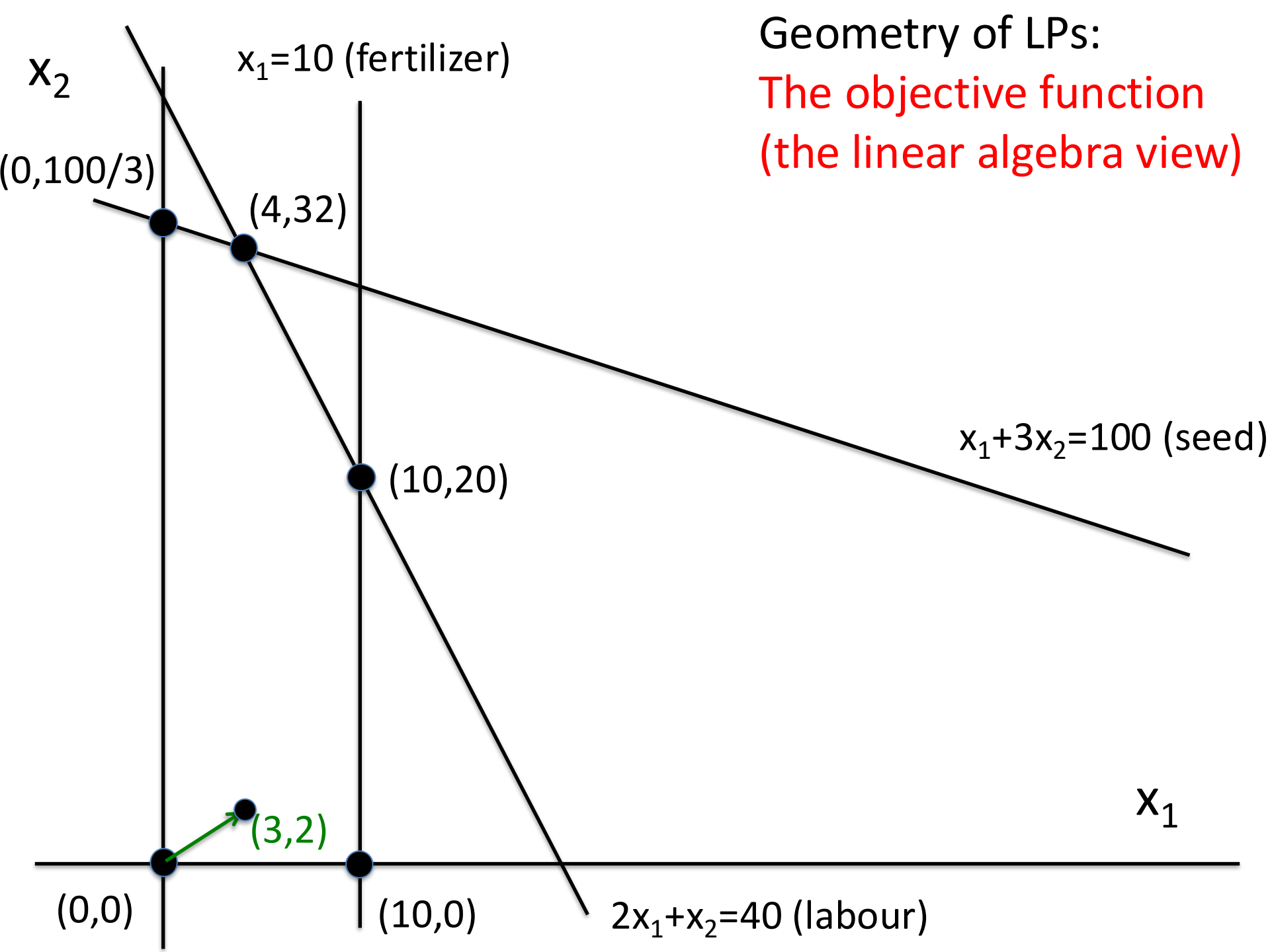

Geometry of LPs:
The objective function
x_2
(4,32)
(0,100/3)
$x_1+3x_2=100$ (seed)
(10,20)
FEASIBLE
SOLUTIONS
(0,0)
(10,0)
x_1
$x_1=10$ (fertilizer)
$2x_1+x_2=40$ (labour)
$3x_1+2x_2= L$

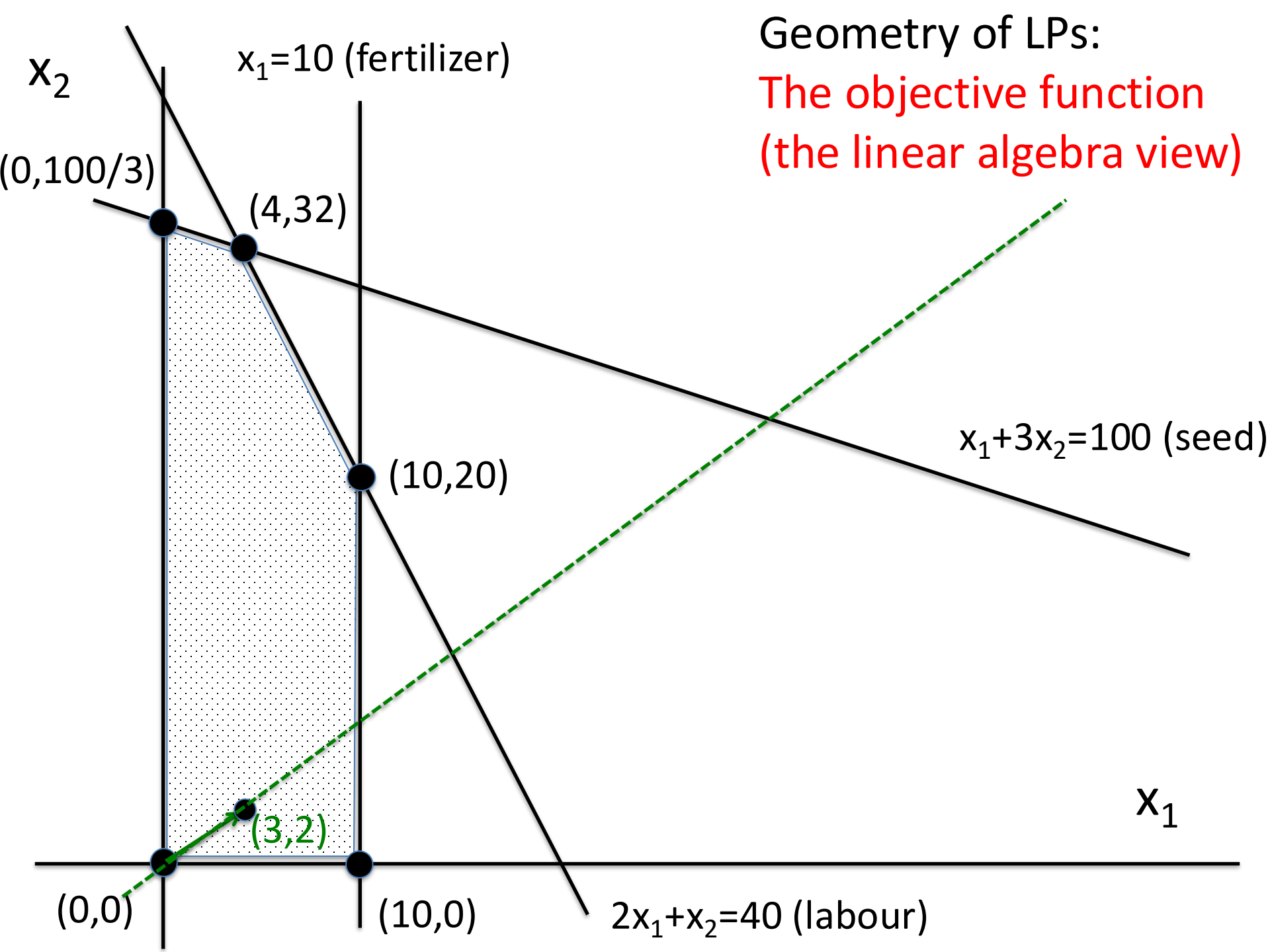

Geometry of LPs:
The objective function
x_2
optimal solution
x_1=4, x_2=32, yielding
profit $3x_1+2x_2$= 76
(4,32)
(0,100/3)
x_1+3x_2=100 (seed)
(10,20)
FEASIBLE
SOLUTIONS
$3x_1+2x_2$= 76
(0,0)
(10,0)
x_1
x_1=10 (fertilizer)
$2x_1+x_2$=40 (labour)

