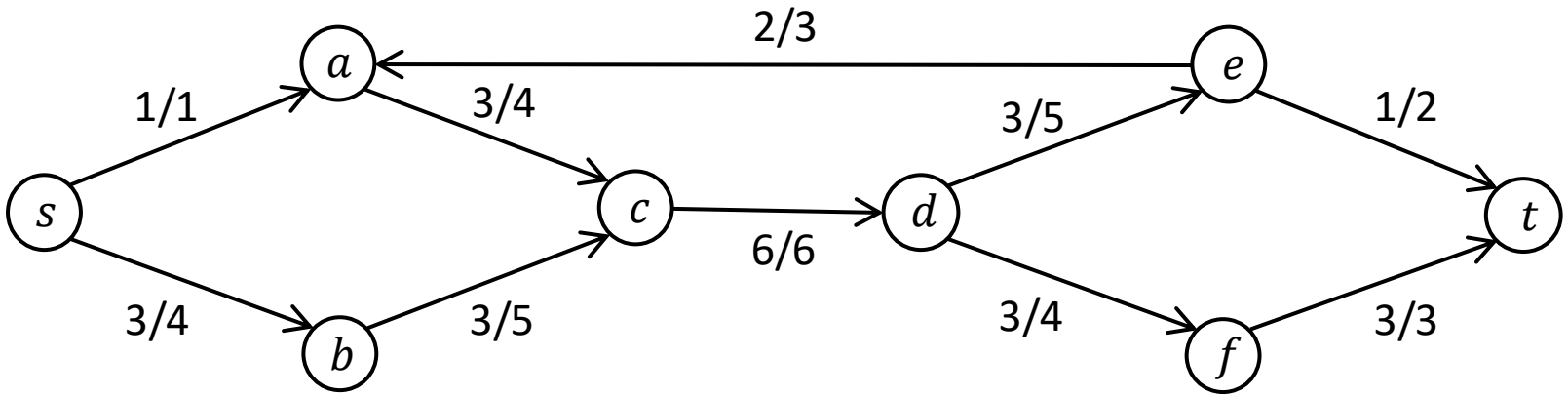
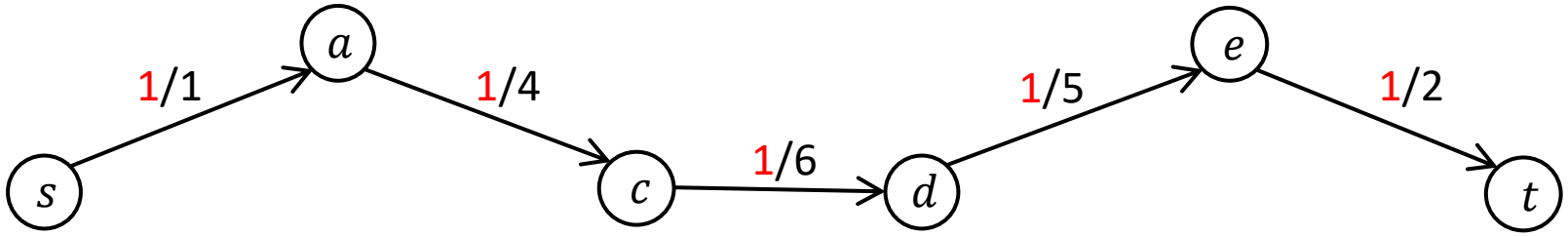
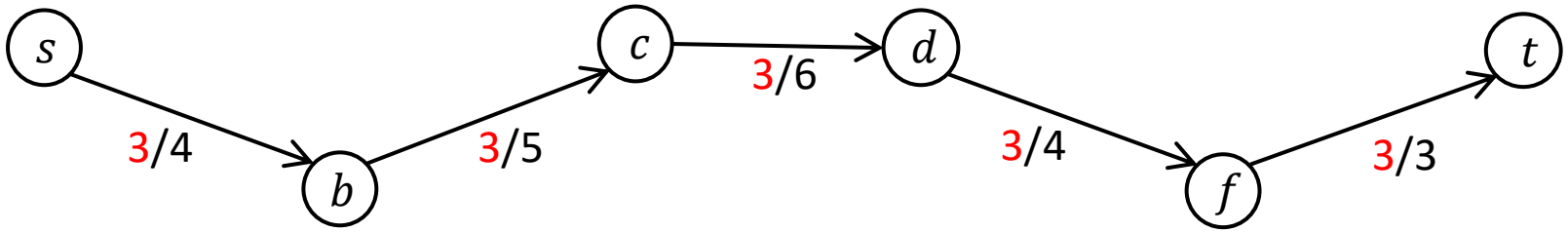
Flow decomposition


