Homework Assignment #3
(worth 15% of the course grade)
Due: August 3, 2020, by 10 am

You must submit your assignment through the Crowdmark system. You will receive by email an invitation through which you can submit your work in the form of separate PDF documents with your answers to each question of the assignment. To work with a partner, you and your partner must form a group on Crowdmark. Crowdmark does not enforce a limit on the size of groups. The course policy that limits the size of each group to at most two remains in effect: submissions by groups of more than two persons will not be graded.

It is your responsibility to ensure that the PDF files you submit are legible. To this end, I encourage you to learn and use the LaTeX typesetting system, which is designed to produce high-quality documents that contain mathematical notation. You are not required to produce the PDF files you submit using LaTeX; you may produce them any way you wish, as long as the resulting document is legible.

By virtue of submitting this assignment you (and your partner, if you have one) acknowledge that you are aware of the policy on homework collaboration for this course.

For any question, you may use results previously described in class, or in prerequisites of this course, without describing them. You may also use any result that we covered in class, or is in the assigned sections of the official course textbook, by referring to it.

Unless we explicitly state otherwise, you should justify your answers. Your paper will be graded based on the correctness and efficiency of your answers, and the clarity, precision, and conciseness of your presentation.

Question 1. (10 marks) Let $f: \mathbb{N} \to \mathbb{N}$. \textsc{TIME}(f(n)) denotes the set of languages that are decided by Turing machines whose running time is in $O(f(n))$. Thus, $P = \bigcup_{k \in \mathbb{N}} \text{TIME}(n^k)$.

Similarly, we define $\text{EXPTIME} = \bigcup_{k \in \mathbb{N}} \text{TIME}(2^{n^k})$; that is, $\text{EXPTIME}$ is the set of languages that can be decided in exponential time (where the exponent can be any polynomial). Clearly $P \subseteq \text{EXPTIME}$.

We proved that there is a decidable language that cannot be decided in polynomial time, i.e., that is not in $P$. In fact, the decidable language we proved is not in $P$ is in $\text{EXPTIME}$. Could it be that all decidable languages can be decided in exponential time; i.e., that they are all in $\text{EXPTIME}$?

Using a similar argument, prove that this is not the case; i.e., prove that there is a decidable language that is \textit{not} in $\text{EXPTIME}$.

\textbf{Hint:} You may use the fact that for any $k \in \mathbb{N}$, $2^{n^k} \leq 2^{2^n}$ for all sufficiently large $n$.

Question 2. (10 marks) Let $G = (V, E)$ be an undirected graph and $\text{wt} : V \to \mathbb{Z}^+$ be a node weight function. The \textit{weight} of a set of nodes $V' \subseteq V$, denoted $\text{wt}(V')$, is the sum of their weights: $\text{wt}(V') = \sum_{u \in V'} \text{wt}(u)$.

\textsc{MaxWeightClique} is the following optimization problem:

\textbf{INPUT:} $\langle G, \text{wt} \rangle$, where $G = (V, E)$ is an undirected graph and $\text{wt} : V \to \mathbb{Z}^+$ is a node weight function.
Output: \( \langle V' \rangle \), where \( V' \) is a maximum weight clique of \( G \); i.e., a set \( V' \subseteq V \) such that (a) \( V' \) is a clique of \( G \) (any two nodes in \( V' \) are connected by an edge in \( E \)), and (b) for any clique \( V'' \) of \( G \), \( \text{wt}(V') \geq \text{wt}(V'') \).

Formulate a decision problem \( \Pi \) that is polytime equivalent to MAXWEIGHTCLIQUE: there is a polytime algorithm for each of the two problems that uses an oracle (a subroutine given as a “black box”) that solves the other problem in one step. Prove that your problem \( \Pi \) has this property.

**Question 3.** (10 marks) Let \( G = (V, E) \) be an undirected graph, and \( k \) be a positive integer. \( G \) is \( k \)-colourable if there is a function \( f : V \rightarrow \{1, 2, \ldots, k\} \) such that if \( \{u, v\} \in E \) then \( f(u) \neq f(v) \). That is, it is possible to assign one of up to \( k \) “colours” to each of \( G \)’s nodes so that no two adjacent nodes are assigned the same colour.

The 3-colourability decision problem, abbreviated 3Col, is as follows:

**Instance:** \( \langle G \rangle \), where \( G = (V, E) \) is an undirected graph.

**Question:** Is \( G \) 3-colourable?

Give an explicit polytime reduction showing that 3Col \( \leq_p \) 3Sat, and prove that your reduction is correct.

**Question 4.** (10 marks) A 3-CNF propositional formula \( F \) is uniform if and only if every clause contains only positive or only negative literals, i.e., it is of the form \((x \lor y \lor z)\) or \((\neg x \lor \neg y \lor \neg z)\), where \( x, y, z \) are propositional variables.

Prove that the satisfiability problem even for uniform 3-CNF formulas is \( \text{NP} \)-complete.

**Hint:** Reduce 3Sat to this problem, by adding new variables and new clauses to make the given 3-CNF formula uniform.