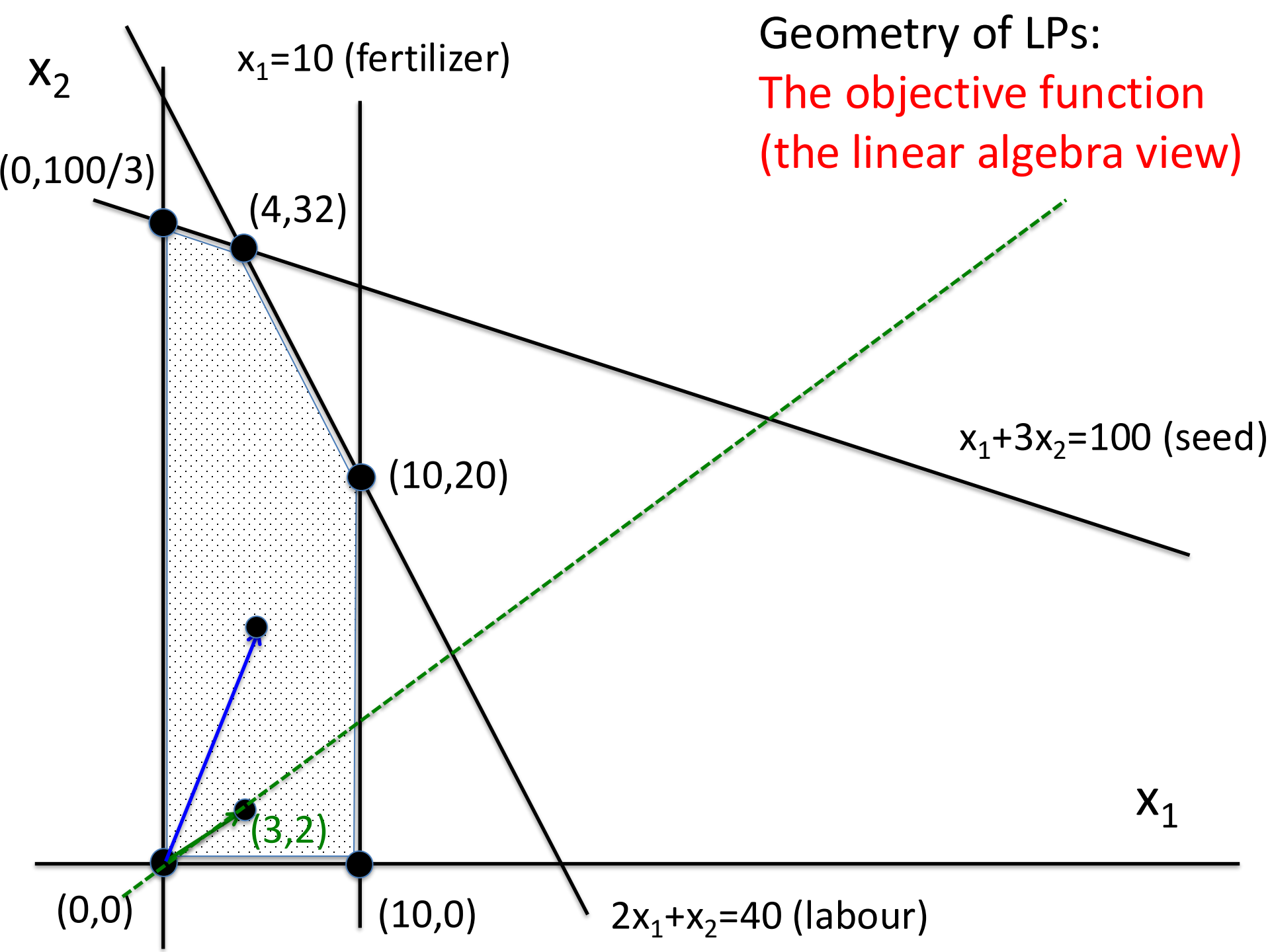
Geometry of LPs:
The objective function

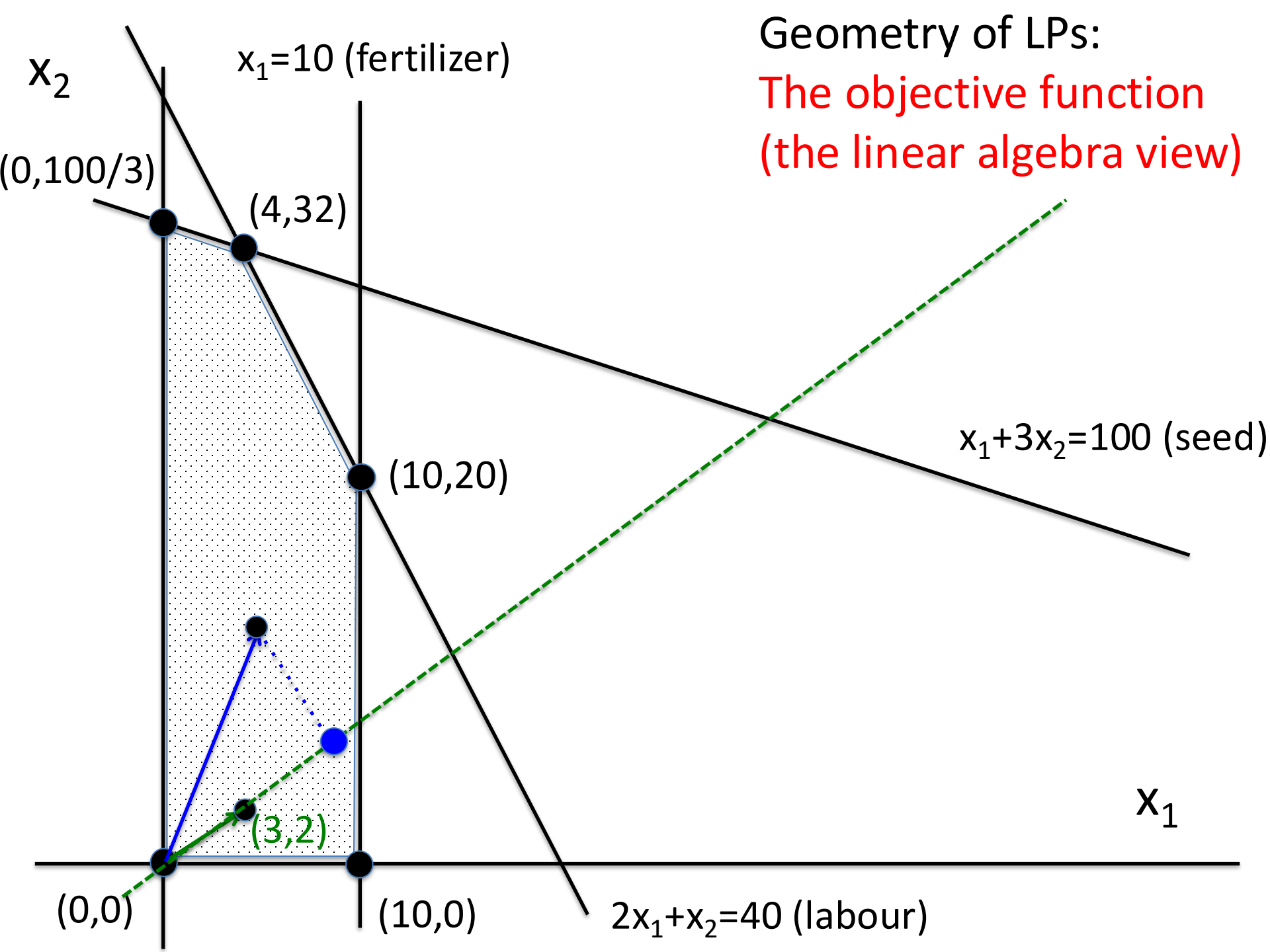


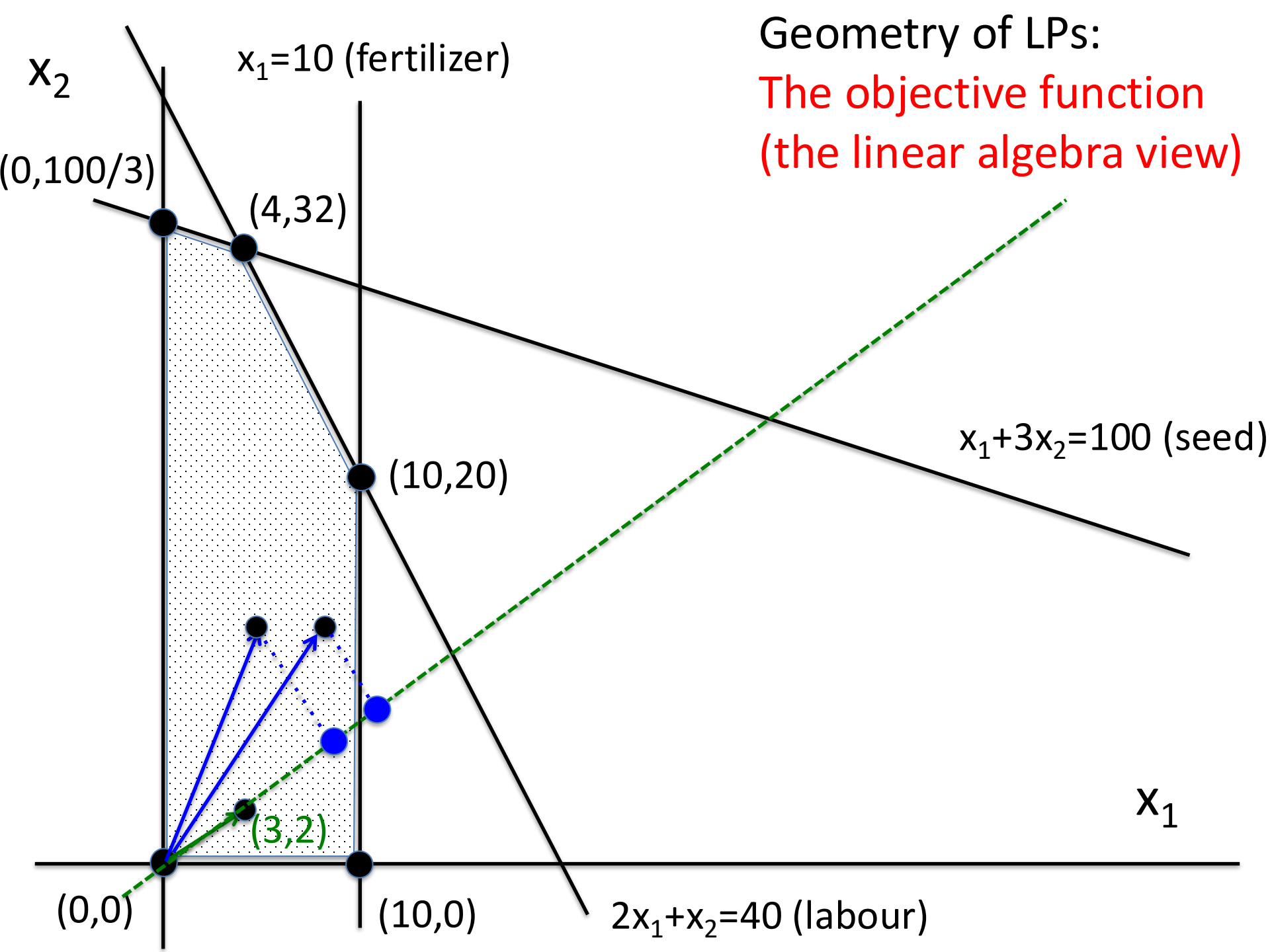
Fact: If optimal solutions exist, then there is an optimal solution that is a basic feasible solution (a vertex of the feasible region)

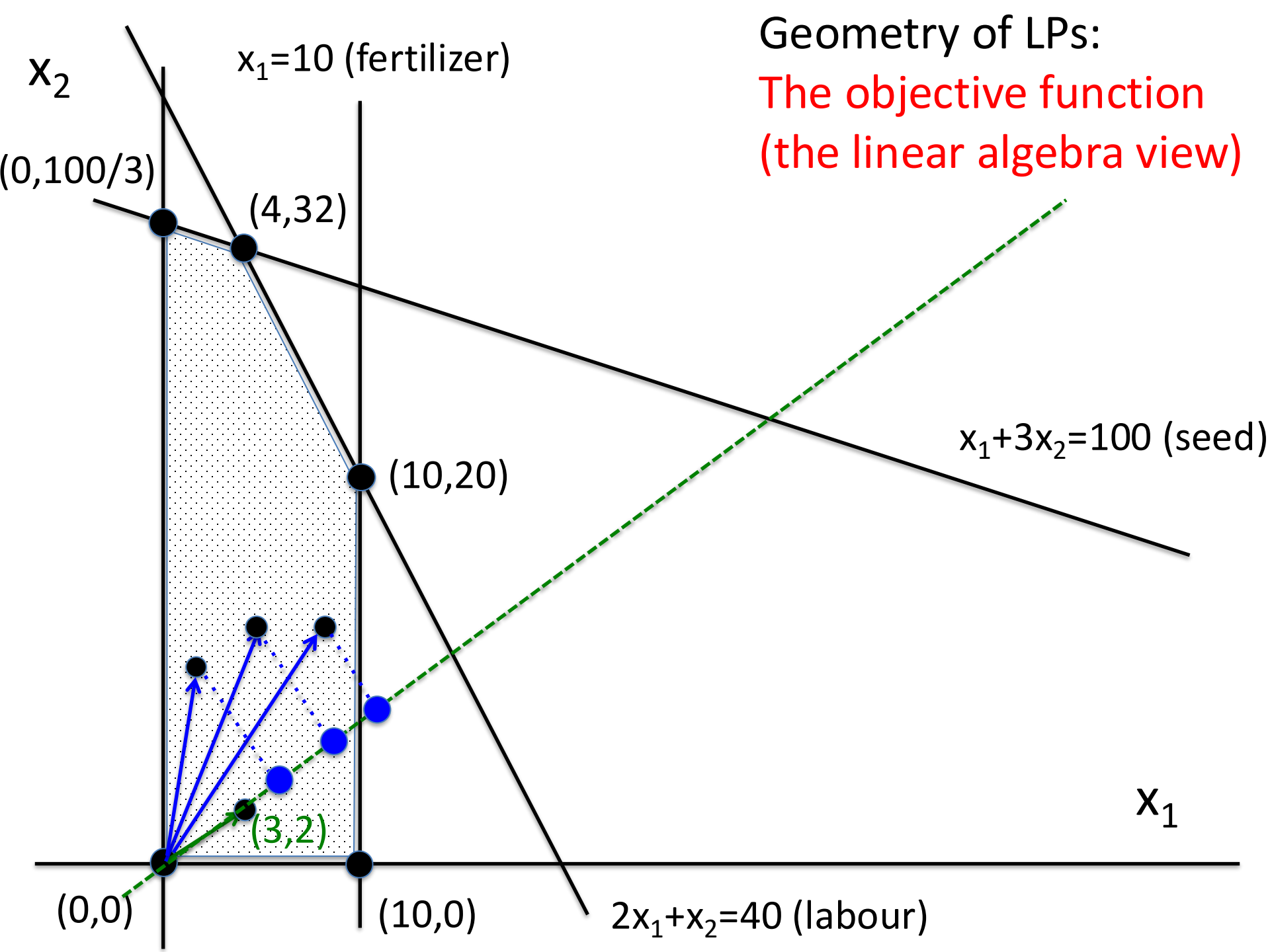

Geometry of LPs:
The objective function
(the linear algebra view)
x_2
x_1=10 (fertilizer)
(0,100/3)
(4,32)
$x_1+3x_2=100$ (seed)
(10,20)
(3,2)
x_1
(0,0)
(10,0)
$2x_1+x_2=40$ (labour)

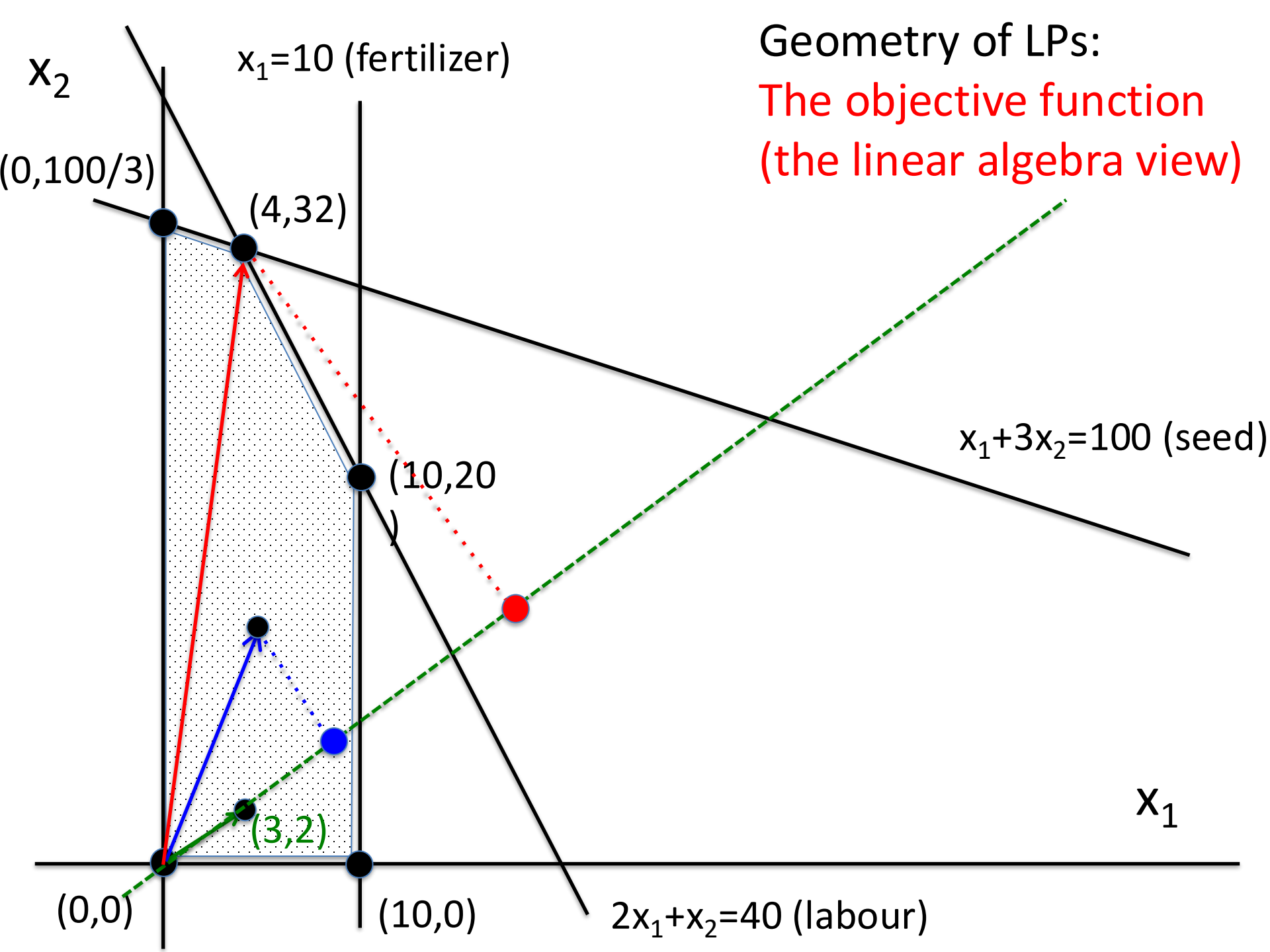

Geometry of LPs:
The objective function
(the linear algebra view)
x_2
x_1=10 (fertilizer)
(0,100/3)
(4,32)
(10,20)
$x_1+3x_2=100$ (seed)
(3,2)
x_1
(0,0)
(10,0)
$2x_1+x_2=40$ (labour)


Geometry of LPs:
The objective function
(the linear algebra view)
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x1
(3,2)
(0,0)
(10,0)
2x1+x2=40 (labour)

