Computer Science C63 — March 25, 2022
Scarborough Campus — University of Toronto

Homework Assignment #7
Due: April 8, 2022, by 11:59 pm

• ***You must submit your assignment through the Crowdmark system.*** *You will receive by email an invitation through which you can submit your work in the form of separate PDF docucments with your answers to each question of the assignment. To work with a partner, you and your partner must form a group on Crowdmark. Crowdmark does not enforce a limit on the size of groups.* ***The course policy that limits the size of each group to at most two remains in effect:*** *submissions by groups of more than two persons will not be graded.*
• *It is your responsibility to ensure that the PDF files you submit are legible. To this end, I encourage you to learn and use the LaTex typesetting system, which is designed to produce high-quality documents that contain mathematical notation. You are not required to produce the PDF files you submit using LaTex; you may produce it any way you wish, as long as the resulting document is legible.*
• *By virtue of submitting this assignment you (and your partner, if you have one) acknowledge that you are aware of the policy on homework collaboration for this course.*[a]
• *For any question, you may use facts previously proved in this course, its prerequisites, or in the assigned sections of the textbook.*
• *Unless we explicitly state otherwise, you should justify your answers. Your paper will be graded based on the correctness a of your answers, and the clarity, precision, and conciseness of your presentation.*

[a] "In each homework assignment you may collaborate with at most one other student who is currently taking CSCC63. If you collaborate with another student on an assignment, you and your partner must submit only one copy of your solution, with both of your names. The solution will be graded in the usual way and both partners will receive the same mark. Collaboration involving more than two students is not allowed. **For help with your homework you may consult only the instructor, TAs, your homework partner (if you have one), your textbook, and your class notes. You may not consult any other source.**"

Question 1. (10 marks) UTSC has many, possibly overlapping, student clubs: Computer Science student club, Marxist club, gardening club, theatre club, cross-country running club, and so on. We want to create a council of representatives consisting of ***exactly one*** member from each club. We call such a council ***fair***, because every club is represented but none is over-represented. The Fair Council problem is to determine whether a fair council exists, given the membership of every club. More precisely:

INSTANCE: $\langle S, \mathcal{C} \rangle$, where $S = \{s_1, \ldots, s_n\}$ is a set (the students) and $\mathcal{C} = \{C_1, \ldots, C_m\}$ is a set of subsets of S (the clubs).
QUESTION: Is there a fair council of representatives for $\mathcal{C}$? That is, is there a set $R \subseteq \bigcup_{j=1}^{m} C_j$ such that for each $j \in 1..m$, $|R \cap C_j| = 1$?

Prove that the Fair Council problem is **NP**-complete.

Question 2. (15 marks) Prove that the Clique-and-Independent-Set problem defined below is **NP**-complete.

INSTANCE: $\langle G, b \rangle$, where $G = (V, E)$ is an undirected graph and b is a positive integer.
QUESTION: Does G have a clique A and an independent set of nodes B such that the sum of the number of nodes in A and B is b? (Note that the clique and the independent set may have a node in common, in which case that node counts twice towards the sum b.)

[continued on the next page]

Question 3. (20 marks) A ***Hamiltonian path*** of a (directed or undirected) graph $G = (V, E)$ is a path that visits every node of G exactly once. If $s, t \in V$, a ***Hamiltonian*** (s, t)***-path*** of G is a Hamiltonian path that starts at s and ends at t. Prove that the following problems are **NP**-hard. (They are, in fact, **NP**-complete, but for this question don't bother with the argument that they are in **NP**.)

a. The (s, t)-Directed Hamiltonian Path problem.

INSTANCE: $\langle G, s, t \rangle$, where $G = (V, E)$ is a directed graph, and $s, t \in V$.
QUESTION: Goes G have a Hamiltonian (s, t)-path?

b. The Directed Hamiltonian Path problem.

INSTANCE: $\langle G \rangle$, where $G = (V, E)$ is a directed graph.
QUESTION: Goes G have a Hamiltonian path?

c. The (s, t)-Undirected Hamiltonian Path problem.

INSTANCE: $\langle G, s, t \rangle$, where $G = (V, E)$ is an undirected graph, and $s, t \in V$.
QUESTION: Goes G have a Hamiltonian (s, t)-path?

Note: The Undirected Hamiltonian Path problem is also **NP**-complete, and you should convince yourself that it is, but there is not much new in such a proof, so don't bother including it in your answer.

THAT'S IT WITH HOMEWORK, FOLKS!