Computer Science C63 — March 19, 2025
Scarborough Campus — University of Toronto

Homework Assignment #7
Due: April 2, 2025, by 11:59 pm

- ***You must submit your assignment through the Crowdmark system.*** *You will receive by email an invitation through which you can submit your work in the form of separate PDF docucments with your answers to each question of the assignment. To work with a partner, you and your partner must form a group on Crowdmark. Crowdmark does not enforce a limit on the size of groups.* ***The course policy that limits the size of each group to at most two remains in effect:*** *submissions by groups of more than two persons will not be graded.*
- *To minimize bias in grading, Crowdmark does not reveal your name to the grader. Please do not subvert this feature by including your name(s) on the files you submit.*
- *It is your responsibility to ensure that the files you submit are legible. To this end, I encourage you to learn and use the LaTex typesetting system, which is designed to produce high-quality documents that contain mathematical notation. You are not required to produce the files you submit using LaTex; you may produce it any way you wish, as long as the resulting document is legible.*
- *By virtue of submitting this assignment you (and your partner, if you have one) acknowledge that you are aware of the policy on homework collaboration for this course.*[a]
- *For any question, you may use facts previously proved in this course, its prerequisites, or in the assigned sections of the textbook.*
- *Unless we explicitly state otherwise, you should justify your answers. Your paper will be graded based on the correctness a of your answers, and the clarity, precision, and conciseness of your presentation.*

[a] "In each homework assignment you may collaborate with at most one other student who is currently taking CSCC63. If you collaborate with another student on an assignment, you and your partner must submit only one copy of your solution, with both of your names. The solution will be graded in the usual way and both partners will receive the same mark. Collaboration involving more than two students is not allowed. **For help with your homework you may consult only the instructor, TAs, your homework partner (if you have one), your textbook, and your class notes. You may not consult any other source.** *Nota Bene:* 'Other source' includes, but is not limited to, the use of AI-based tools, even for (allegedly) 'just improving' your work. What you submit must be entirely your own creation."

Question 1. (20 marks) We are given n jobs, each of which requires the use of a machine during a specified set of intervals of time. We have only one machine that can be used for these jobs, so if two jobs specify intervals that overlap, at most one of the two jobs can be completed: For a job to be completed, it must have exclusive access to the machine during <u>all</u> its intervals. We want to determine if there is a set of k jobs that can be completed.

More precisely, we want to solve the following decision problem:

Instance: $\langle J_1, J_2, \ldots, J_n, k\rangle$, where $J_i = \{[s_{i1}, f_{i1}], [s_{i2}, f_{i2}], \ldots, [s_{i\ell_i}, f_{i\ell_i}]\}$, for integers $s_{ij} < f_{ij}$ specifying, respectively, the start and finish times of the j-th interval of job i), and k is a positive integer.

Question: Is there a subset of k jobs among $J_1, J_2, \ldots, J_n$ so that no two of them contain intersecting intervals?

Prove that this problem is **NP**-complete.

Note: Those of you who have taken (or are currently taking) CSCC73 know that the special case of this problem where each job has exactly one interval is solvable in polynomial time.

[Continued on the next page]

Question 2. (35 marks)
a. (20 marks) A rural area contains villages, some of which are connected to others by roads. We represent this as an undirected graph $G = (V, E)$, with the villages as nodes and an edge between two villages if and only if there is a road that connects them directly (without going through other villages). We are given the lengths of the roads in the form of a function $\ell\colon E \to \mathbb{Z}^+$.

The government wants to build hospitals in some of the villages to serve the whole area and allocated a budget B for this purpose. It costs an amount $c(u)$ to build a hospital in village u. Health care safety policy stipulates that no village should be farther than distance L from the nearest hospital. So the subset S of villages where hospitals are built must satisfy the following two constraints:

Budget: $\sum_{v \in S} c(v) \leq B$.

Proximity: For each $u \in V$ there is some $v \in S$ such that the length of a shortest path from u to v in G is at most L.[1]

The question is whether such a set S of villages exists. More precisely, the ***Hospital Placement Decision*** problem, HP-Dec is as follows:

Instance: $\langle G, \ell, c, B, L \rangle$, where

- $G = (V, E)$ is an undirected graph (representing the set V of villages and the set of roads E connecting them),
- $\ell\colon E \to \mathbb{Z}^+$ is a function (specifying the length $\ell(\{u, v\})$ of the road connecting villages u and v),
- $c\colon V \to \mathbb{Z}^+$ is a function (specifying the cost $c(u)$ of building a hospital in village u),
- $B \in \mathbb{Z}^+$ is a positive integer (the budget allocated for building hospitals), and
- $L \in \mathbb{Z}^+$ is a positive integer (the maximum distance between a village and its closest hospital).

Question: Is there a set $S \subseteq V$ that satisfies the budget and proximity constraints listed above?

Prove that HP-Dec is **NP**-complete.

Hint: Consider the special case where $c(u) = 1$ for every node u (so building a hospital costs the same in every village), $\ell(e) = 1$ for every edge e, and $L = 1$ (so the proximity constraint implies that every village must have a hospital in it or in one of its neigbours). Even this special case is **NP**-complete.

b. (15 marks) The problem that the government really wants to solve is not the above decision problem, but an optimization problem: In which villages should the hospitals be built so as to minimize the sum of costs, while ensuring that no village is farther than L from its nearest hospital? More precisely, the ***Hospital Placement Optimization*** problem, HP-Opt, is:

Input: $\langle G, \ell, c, L \rangle$, where $G = (V, E)$, $\ell\colon E \to \mathbb{Z}^+$, $c\colon V \to \mathbb{Z}^+$, and $L \in \mathbb{Z}^+$ are as in the specification of HP-Dec (but note the absence of the budget B.)

Output: A set $S \subseteq V$ that (a) satisfies the proximity constraint and (b) ***minimizes*** $\sum_{v \in S} c(v)$ — i.e., for any set S' that satisfies the proximity constraint, $\sum_{v \in S} c(v) \leq \sum_{v \in S'} c(v)$.

Give a Cook reduction (i.e., a polynomial-time Turing reduction) of the Hospital Placement Optimization problem to the Hospital Placement Decision problem. In other words, describe a polynomial-time algorithm for HP-Opt that uses a black box HP-Dec-oracle that solves HP-Dec (the original version, not the special case described in the hint above), at the cost of one unit of time per use of the black box. Describe clearly your reduction in high-level pseudocode that involves calls to the black box, justify why it is a polynomial-time reduction, and why it is correct.

THAT'S IT WITH HOMEWORK, FOLKS!

[1]The length of a path is the sum of the lengths of the edges on that path. As you learned — or will learn — in CSCC73, given G and ℓ we can compute, in polynomial time, shortest paths between every pair of nodes. You can use this fact if you wish, though it is possible to do this problem without it.