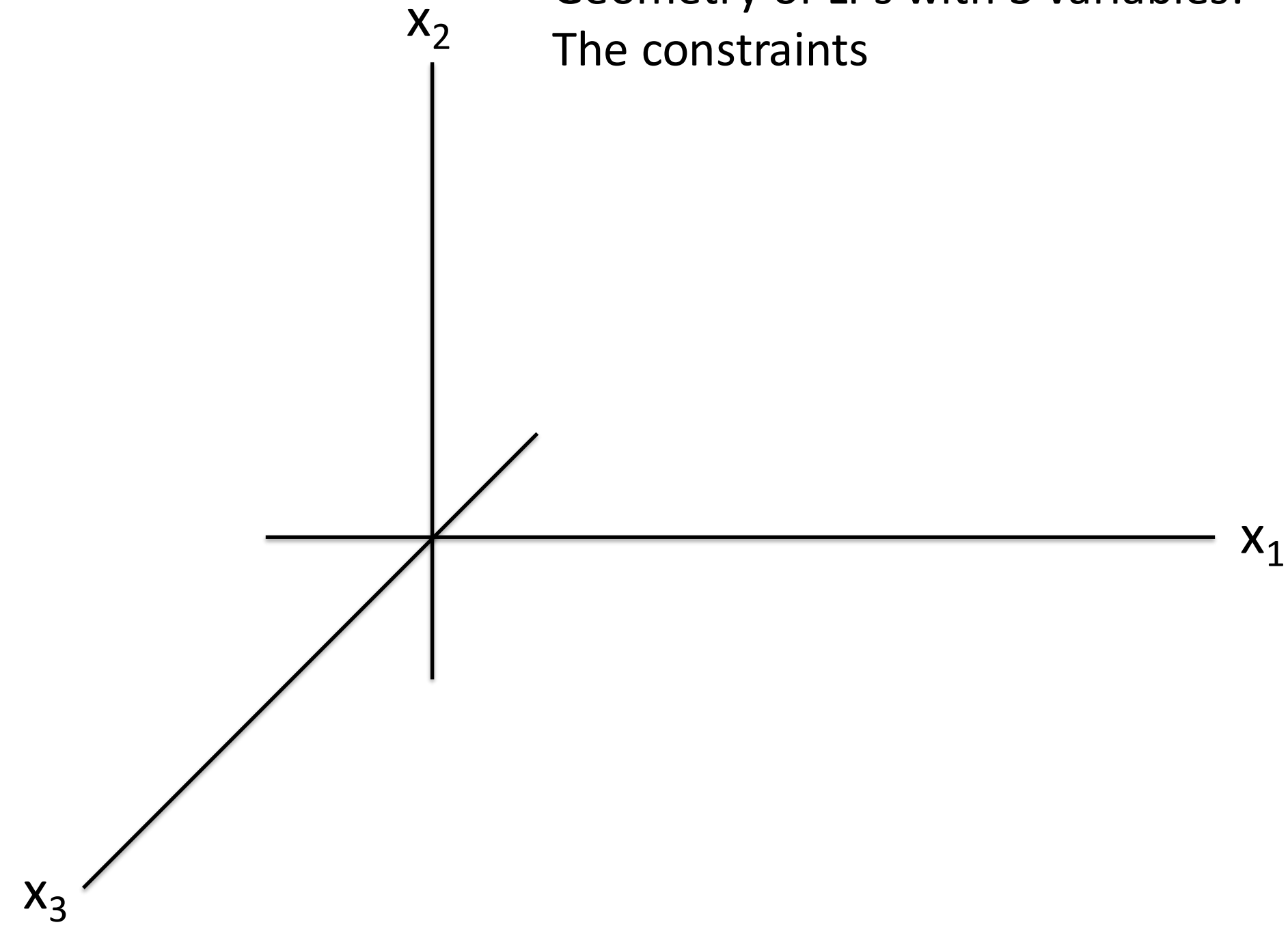

Geometry of LPs:
The objective function
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(4,32)
x1+3x2=100 (seed)
(10,20)
x1
(3,2)
(0,0)
(10,0)
2x1+x2=40 (labour)


Geometry of LPs:
The objective function
(the linear algebra view)
x2
x1=10 (fertilizer)
(0,100/3)
(4,32)
(10,20)
x1+3x2=100 (seed)
x1
(3,2)
(0,0)
(10,0)
2x1+x2=40 (labour)

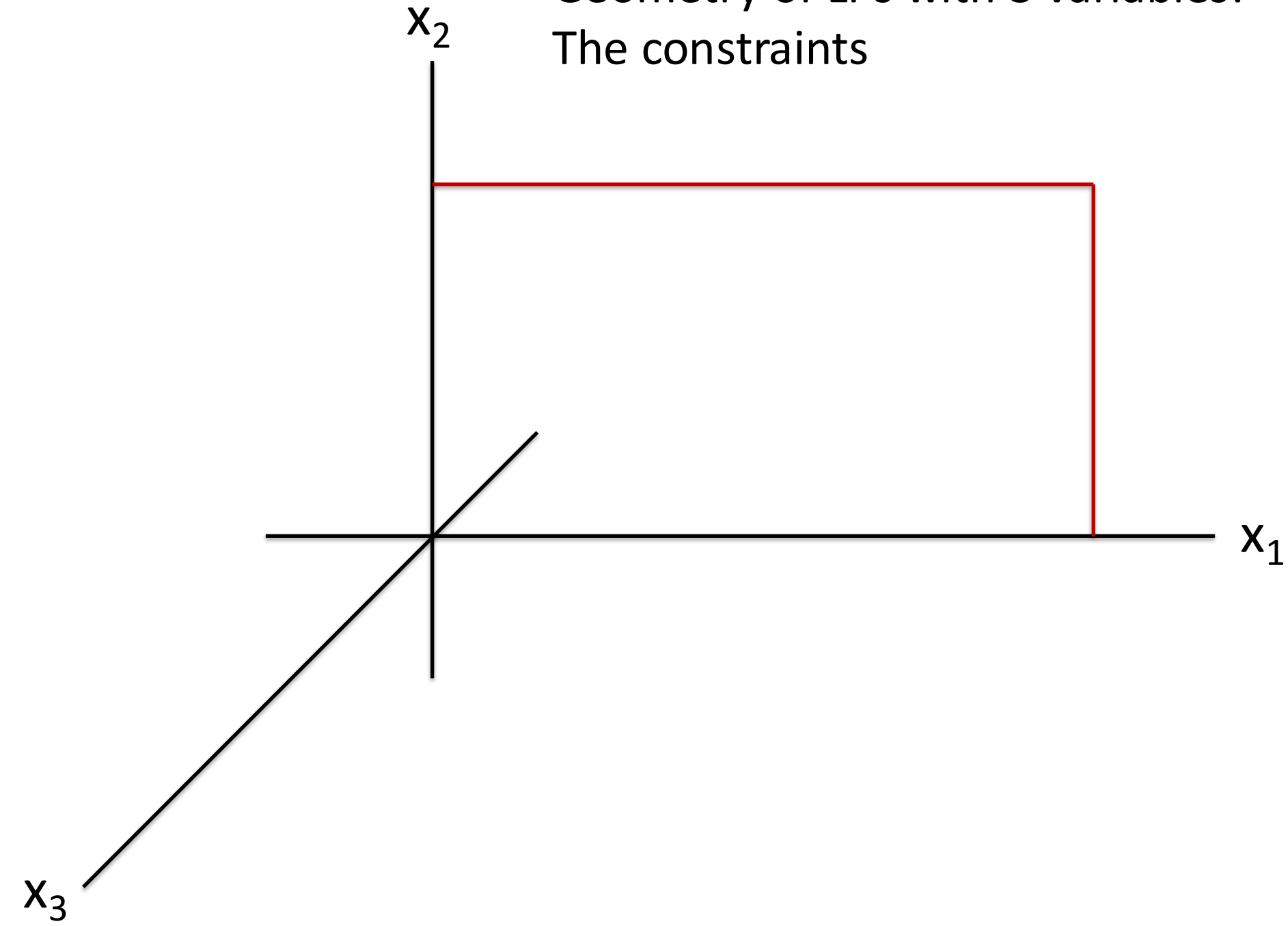

Geometry of LPs:
The objective function
(the linear algebra view)
x2
x1=10 (fertilizer)
(0,100/3)
(4,32)
(10,20)
x1+3x2=100 (seed)
x1
(3,2)
(0,0)
(10,0)
2x1+x2=40 (labour)


Geometry of LPs:
The objective function
(the linear algebra view)
x2
x1=10 (fertilizer)
(0,100/3)
(4,32)
(10,20
)
x1+3x2=100 (seed)
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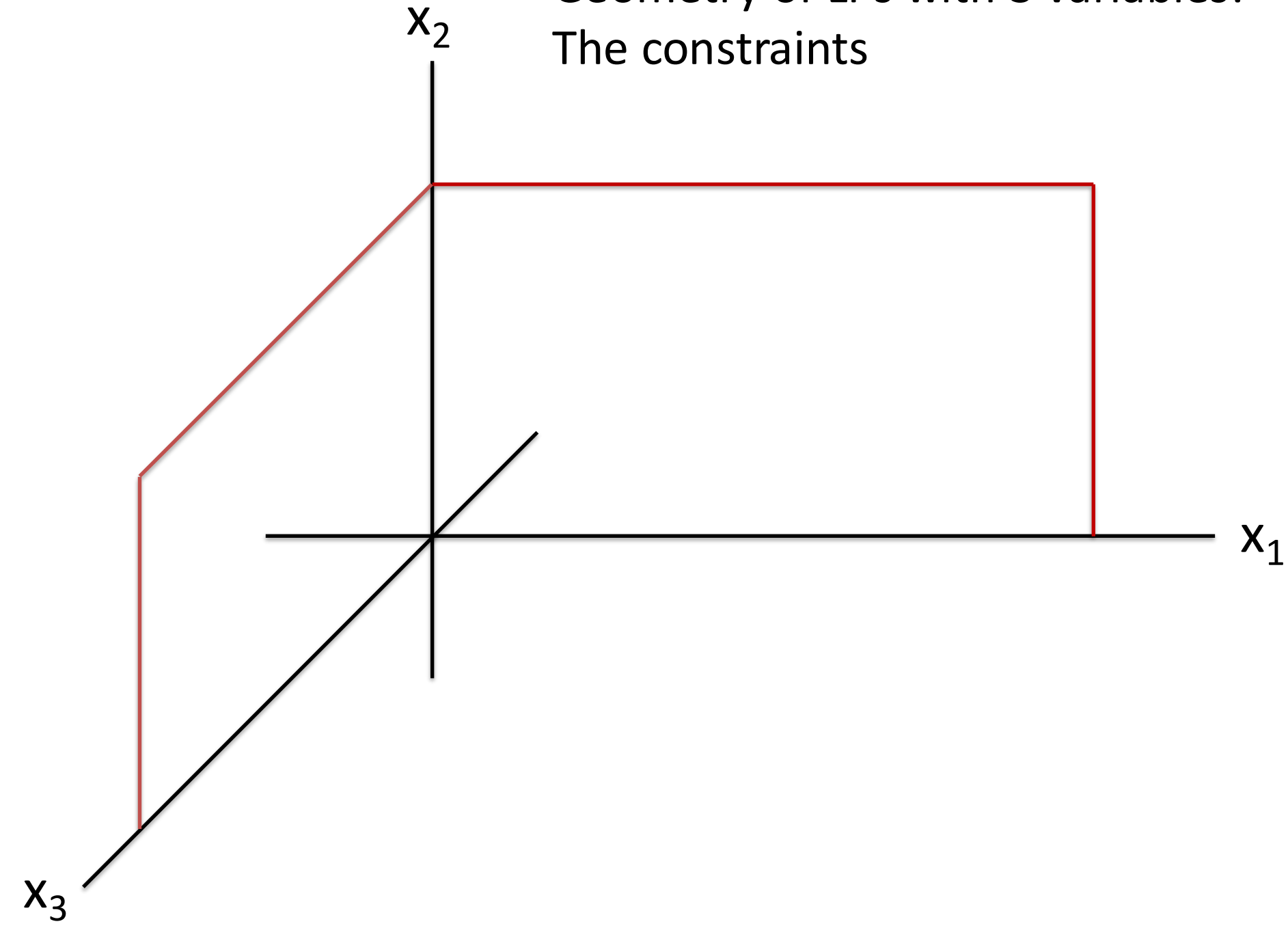
Geometry of LPs with 3 variables: The constraints



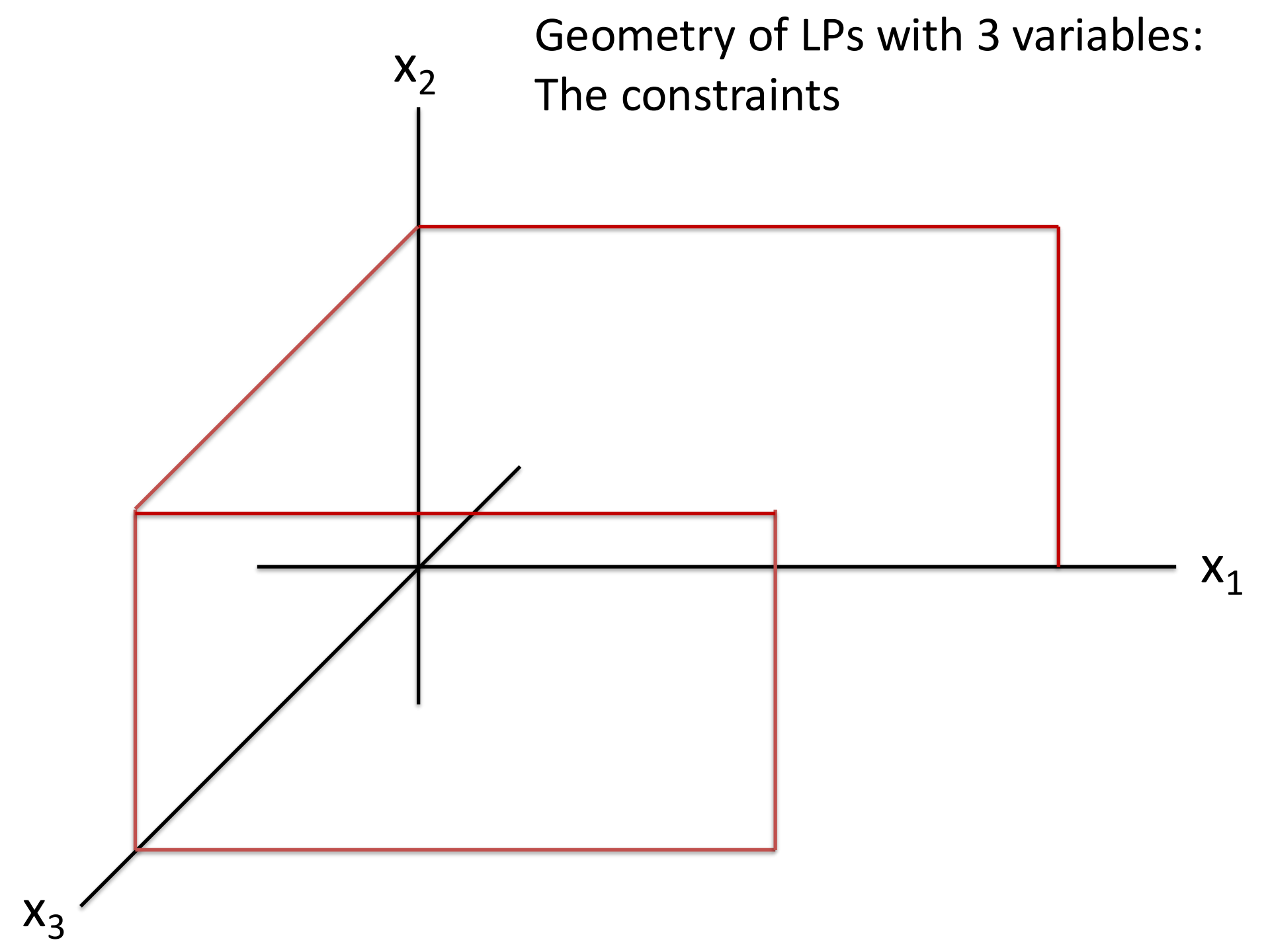
Geometry of LPs with 3 variables: The constraints