2/3
a
e
1/1
3/4
3/5
1/2
s
c
d
t
6/6
3/4
3/5
3/4
3/3
b
f

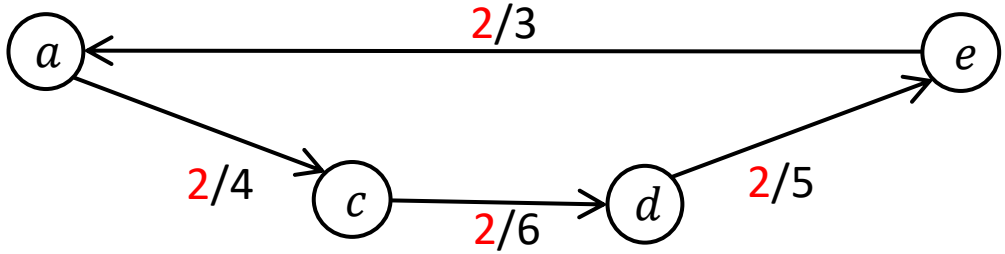
$p =$


a
e
1/1
1/4
1/5
1/2
s
c
1/6
d
t

$q =$


s
c
d
t
3/6
3/4
3/5
3/4
3/3
b
f

$c =$


2/3
a
e
2/4
2/5
c
d
2/6

If p is a simple s $\rightsquigarrow$ t path or a simple cycle define the unit flow on p:

$$f_p(e) = \begin{cases} 1, & \text{if } e \text{ is on } p \\ 0, & \text{if } e \text{ is not on } p \end{cases}$$

Theorem 5.16 (Flow Decomposition): For any flow f, there is a set P of simple s $\rightsquigarrow$ t paths and a set C of simple cycles, and positive constants d_p for every $p \in P$ and d_c for every $c \in C$ so that

$$f = \sum_{p \in P} d_p \cdot f_p + \sum_{c \in C} d_c \cdot f_c$$

If f is integral, then d_p and d_c can be chosen as integers.

In our example, $d_p = 1,\ d_q = 3,\ d_c = 3$.

FlowDecomp(f)

if $\forall e\ f(e) = 0$ **then return** $(\emptyset, \emptyset)$

if $\exists e \in out(s)$ s.t. $f(e) > 0$ **then** $u_0 := s$

else u_0:= any node s.t. for some $e \in out(u_0)\ f(e) > 0$

$k := 0$; $\pi := u_0$

while true do

  $k := k + 1$; $u_k :=$ any node s.t. $f(u_{k-1}, u_k) > 0$

  append u_k to π

  if $u_k = t$ **then** ▶ found path

    b := min traffic on any edge on π

    for each e on π **do** $f(e) := f(e) - b$

    (P, C): =FlowDecomp(f)

    return $(P \cup \{(\pi, b)\}, C)$

  elsif $\exists i \in [0..k-1]$ s.t. $u_k = u_i$ **then** ▶ found cycle

    $\sigma := u_i, u_{i+1}, \ldots, u_k$

    $b :=$ min traffic on any edge on σ

    for each e on σ **do** $f(e) := f(e) - b$

    $(P, C) :=$ FlowDecomp(f)

    return $(P, C \cup \{(\sigma, b)\})$

  else continue ▶ $\exists u$ s.t. $f(u_k, u) > 0 \Rightarrow$ extend π

MaxDisjointPaths(G, s, t)

$\mathcal{F}$:= flow network obtained from G by removing edges into s and out of t
and giving each remaining edge capacity = 1

f := FF-MaxFlow($\mathcal{F}$)

(P, C) := FlowDecomp(f)

return paths in P

Theorem 5.17:

(a) If f is an integral flow of $\mathcal{F}$, FlowDecomp(f) returns a set of $\mathcal{V}(f)$ edge-disjoint simple s $\rightsquigarrow$ t paths in G.

(b) For any set P of edge-disjoint simple s $\rightsquigarrow$ t paths in G, there is a flow of $\mathcal{F}$ whose value is $|P|$.