Question 1. (20 marks) A happy couple is planning their wedding. They have settled on the list of guests and they know, for any two of their guests, whether each party knows the other. To minimize the chances of drama during the wedding they decided that any two guests assigned to sit at the same table in the banquet must know each other. Note that it is possible for guest \( a \) to know guest \( b \) but for \( b \) not to know \( a \); the requirement is that if \( a \neq b \) are at the same table, each must know the other.

The happy couple want to know if they can achieve this with the number of tables they have ordered for the banquet. So, they want to solve the following WEDDING PLANNING decision problem:

**Instance:** \((A, K, t)\), where \( A \) is a set (the guests), \( K \subseteq A \times A \) is a set of pairs ((\( a, b \)) \( \in K \) means that \( a \) knows \( b \)), and \( t \in \mathbb{Z}^+ \) (the number of tables).

**Question:** Is there a partition of \( A \) into subsets \( A_1, A_2, \ldots, A_m \), \( m \leq t \), such that, for all \( i \in [1..m] \), if \( a, b \in A_i \) and \( a \neq b \) then \((a, b) \in K\)? (Recall that a partition of a set \( A \) is a collection of mutually disjoint subsets of \( A \) whose union is \( A \).)

Prove that the WEDDING PLANNING problem is NP-complete.

Question 2. (30 marks) Scarborough Consulting Services, better known as ScarCon, is a start-up that provides a set of consulting services \( S \), and has a payroll budget \( b \). ScarCon wants to hire employees from a set \( A \) of applicants. Each applicant \( a \) is qualified to provide some subset of the services and requires a certain salary.

a. ScarCon wants to know if they can hire a subset of the applicants who can collectively provide all the services in \( S \) and whose total salary requirements do not exceed the payroll budget \( b \). So, they want to solve the HIRING decision problem, stated below:
Instance: \((S, A, Q, R, b)\), where \(S\) is the set of services; \(A\) is the set of applicants; \(Q\) is a function that maps each applicant \(a\) to a subset \(Q(a)\) of \(S\) (the services for which applicant \(a\) is qualified); \(R\) is a function that maps each applicant \(a\) to a positive integer \(R(a)\) (\(a\)'s salary requirement); and \(b \in \mathbb{Z}^+\) (the payroll budget).

Question: Is there a set \(H \subseteq A\) such that \(\bigcup_{a \in H} Q(a) = S\) and \(\sum_{a \in H} R(a) \leq b\)?

Prove that Hiring is \(\text{NP}\)-complete. (Hint: Consider the special case where \(R(a) = 1\) for all \(a \in A\).)

b. (15 marks) The problem that ScarCon really want to solve, however, is not the above decision problem but an optimization problem: Whom should they hire in order to minimize its payroll while being able to provide all its services. More precisely, the Hiring Optimization problem is:

Instance: \((S, A, Q, R)\), where \(S\), \(A\), \(Q\), and \(R\) are as in the Hiring decision problem.

Output: A qualified set of applicants with minimum payroll, if one exists; otherwise the empty set. That is, output a set \(H \subseteq A\) such that \(\bigcup_{a \in H} Q(a) = S\) and for any set of applicants \(H' \subseteq A\), if \(\bigcup_{a \in H'} Q(a) = S\) then \(\sum_{a \in H} R(a) \leq \sum_{a \in H'} R(a)\); if no such set \(H\) exists, then output \(\emptyset\).

Give a Cook reduction (i.e., a polynomial-time Turing reduction) of the Hiring Optimization problem to the Hiring decision problem. Describe clearly your reduction in high-level pseudocode, and justify its correctness (clearly stating any invariants!) and running time.

THAT’S IT WITH HOMEWORK, FOLKS!