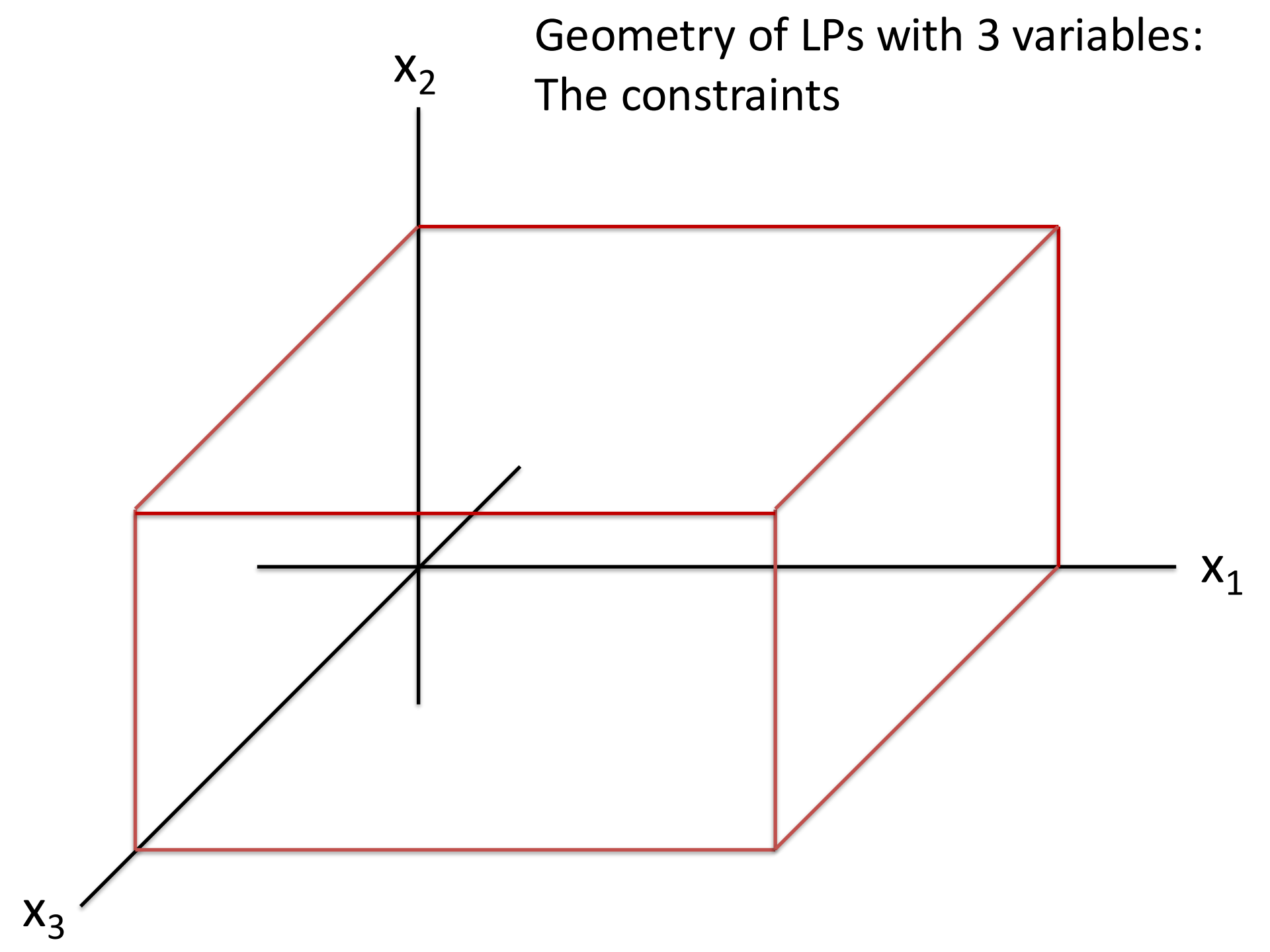
Geometry of LPs with 3 variables: The constraints



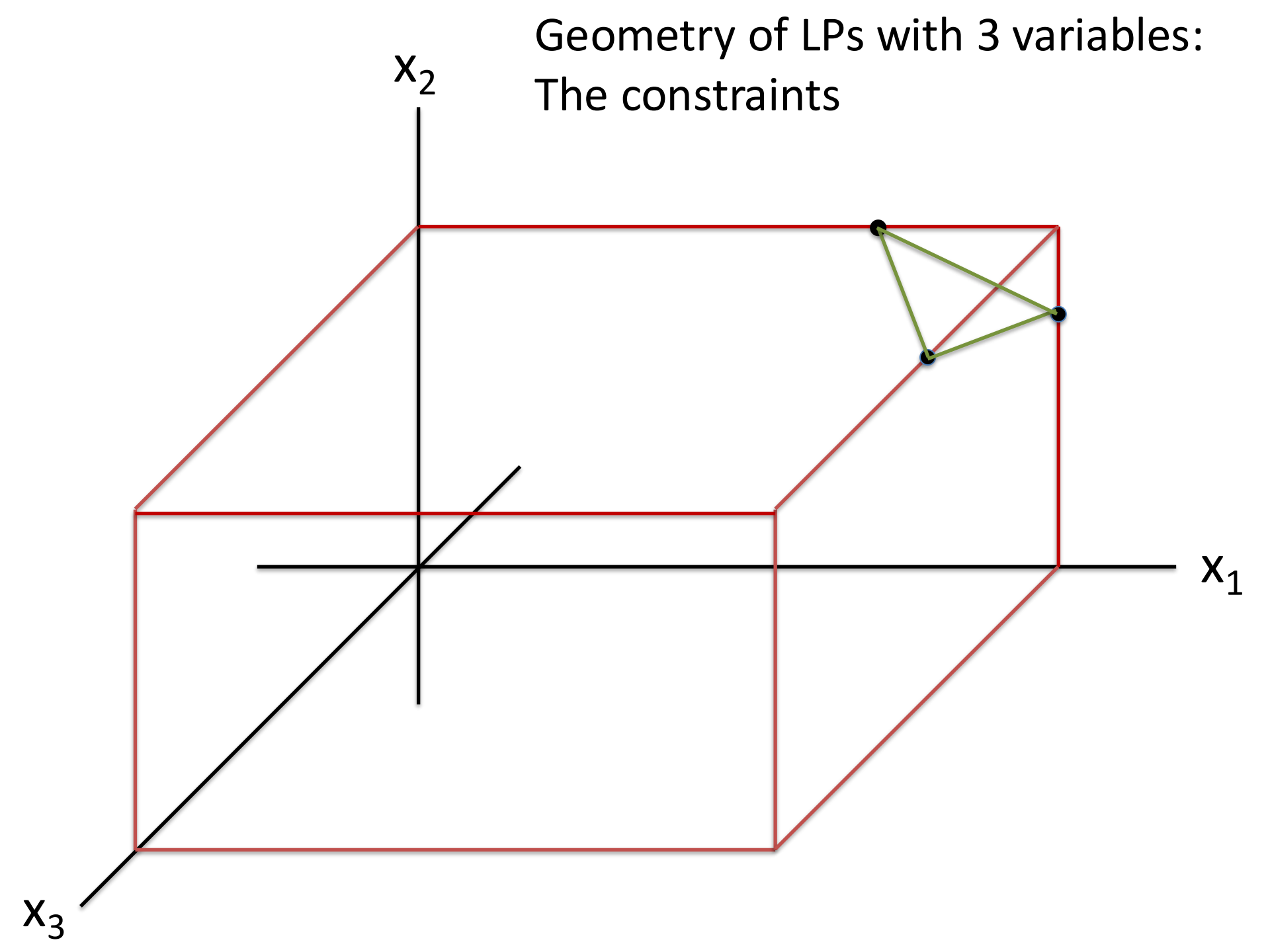
Geometry of LPs with 3 variables: The constraints



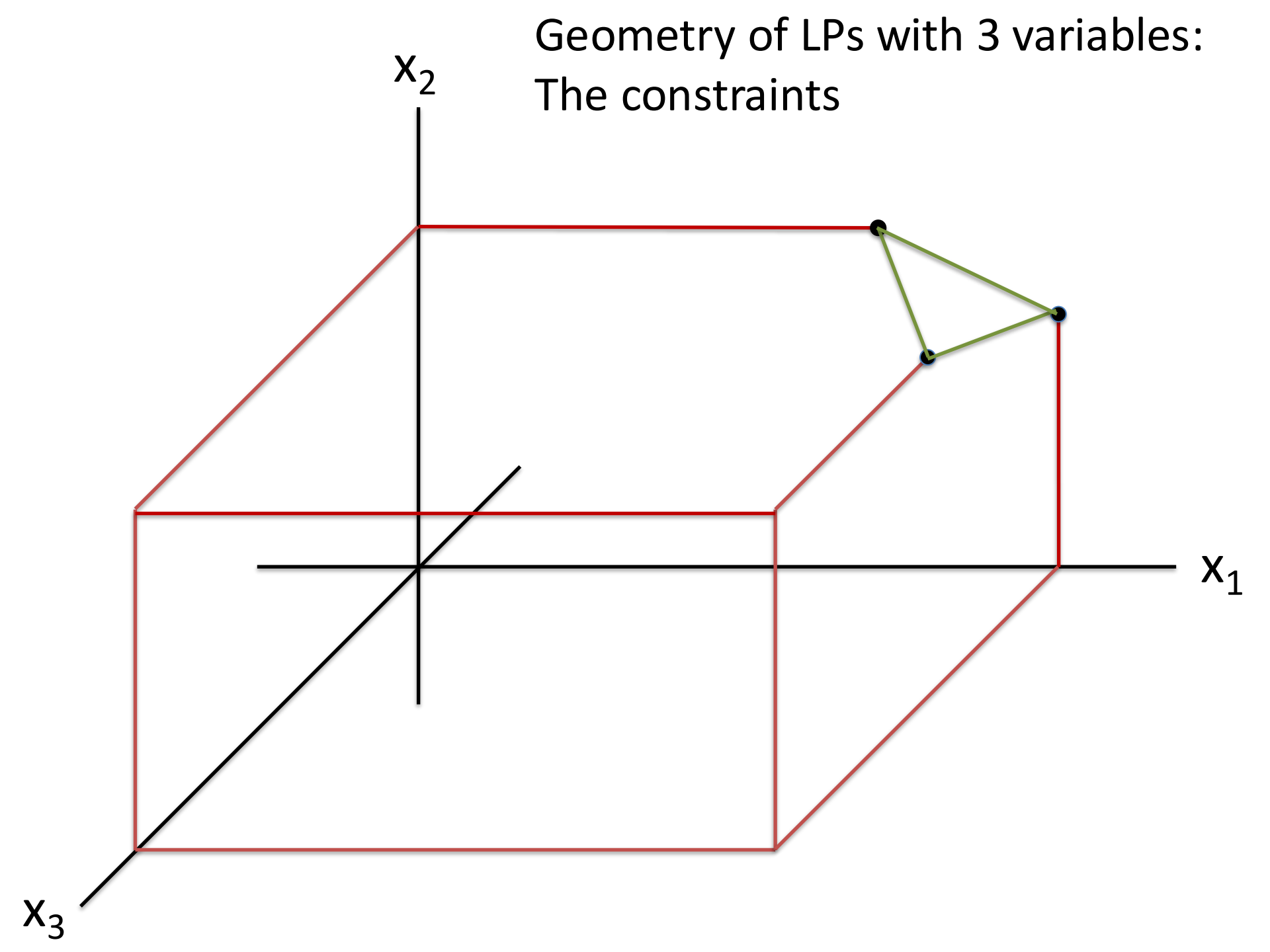
Geometry of LPs with 3 variables: The constraints



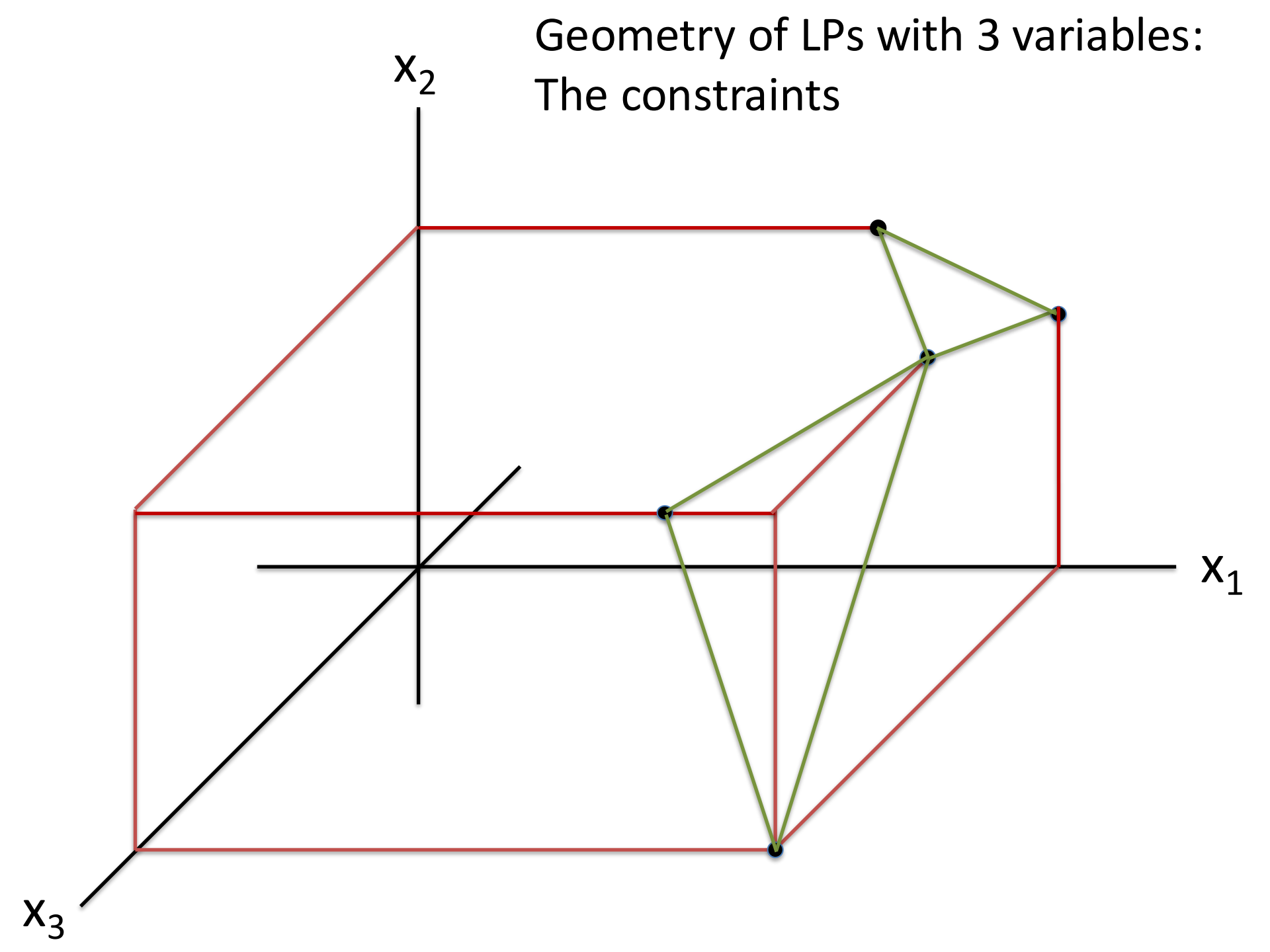
Geometry of LPs with 3 variables: The constraints



