Definition of the “yields” relation ⊢

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Let $M = (Q, \Sigma, \Gamma, \delta, q_0, h_A, h_R)$ be a Turing machine. Without loss of generality, we assume that $Q \cap \Gamma = \emptyset$, so that state symbols cannot be confused with tape symbols.

Notational conventions:

- Lower case characters near the beginning of the alphabet ($a, b, c, \ldots$) denote tape symbols (elements of $\Gamma$).
- Lower case characters near the end of the alphabet ($w, x, y, z, \ldots$) denote strings of tape symbols (elements of $\Gamma^*$).
- $p, q$ (decorated with accents, subscripts, superscripts etc.) denote states (elements of $Q$).
- $\bot$ is the blank symbol (element of $\Gamma$).

A configuration of $M$ is a string of the form $xqy$, where $x, y \in \Gamma^*$ and $q \in Q$, where $y$ does not end with the blank symbol $\bot$. This describes the complete state of the Turing machine at some point in its computation: The machine is in state $q$, its tape contains the string $xy$ starting in cell 1 (the leftmost cell) followed by an infinite number of blanks; and the tape head is positioned over cell $|x| + 1$, i.e., the first symbol of $y$, if $y \neq \epsilon$, or the leftmost of the infinite sequence of trailing blanks, if $y = \epsilon$.

We define the relation $\vdash_M$ between configurations (written simply $\vdash$, if $M$ is clear from the context) to hold if $M$ can move from one configuration to the other in a single step, based on its transition function. More precisely, let $C = xqy$. Then,

Case 1. $y = ay'$, for some $a \in \Gamma$. (Thus, $y \neq \epsilon$, and if $a = \bot$ then $y' \neq \epsilon$.)

- if $\delta(q, a) = (p, b, R)$ then
  \[ C \vdash_M C' \iff C' = xbp'y'. \]

- if $\delta(q, a) = (p, b, L)$ and $x = x'c$ for some $c \in \Gamma$ (thus $x \neq \epsilon$ and the head is not on cell 1) then
  \[ C \vdash_M C' \iff \begin{cases} C' = x'pcby', & \text{if } b \neq \bot \text{ or } y' \neq \epsilon \\ C' = x'pc, & \text{if } b = \bot \text{ and } y' = \epsilon \text{ and } c \neq \bot \\ C' = x'p, & \text{if } b = \bot \text{ and } y' = \epsilon \text{ and } c = \bot \end{cases} \]

- if $\delta(q, a) = (p, b, L)$ and $x = \epsilon$ (thus the head is on cell 1) then
  \[ C \vdash_M C' \iff \begin{cases} C' = pb'y', & \text{if } b \neq \bot \text{ or } y' \neq \epsilon \\ C' = p, & \text{if } b = \bot \text{ and } y' = \epsilon \end{cases} \]

Case 2. $y = \epsilon$. (Thus, in $C$ the tape head is on the leftmost of the infinitely many trailing blanks.)

- if $\delta(q, \bot) = (p, b, R)$ then
  \[ C \vdash_M C' \iff C' = xbp. \]
• if $\delta(q, \sqcup) = (p, b, L)$ and $x = x'c$ for some $c \in \Gamma$ (thus $x \neq \epsilon$ and the head is not on cell 1) then

$$C \vdash_M C' \iff \begin{cases} C' = x'pcb, & \text{if } b \neq \sqcup \\ C' = x'pc, & \text{if } b = \sqcup \text{ and } c \neq \sqcup \\ C' = x'p, & \text{if } b = \sqcup \text{ and } c = \sqcup \end{cases}$$

• if $\delta(q, \sqcup) = (p, b, L)$ and $x = \epsilon$ (thus the head is on cell 1) then

$$C \vdash_M C' \iff \begin{cases} C' = pb, & \text{if } b \neq \sqcup \\ C' = p, & \text{if } b = \sqcup \end{cases}$$