On countable and uncountable sets

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A set A is ***countable*** or ***enumerable*** if it is finite or there is a bijection $f : \mathbb{N} \to A$ (i.e., A can be placed in a one-to-one correspondence with the natural numbers). A set is ***uncountable*** if it is not countable.

From the definition of countable set it follows immediately that A is countable if and only if there is a finite or infinite sequence $S = a_0, a_1, a_2, \ldots$ such that every element of A appears once and only once in S. We call such a sequence an ***enumeration*** of the set A.

In lecture we saw examples of countable and uncountable sets. In particular, we saw two proof techniques, both due to Cantor: ***Dovetailing*** (merging countably many infinite sequences into a single infinite sequence), which can be used to prove that a set is countable; and ***diagonalization***, which can be used to prove that a set is not countable.

Following is a list of the results of this nature that we have proved using these techniques.

Theorem 1.1: *$\mathbb{Z}$ is countable.*

Theorem 1.2: *$\mathbb{N} \times \mathbb{N}$ is countable.*

Theorem 1.3: *$\mathbb{N}^k = \underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{k \text{ times}}$ is countable.*

Theorem 1.4: *The set of <u>infinite</u> binary strings is uncountable.*

Theorem 1.5: *The set of functions from $\mathbb{N}$ to $\mathbb{N}$ is uncountable.*

Theorem 1.6: *The set of <u>finite</u> strings over a (finite) alphabet is countable.*

We can leverage the fact that we know that certain sets are countable (or uncountable) to show that other sets have this property.

For example, consider the set of "step functions" from $\mathbb{N}$ to $\mathbb{N}$. A function $f : \mathbb{N} \to \mathbb{N}$ is called a ***step function*** if there is some $n_f \in \mathbb{N}$ and $a, b \in \mathbb{N}$ such that for all n, $0 \leq n \leq n_f$, $f(n) = a$ and for all $n > n_f$, $f(n) = b$. Is the set of step functions from $\mathbb{N}$ to $\mathbb{N}$ countable or uncountable?

Notice that a step function $f : \mathbb{N} \to \mathbb{N}$ can be uniquely and fully described by a triple of natural numbers (n_f, a, b). Sometimes we say that this triple ***encodes*** the function f: Given a step function there is a unique triple that corresponds to it; and conversely, given a triple, there is a unique step function defined by it. Since triples of natural numbers can be encoded by (correspond one-to-one to) natural numbers (see Theorem 1.3), we conclude that

Theorem 1.7: *The set of step functions from $\mathbb{N}$ to $\mathbb{N}$ is countable.*

The preceding argument exemplifies a useful way to prove that a set is countable: Show that every element of the set can be encoded by (i.e., placed into a one-to-one correspondence with) the elements of a set we have already proved is countable. Now, we can use Theorem 1.7 to show:

Theorem 1.8: *The set of non-step functions from $\mathbb{N}$ to $\mathbb{N}$ is uncountable.*

PROOF. Suppose, for contradiction, that the set of non-step $\mathbb{N} \to \mathbb{N}$ functions is countable, and let $f_0, f_1, f_2, \ldots$ be an enumeration of this set. By Theorem 1.7, the set of $\mathbb{N} \to \mathbb{N}$ step functions is countable, so let $g_0, g_1, g_2, \ldots$ be an enumeration of this set. Then we can enumerate the set of all $\mathbb{N} \to \mathbb{N}$ functions by dovetailing as $f_0, g_0, f_1, g_1, f_2, g_2, \ldots$, contradicting Theorem 1.5. $\square$

As an exercise, try to prove Theorem 1.8 using diagonalization.