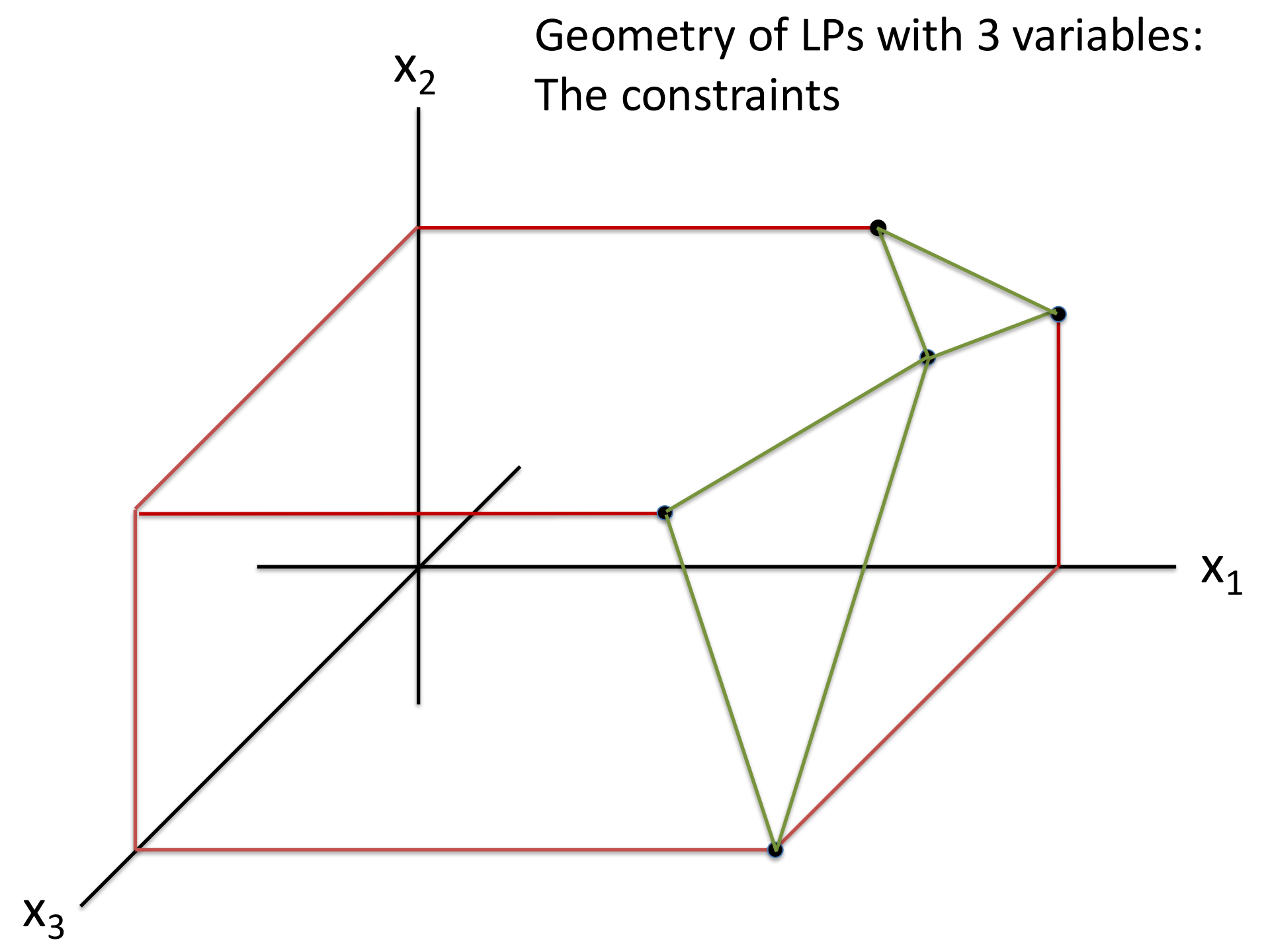
Geometry of LPs with 3 variables: The constraints



