Computer Science C63 — January 12, 2022
Scarborough Campus — University of Toronto

Homework Assignment #1
Due: January 19, 2022, by 11:59 pm

- ***You must submit your assignment through the Crowdmark system.*** *You will receive by email an invitation through which you can submit your work in the form of separate PDF docucments with your answers to each question of the assignment. To work with a partner, you and your partner must form a group on Crowdmark. Crowdmark does not enforce a limit on the size of groups.* ***The course policy that limits the size of each group to at most two remains in effect:*** *submissions by groups of more than two persons will not be graded.*
- *It is your responsibility to ensure that the PDF files you submit are legible. To this end, I encourage you to learn and use the LaTex typesetting system, which is designed to produce high-quality documents that contain mathematical notation. You are not required to produce the PDF files you submit using LaTex; you may produce it any way you wish, as long as the resulting document is legible.*
- *By virtue of submitting this assignment you (and your partner, if you have one) acknowledge that you are aware of the policy on homework collaboration for this course.*[a]
- *For any question, you may use facts previously proved in this course, its prerequisites, or in the assigned sections of the textbook.*
- *Unless we explicitly state otherwise, you should justify your answers. Your paper will be graded based on the correctness a of your answers, and the clarity, precision, and conciseness of your presentation.*

[a] "In each homework assignment you may collaborate with at most one other student who is currently taking CSCC63. If you collaborate with another student on an assignment, you and your partner must submit only one copy of your solution, with both of your names. The solution will be graded in the usual way and both partners will receive the same mark. Collaboration involving more than two students is not allowed. **For help with your homework you may consult only the instructor, TAs, your homework partner (if you have one), your textbook, and your class notes. You may not consult any other source.**"

Question 1. (20 marks) Prove each of the following facts.

(a) Let A be a set and $X = x_0, x_1, x_2, \ldots$ be an infinite sequence such that every element of A appears in X at least once. (Thus, X is an enumeration of A with possible repetition.) Prove that A is countable.

(b) If A is a countable set and $B \subseteq A$ then B is countable.

(c) If A_i is a countable set for every $i \in \mathbb{N}$, then $\cup_{i \in \mathbb{N}} A_i$ is countable. This fact is sometimes phrased as: The countable union of countable sets is countable.

(Question to think about, but not to include in your answer: Use this to prove that the set of finite sequences of natural numbers is countable. Do you see why diagonalization fails if we try to use it to prove that the set of finite sequences of natural numbers set is uncountable?)

(d) The set of all languages over the alphabet $\{0, 1\}$ is uncountable.

(Question to think about, but not to include in your answer: Is the set of all regular languages over the alphabet $\{0, 1\}$ countable or uncountable?)

Question 2. (20 marks) Let $f : \mathbb{N} \to \mathbb{N}$ be a function. We say that

- f is ***increasing*** if for every $i \in \mathbb{N}$, $f(i) \leq f(i + 1)$.

- f is ***decreasing*** if for every $i \in \mathbb{N}$, $f(i) \geq f(i+1)$.
- f is ***monotone*** if it is increasing or it is decreasing.

For each set of functions defined below, state whether it is countable or not, and justify[1] your answer.

(a) The set of functions from $\{0, 1\}$ to $\mathbb{N}$.

(b) The set of increasing functions from $\mathbb{N}$ to $\mathbb{N}$.

(c) The set of decreasing functions from $\mathbb{N}$ to $\mathbb{N}$.

(d) The set of functions from $\mathbb{N}$ to $\mathbb{N}$ that are monotone.

(e) The set of functions from $\mathbb{N}$ to $\mathbb{N}$ that are ***not*** monotone.

[1] "Justify" means "prove", but often signals a short argument.