Homework Assignment #4  
(worth 15% of the course grade)  
Due: August 12, 2020, by 10:00 am

- You must submit your assignment through the Crowdmark system. You will receive by email an invitation through which you can submit your work in the form of separate PDF documents with your answers to each question of the assignment. To work with a partner, you and your partner must form a group on Crowdmark. Crowdmark does not enforce a limit on the size of groups. The course policy that limits the size of each group to at most two remains in effect: submissions by groups of more than two persons will not be graded.
- It is your responsibility to ensure that the PDF files you submit are legible. To this end, I encourage you to learn and use the LaTeX typesetting system, which is designed to produce high-quality documents that contain mathematical notation. You are not required to produce the PDF files you submit using LaTeX; you may produce them any way you wish, as long as the resulting document is legible.
- By virtue of submitting this assignment you (and your partner, if you have one) acknowledge that you are aware of the policy on homework collaboration for this course.
- For any question, you may use results previously described in class, or in prerequisites of this course, without describing them. You may also use any result that we covered in class, or is in the assigned sections of the official course textbook, by referring to it.
- Unless we explicitly state otherwise, you should justify your answers. Your paper will be graded based on the correctness and efficiency of your answers, and the clarity, precision, and conciseness of your presentation.

**Question 1.** (10 marks) The disjoint set problem, abbreviated DisjSets, is the following decision problem:

**Instance:** $\langle U, C, k \rangle$, where $U$ is a set (the “universe”), $C \subseteq \mathcal{P}(U)$ is a collection of subsets of $U$, and $k$ is a positive integer.

**Question:** Are there $k$ disjoint sets in $C$?

For example, if $U = \{ n \in \mathbb{Z} : 1 \leq n \leq 10 \}$, $C = \{ \{1, 3, 5\}, \{2, 3, 7\}, \{2, 7, 9\}, \{1, 2\} \}$, then the answer for instance $\langle U, C, 2 \rangle$ is yes, but the answer for instance $\langle U, C, 3 \rangle$ is no.

Prove that DisjSets is NP-complete.

**Hint:** Use the fact that IndependentSet is NP-complete.

**Question 2.** (10 marks) An $(s, t)$ Hamiltonian path in a directed graph $G = (V, E)$, where $s, t \in V$ and $s \neq t$, is a path in $G$ that starts in $s$, ends in $t$, and visits every node exactly once. The $(s, t)$-Directed Hamiltonian Path (DHP) problem is the following decision problem:

**Instance:** $\langle G, s, t \rangle$, where $G = (V, E)$ is a directed graph, and $s, t \in V$ such that $s \neq t$.

**Question:** Does $G$ have an $(s, t)$ Hamiltonian path?

Prove that $(s, t)$-DHP is an NP-complete problem.
Note. Having proved that \((s,t)\)-DHP is \(\text{NP}\)-complete, one can prove that the corresponding problem in undirected graphs is also \(\text{NP}\)-complete. You can do this by reducing \((s,t)\)-DHP to its undirected counterpart using the same trick that we used to reduce DHC (the Directed Hamiltonian Cycle problem) to UHC (its undirected counterpart). You should do this as an exercise, but don’t include it with your answer to this question.

**Question 3.** (10 marks) Vassos has \(n\) topics \(P_1, P_2, \ldots, P_n\) that he wants to present to his class. Each topic \(P_i\) has an associated duration \(d_i\) — the amount of time that he needs to present it. He must present these topics in \(k\) lectures, each of duration at most \(T\). (He does not need to completely fill each lecture.)

The topics can be presented in any order: any one of them could be in any lecture. Vassos’ problem is to determine whether he can present all \(n\) topics in the \(k\) lectures without exceeding the length of any lecture.

For example, if he has 10 topics to present, 5 of which take 10 minutes each, 2 take 15 minutes each, 1 takes 20 minutes, and 2 take 25 minutes each (for a total of 150 minutes) he can present all of them in three 50-minute lectures. On the other hand, if he has 8 topics to present, 1 of which takes 5 minutes, 1 takes 15 minutes, 4 take 20 minutes each, and 2 take 25 minutes each (again for a total of 150 minutes) it is impossible to present them all in three 50-minute lectures.

Vassos’s dilemma (VDil) is the following decision problem:

**Instance:** \((D, k, T)\), where \(D = d_1, d_2, \ldots, d_n\) is a sequence of positive integers (the durations of the topics that Vassos wants to present); \(k \geq 2\) is a positive integer (the number of lectures in which to present these topics); and \(T\) is a positive integer (the maximum duration of each lecture).

**Question:** Can Vassos present these \(n\) topics (in any order) in \(k\) lectures each of duration at most \(T\)?

Prove that VDil is an \(\text{NP}\)-complete problem.

**Hint:** Reduce a problem we have proved to be \(\text{NP}\)-complete to VDil when \(k = 2\). This is simpler to do, and proves a stronger result: VDil is \(\text{NP}\)-complete even in the special case \(k = 2\). (It is easy to see that for \(k = 1\) the problem is in \(\text{P}\).)

**Question 4.** (10 marks) [EXTRA CREDIT] Prove that the 3-colourability problem from Assignment 3 is \(\text{NP}\)-complete, using a reduction from 3Sat that uses the “gadgets” described in Figure 1. Do not prove this using a different reduction!

**Intuition and hints:** Given a 3-CNF formula \(\phi\) you want to construct a graph \(G_\phi\) that is 3-colourable if and only if \(\phi\) is satisfiable.

For each variable \(x\) in \(\phi\) introduce a “variable gadget” as shown in Figure 1(a). For the entire graph introduce a single “colour palette gadget” as shown in Figure 1(b). For each clause \(\ell \lor \ell' \lor \ell''\) of \(\phi\) introduce a “clause gadget” as shown in Figure 1(c). Your goal is to assemble the graph \(G_\phi\) from such gadgets.

The colour palette is a clique of size 3 and its three nodes must therefore have three different colours, which we think of as \(T\) (“true”), \(F\) (“false”), and \(N\) (“neutral”).

We want to force each variable gadget node to be coloured \(T\) or \(F\). How should you connect it to the palette gadget to ensure this? Note that once we have forced the two nodes of each variable gadget to be coloured \(T\) or \(F\), the edge between them ensures that they won’t both be assigned the same colour, which is what we want: a variable and its negation must not have the same truth value.

Next consider the “clause gadget” shown in Figure 1(c). The nodes \(\ell\), \(\ell'\), and \(\ell''\) represent literals, which are nodes of the variable gadgets. If these nodes are assigned colours \(T\) or \(F\) (in any combination) and we force the node labeled \(\ell \lor \ell' \lor \ell''\) to be coloured \(T\), the remaining nodes of the gadget can be coloured consistently (meaning that no adjacent nodes are assigned the same colour) using our three colours \(T\), \(F\), and \(N\) if and only if at least one of \(\ell\), \(\ell'\), and \(\ell''\) is coloured \(T\). (This fact is something that you must prove as part of the argument that your reduction works.) So, in some sense, this gadget can be made to act like a logical-or of the truth values (i.e., colours) of \(\ell\), \(\ell'\), and \(\ell''\).

\(^1\)Smart-aleck answers such as “yes, by speaking faster and faster, as he is known to do”, receive a chuckle but no credit.
Given the above discussion you should be able to describe how to combine these gadgets to construct from $\phi$ a graph $G_\phi$ that is 3-colourable if and only if $\phi$ is satisfiable.