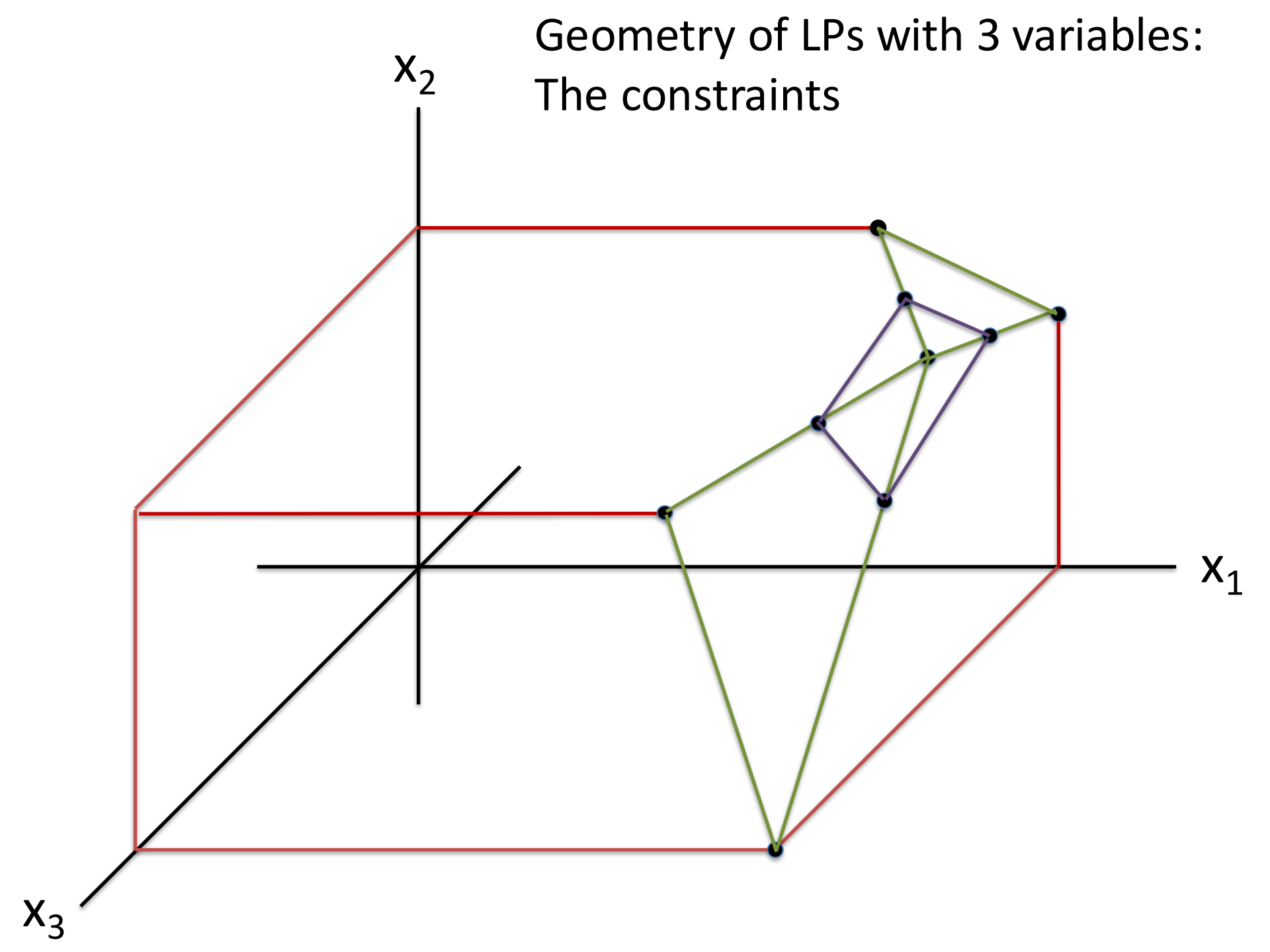
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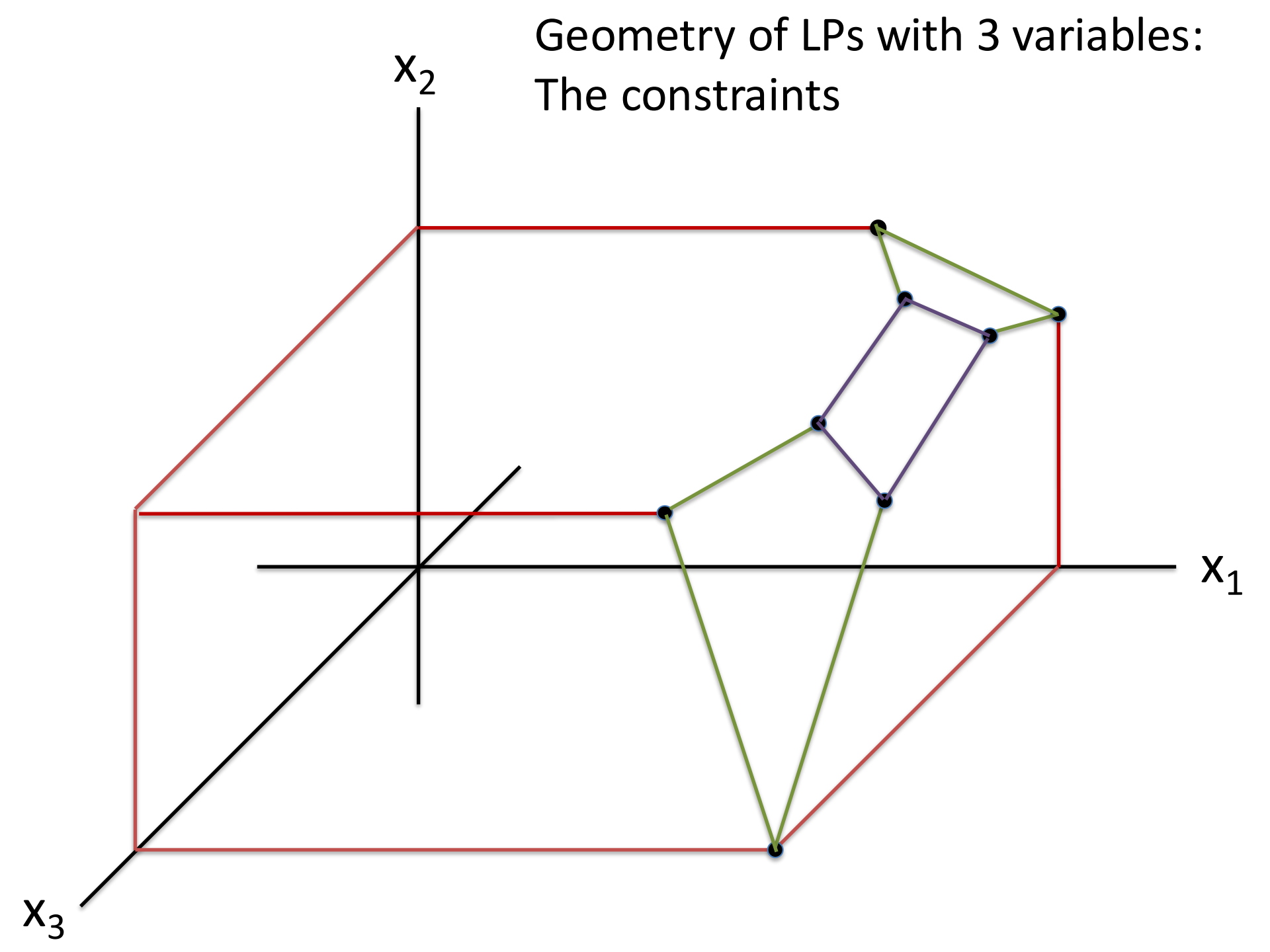
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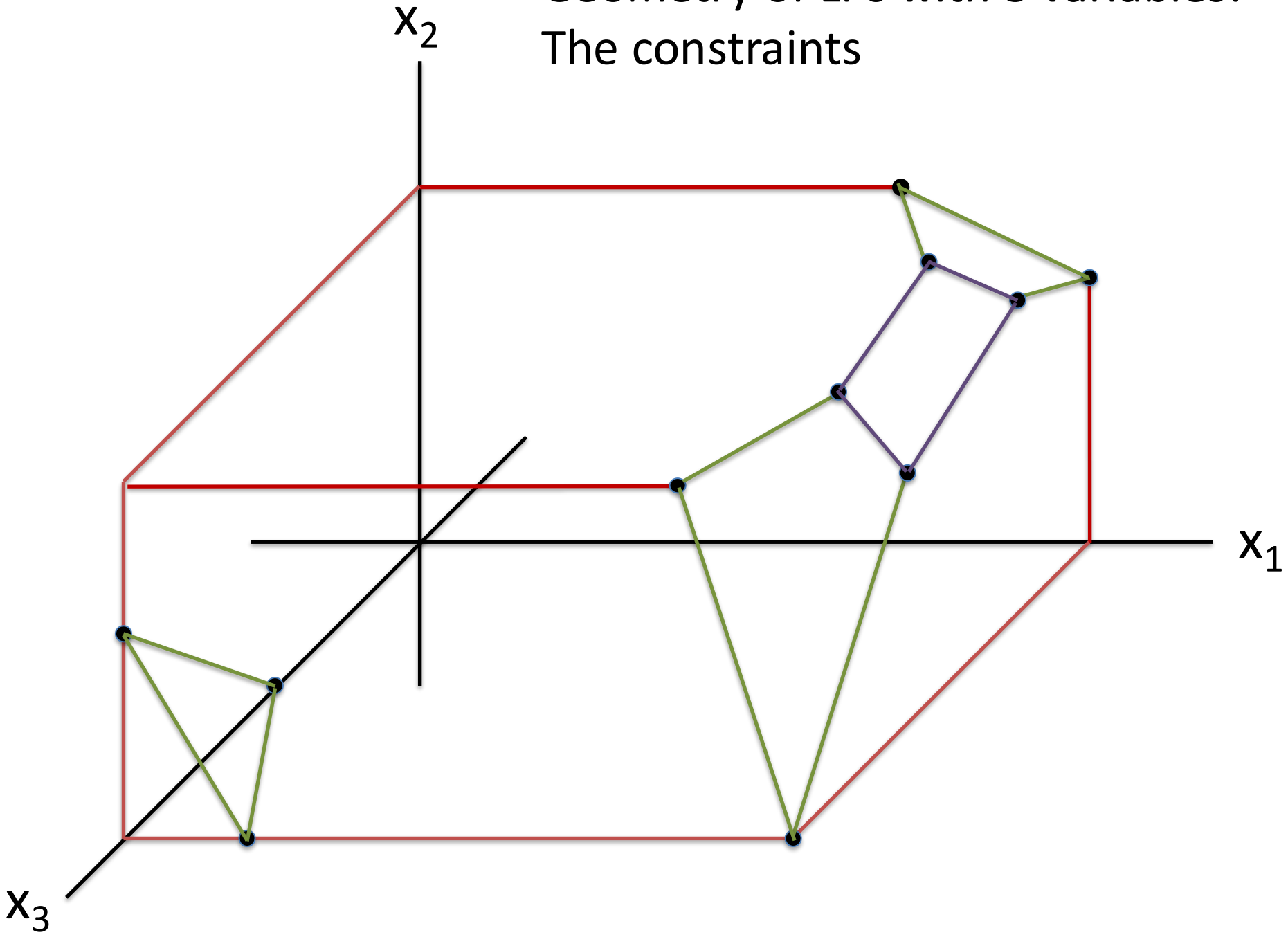
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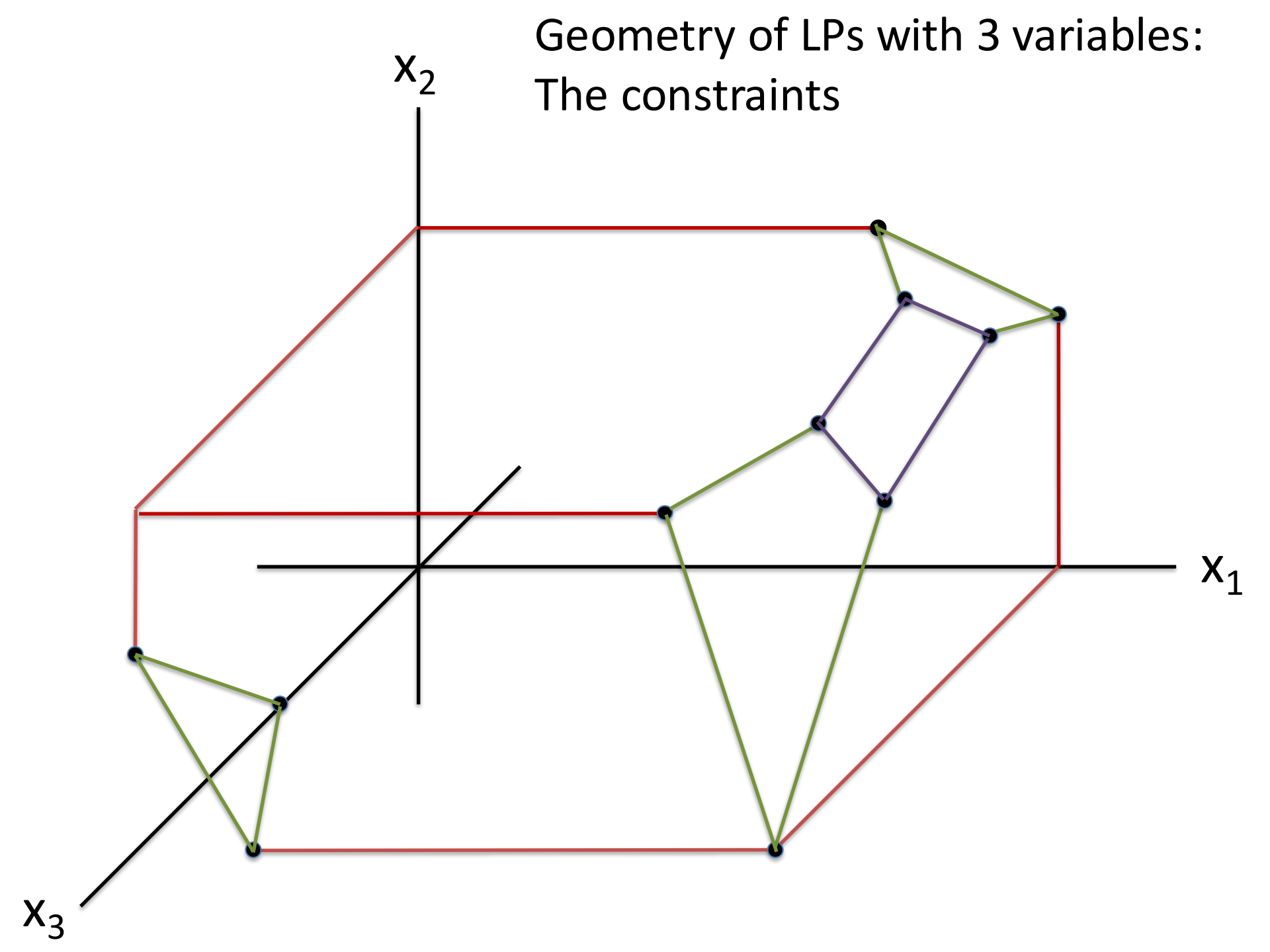
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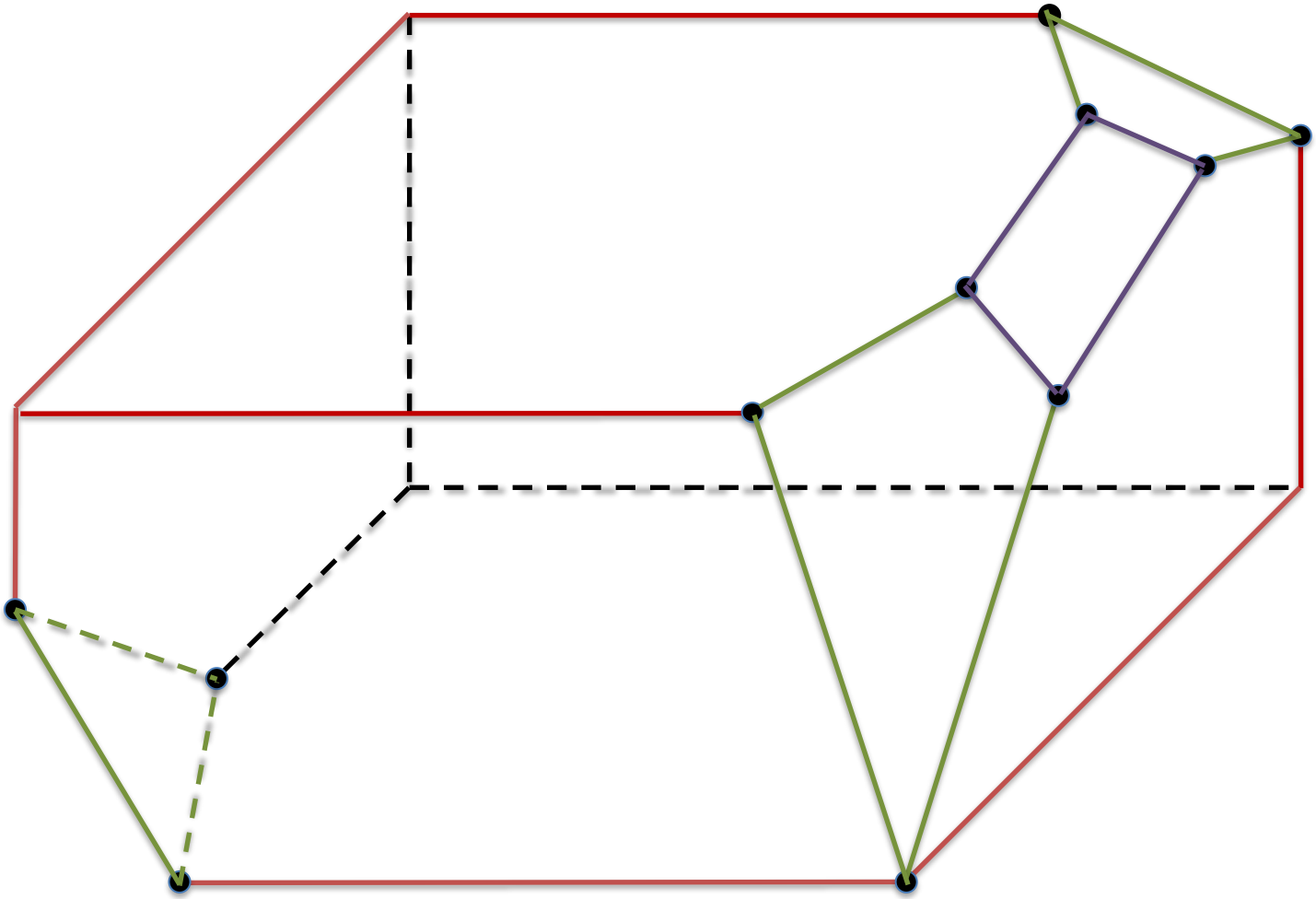
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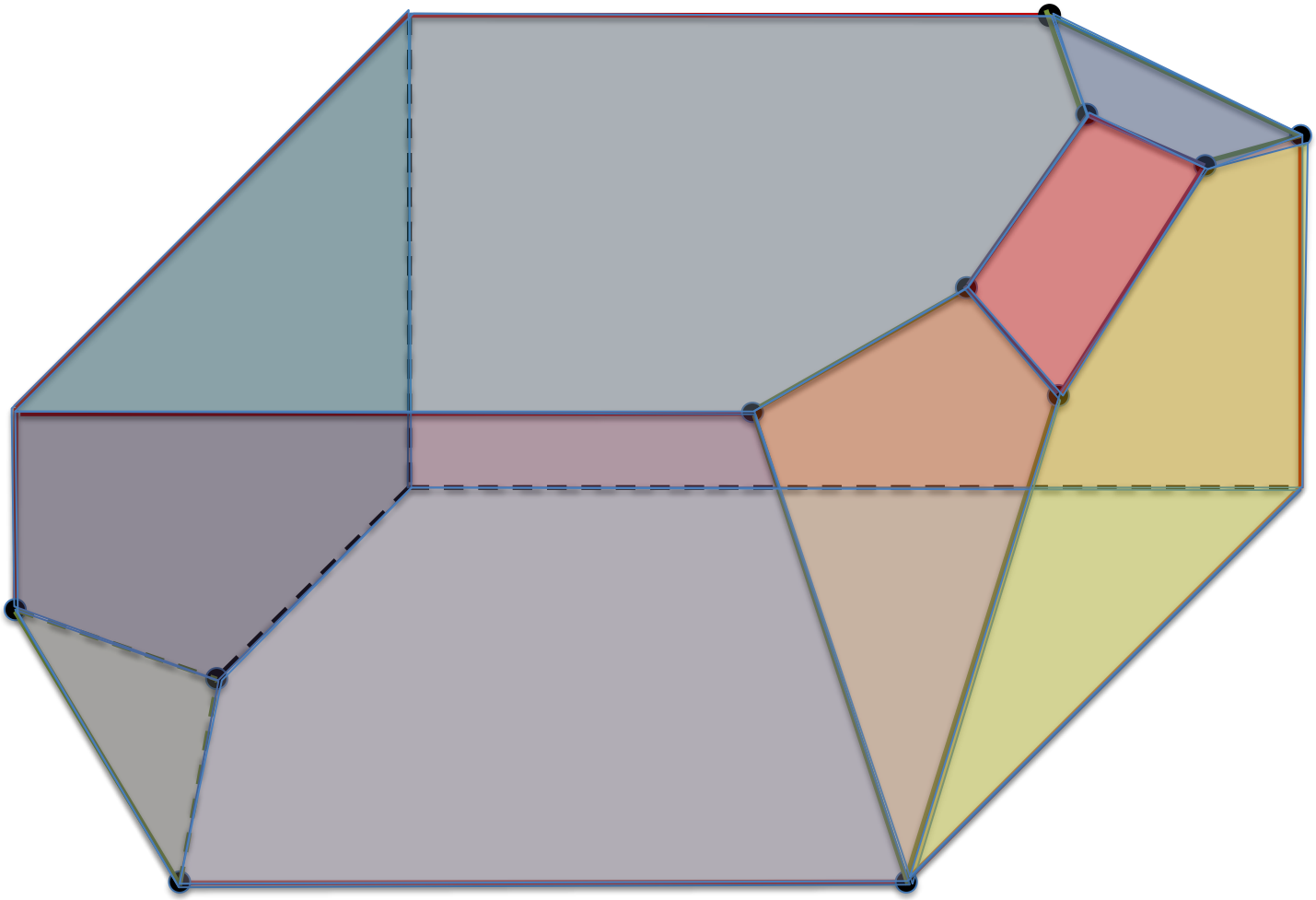


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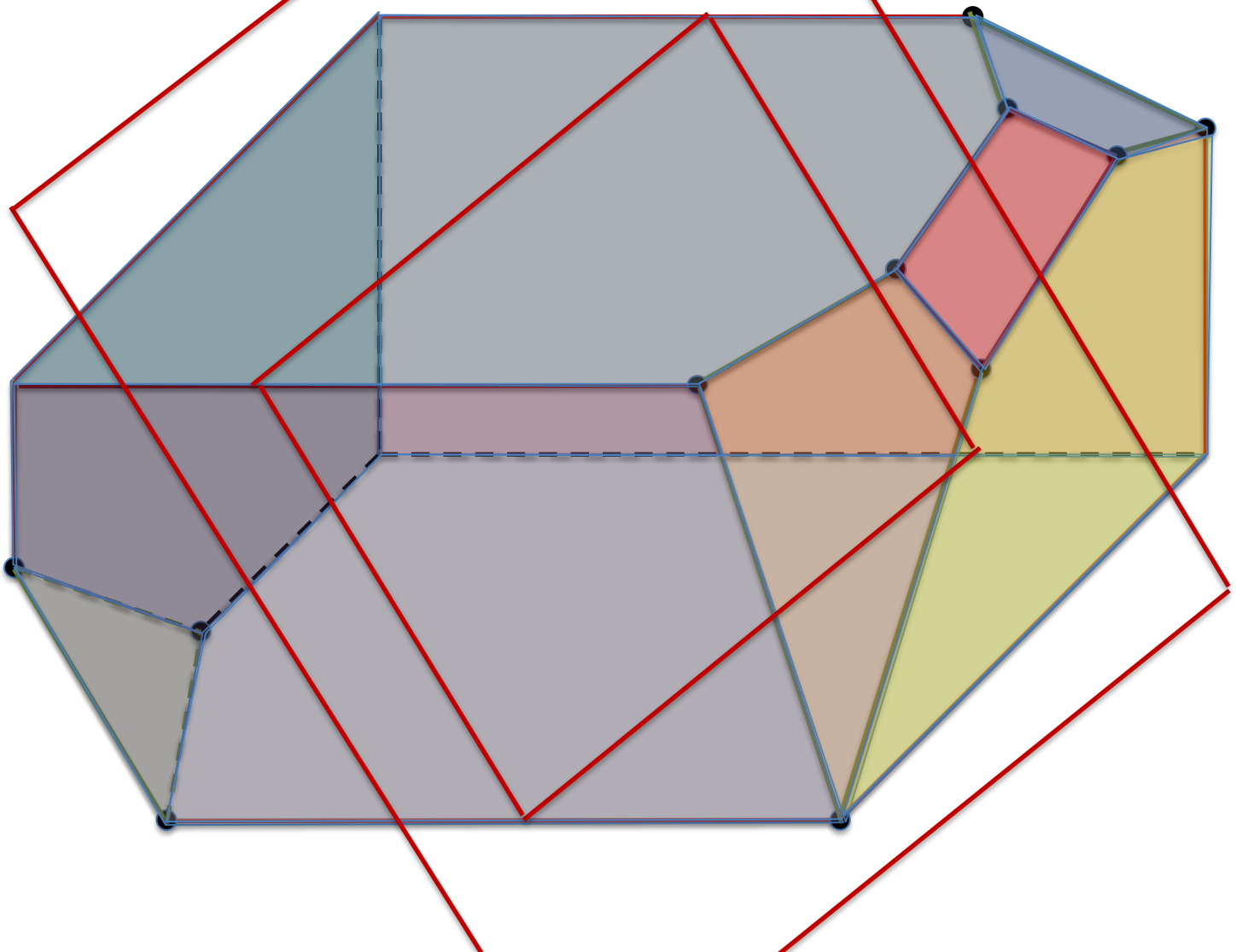


