Homework Assignment #5
Due: March 15, 2023, by 11:59 pm

- You must submit your assignment through the Crowdmark system. You will receive by email an invitation through which you can submit your work in the form of separate PDF documents with your answers to each question of the assignment. To work with a partner, you and your partner must form a group on Crowdmark. Crowdmark does not enforce a limit on the size of groups. The course policy that limits the size of each group to at most two remains in effect: submissions by groups of more than two persons will not be graded.
- It is your responsibility to ensure that the PDF files you submit are legible. To this end, I encourage you to learn and use the LaTeX typesetting system, which is designed to produce high-quality documents that contain mathematical notation. You are not required to produce the PDF files you submit using LaTeX; you may produce it any way you wish, as long as the resulting document is legible.
- By virtue of submitting this assignment you (and your partner, if you have one) acknowledge that you are aware of the policy on homework collaboration for this course.
- Unless we explicitly state otherwise, you should justify your answers. Your paper will be graded based on the correctness of your answers, and the clarity, precision, and conciseness of your presentation.

Question 1. (20 marks) CLIQUE is the following decision problem:

Instance: \((G, k)\), where \(G = (V, E)\) is an undirected graph and \(k\) is a positive integer.

Question: Does \(G\) have a clique of size \(k\)? That is, is there a set \(V' \subseteq V\) such that \(|V'| = k\) and for all \(u, v \in V'\), \(\{u, v\} \in E\)?

Describe a polytime mapping reduction from CLIQUE to CNF-Sat, and prove that it is correct.

**NB:** We know that such a reduction exists by the Cook-Levin Theorem and the fact that CLIQUE \(\in \text{NP}\). The question here is not to argue that such a reduction exists, but to show one explicitly.

**Hint:** Given the graph \(G\) and positive integer \(k\), construct a CNF formula that is true if and only if \(G\) has a clique of size \(k\). Think of the clique as a sequence, rather than a set, of nodes so that you can refer to the first, second, etc. node in the clique.

For your formula introduce a propositional variable \(x_u^i\) for each node \(u\) of \(G\) and each position \(i\) in the sequence, \(1 \leq i \leq k\). Intuitively a truth assignment makes variable \(x_u^i\) true iff \(u\) is the \(i\)-th node in clique of \(G\). Write your formula in a modular way with subformulas expressing specific facts, analogous to (but much simpler than) the way we constructed the formula in the proof of the Cook-Levin Theorem. In your answer be sure to explain the role of each piece of your formula.

Question 2. (20 marks) Let us call a CNF formula frugal if each variable appears at most three times. Note that in formula \((x \lor y) \land (\neg x \lor y \lor \neg z) \land (x \lor \neg y \lor \neg z) \land \neg x\), variable \(x\) appears four times (twice as the positive literal \(x\) and twice as the negative literal \(\neg x\)), \(y\) appears three times, and \(z\) appears twice.
The FrugalSat problem is:
INSTANCE: \( \langle \phi \rangle \), where \( \phi \) is a frugal CNF formula.
QUESTION: Is \( \phi \) satisfiable?
Prove that FrugalSat is \( \text{NP} \)-complete.

**Hint:** Add new variables and new clauses to make the given CNF formula frugal, without changing whether it is satisfiable.

A related problem for you to think about but not to hand in with your answer: Let us call a CNF formula *measly* if each variable appears at most twice. The MeaslySat problem is:
INSTANCE: \( \langle \phi \rangle \), where \( \phi \) is a measly CNF formula.
QUESTION: Is \( \phi \) satisfiable?
Prove that MeaslySat is in \( \text{P} \).

**Hint:** Keep simplifying the formula to reduce the number of occurrences of each variable until it is obvious that the formula is satisfiable or that it is not.