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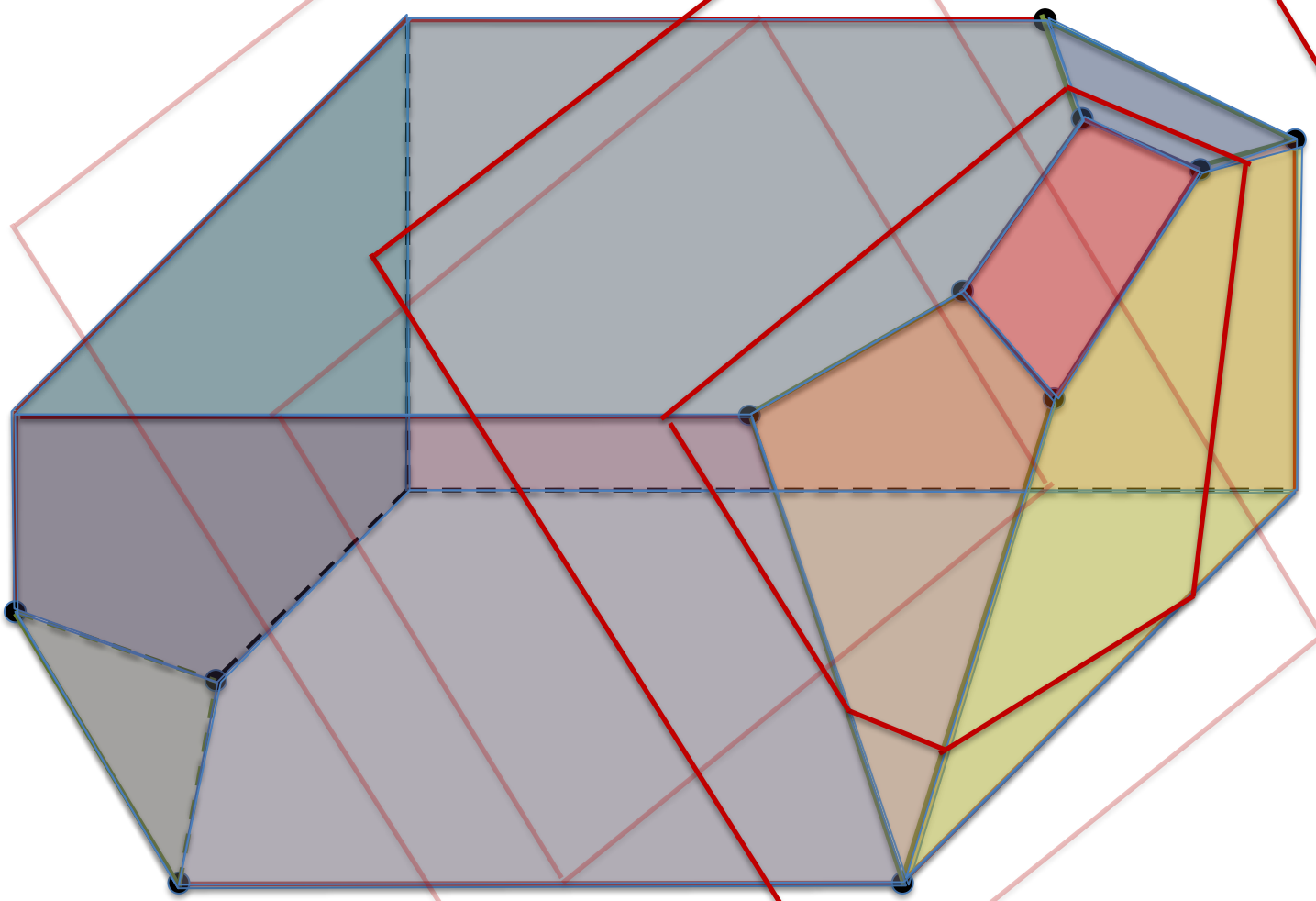
Feasible region:
a convex polyhedron



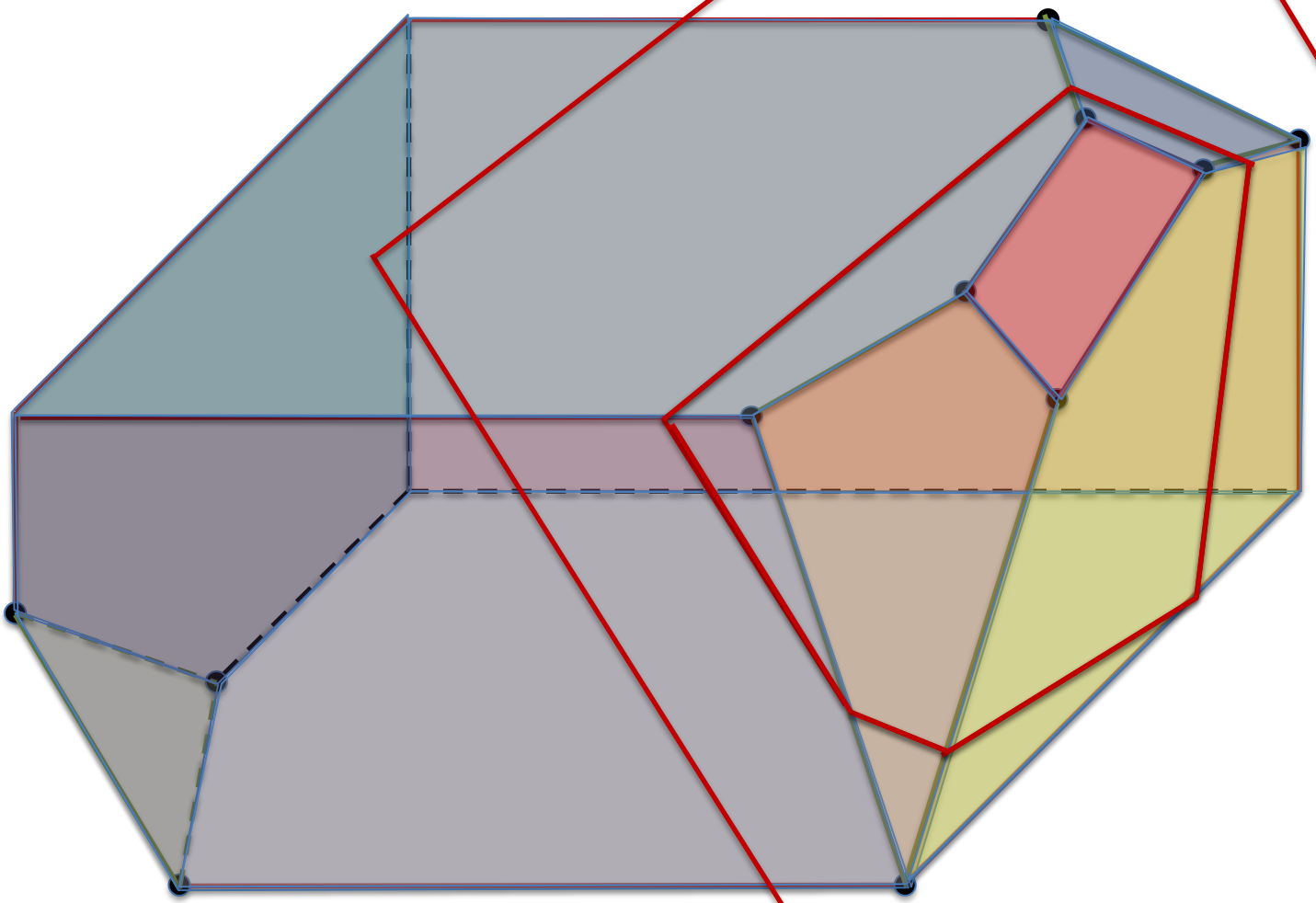
Geometry of LPs with 3 variables:
The objective function: set of parallel planes



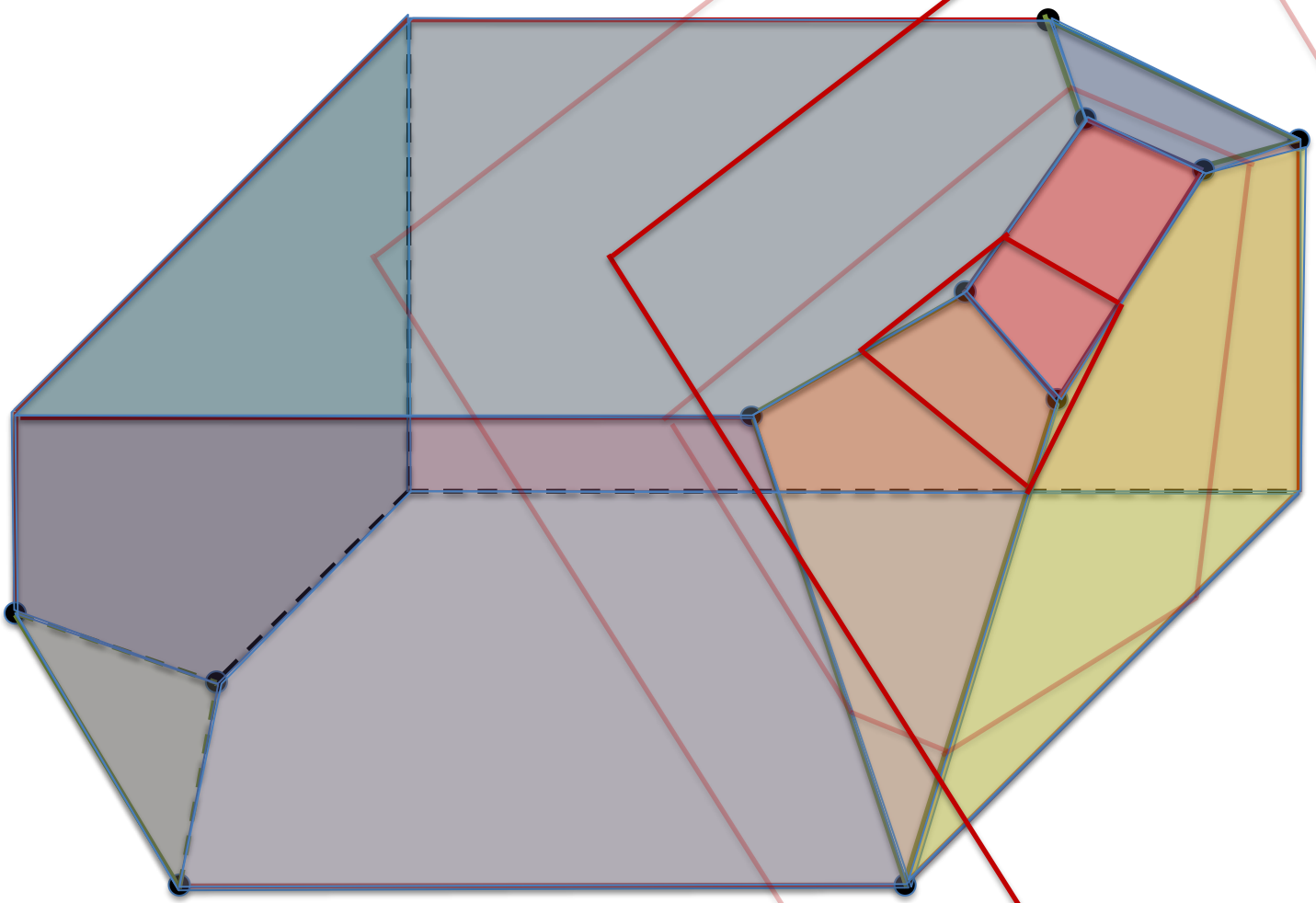
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The objective function: set of parallel planes



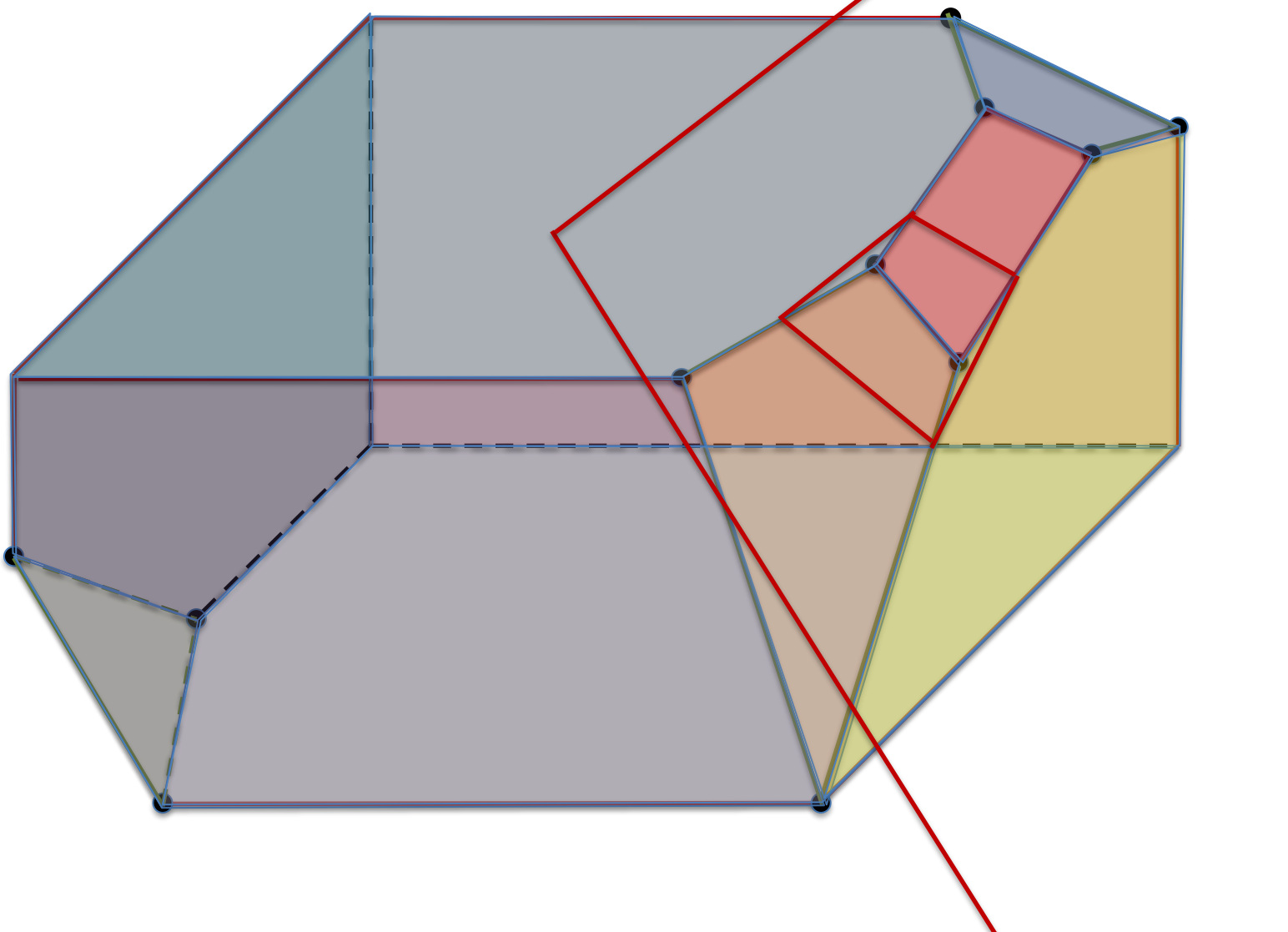
Geometry of LPs with 3 variables:
The objective function: set of parallel planes



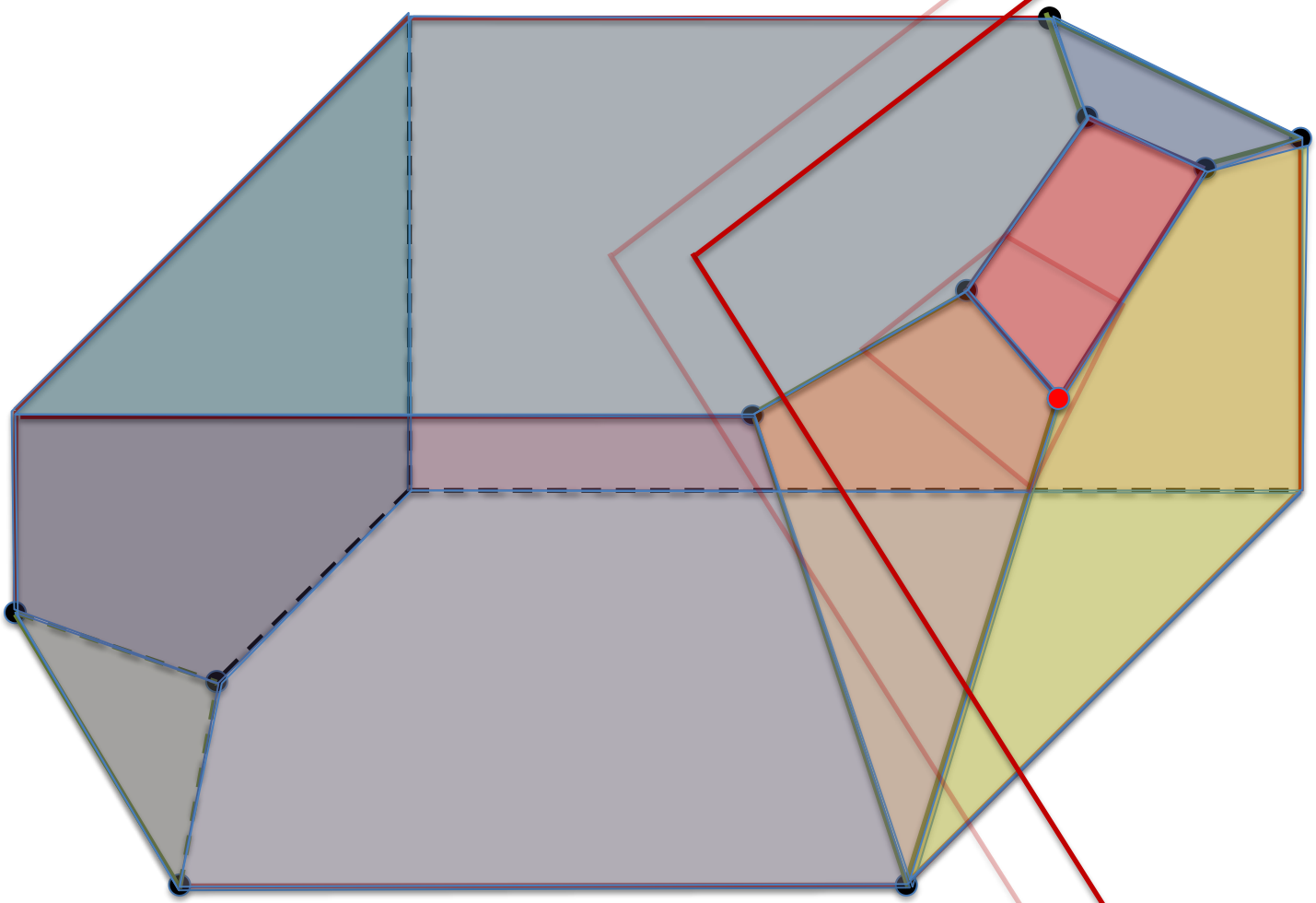
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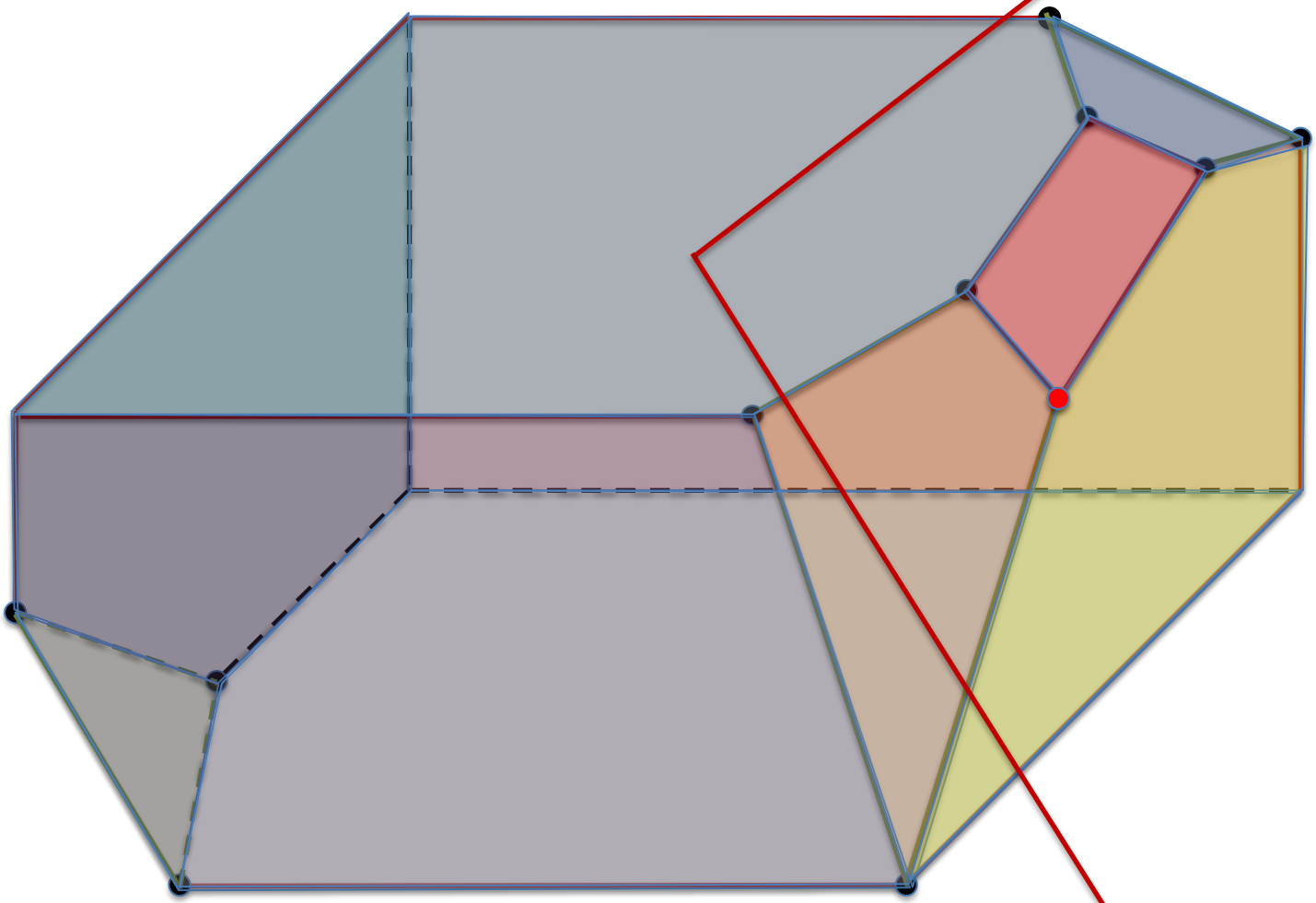
Geometry of LPs with 3 variables:
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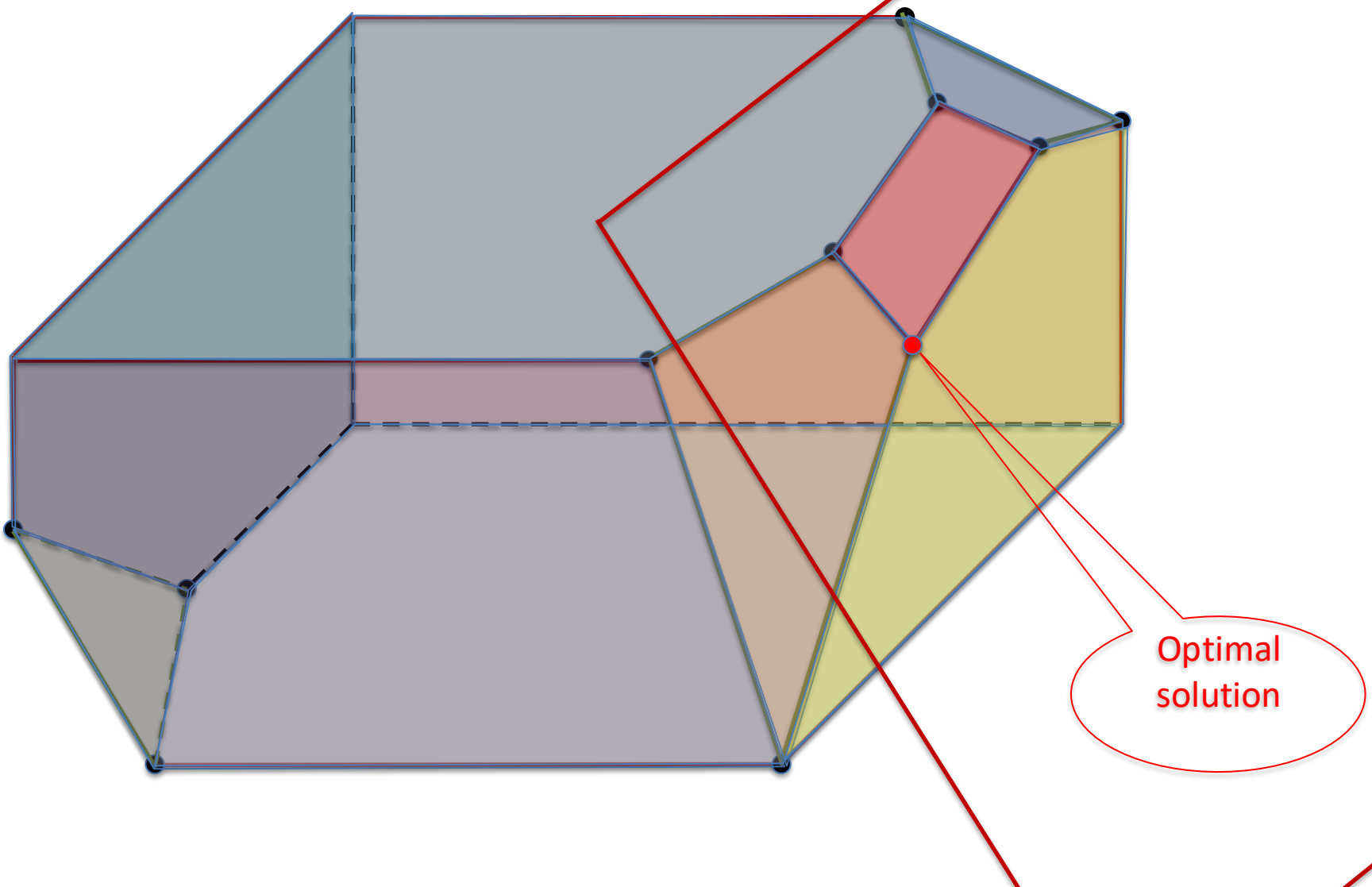
Geometry of LPs with 3 variables:
The objective function: set of parallel planes



Geometry of LPs with 3 variables:
The objective function: set of parallel planes



Geometry of LPs with 3 variables:
The objective function: set of parallel planes



Geometric interpretation of of LPs

	2 variables	3 variables	n variables
constraint	half-plane	half-space	half-hyperspace
feasible region	convex polygon	convex polyhedron	convex polytope
objective function	parallel lines	parallel planes	parallel hyperplanes
basic feasible solution	polygon vertex	polyhedron vertex	polytope vertex

LP terminology

Objective function: Function being minimized or maximized.
Solution: Assignment of real values to the variables.
Feasible solution: Solution that satisfies all the constraints.
Feasible region: The set of feasible solutions; a convex polytope.
Basic feasible solution: A vertex of the feasible solution polytope.
Optimal solution: A feasible solution that minimizes or maximizes the objective function; it is not necessarily unique and it might not exist (see below: infeasible and unbounded LP).
Value of a solution: Value of objective function at a solution; sometimes called "cost" of a solution (for minimization problems)
Optimal value: Value of optimal solution; sometimes called "optimal cost" (for minimization problems).
Feasible LP: LP that has feasible solutions.
Infeasible LP: LP with no feasible solutions.
Bounded LP: LP with an optimal solution.
Unbounded LP: LP that is feasible but has no optimal solution.