Three-dimensional matching

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The ***three-dimensional matching*** problem 3DM, also known as ***tripartite matching***, is the following decision problem:

Instance: $\langle A, B, C, T\rangle$, where A, B, C are finite sets of the same cardinality, and $T \subseteq A \times B \times C$.

Question: Is there a subset $M \subseteq T$ so that $|M| = |A|$ and the triples in M are disjoint in every component: if (a, b, c) and (a', b', c') are distinct triples in M then $a \neq a'$ and $b \neq b'$ and $c \neq c'$? Such a subset M of T is called a ***(tripartite) matching*** of (A, B, C, T).

Note that, by definition, for each element $a \in A$ there is exactly one triple in a matching M that contains a; and similarly for each element of B and C. We say that each element of A, B, and C is covered (with no overlap) by M. Conversely, if a set M' contains exactly n triples from $A \times B \times C$ and each element of A, B, and C is contained in one of these triples, then M' is a matching (it cannot contain overlapping triples in any of the three dimensions).

Theorem 9.5 3DM *is **NP**-complete.*

PROOF. It is straightforward to show that 3DM $\in$ **NP**: A nondeterministic Turing machine can, in polynomial time, "guess" a list of n triples, where $n = |A| = |B| = |C|$ and then check that (a) all the triples on this list are in T, and (b) for each element of A, B, and C there is a triple on the list that contains it.

We prove that 3DM is **NP**-hard by showing that 3SAT $\leq^p_m$ 3DM. Given a 3-CNF formula F we show how to construct, in polynomial time, an instance (A, B, C, T) of 3DM so that

$$F \text{ is satisfiable if and only if } (A, B, C, T) \text{ has a matching.} \qquad (*)$$

First we explain the reduction. Let $x_1, x_2, \ldots, x_n$ be the variables that appear in F, and let F consist of m clauses $C_1, C_2, \ldots, C_m$, where C_j is the disjuction of three literals ℓ^1_j, ℓ^2_j, and ℓ^3_j. So, each literal ℓ^t_j is either x_i (positive literal) or $\overline{x}_i$ (negative literal) for some variable x_i.

The instance (A, B, C, T) of 3DM that we will construct from F has triples that we will put in three groups, each serving a specific purpose:

Group I triples: For each variable x_i and clause C_j we add to T two triples $(a_{ij}, b_{ij}, x^1_{ij})$ and $(a_{ij\oplus 1}, b_{ij}, x^0_{ij})$, where $j \oplus 1 = (j \bmod m) + 1$ (i.e., increment that "circles back" to 1 after m). Intuitively, if the matching contains the triple $(a_{ij}, b_{ij}, x^1_{ij})$, then the variable x_i is assigned the value 1 (true); and if it contains $(a_{ij\oplus 1}, b_{ij}, x^0_{ij})$, then x_i is assigned the value 0 (false). Note that the triples that correspond to variable x_i for different clauses j are joined in a "crown" shape as shown here (pages 1-7) for an example were we have four clauses ($m = 4$); the shaded triples correspond to variable x_1 being set to 1, and the clear ones correspond to x_1 being set to 0. Because of this pattern of interconnection, a matching must choose either all the shaded or all the clear triples; in this way the variable x_i has a consistent value for all clauses j. For the Group I triples, the a_{ij}s belong to set A, the b_{ij}s belong to set B, and the x^1_{ij}s and x^0_{ij}s belong to set C. Inspired by the visualization of how these triples are interconnected we will refer to the third component of each of them as its ***tip***.

Group II triples: Next we define $3m$ triples, one for each literal appearing in a clause. Consider clause $C_j = \ell^1_j \vee \ell^2_j \vee \ell^3_j$. We will define three triples for C_j, one for each literal ℓ^t_j. For these three triples we introduce two new elements $a_j \in A$ and $b_j \in B$. (These are not to be confused with the a_{ij}s and b_{ij}s defined for the Group I triples.)

- If $\ell_j^t = x_i$, i.e., ℓ_j^t is a positive literal for variable x_i, then we add to T a triple (a_j, b_j, x_{ij}^0).
- If $\ell_j^t = \overline{x}_i$, i.e., ℓ_j^t is a negative literal for variable x_i, then we add to T a triple (a_j, b_j, x_{ij}^1).

Note carefully that a positive literal's triple has x_{ij}^0 as its tip, while the negative literal's triple has x_{ij}^1 as its tip. Thus, each clause C_j contributes three such triples, one for each of its literals, all involving the two elements a_j and b_j and having as their third component one of the tips of Group I triples. For an illustration see here (pages 8-12).

The interpretation of the Group II triples is as follows: Because the three triples that correspond to clause C_j share a_j and b_j, and these are the only triples that contain these elements, a matching must include exactly one of them. We want to think of the corresponding literal of C_j as one that satisfies the clause. Thus, if the triple (a_j, b_j, x_{ij}^1) is selected, which according to the definition means that the corresponding literal is $\overline{x}_i$, the matching must include the Group I triples with tip x_{ij}^0 (to avoid conflict with the Group I triple with tip x_{ij}^1). And this, according to our interpretation of the Group I triples, means that x_i is assigned 0 and thus satisfies the literal $\overline{x}_i$. By a similar reasoning, if the triple (a_j, b_j, x_{ij}^0) is selected, x_i is assigned 1 and satisfies the literal x_i.

Group III triples: Group I and II triples involve $mn + m$ elements of A and $mn + m$ elements of B but $2mn$ elements of C (the tips). Therefore, these triples can cover all elements of A and B but will leave $2mn - (mn + m) = m(n - 1)$ elements of C uncovered. To make all three sets have the same cardinality we add to A (respectively B) $m(n-1)$ new elements denoted $\widehat{a}_k$ (respectively $\widehat{b}_k$), for $k \in [1..m(n-1)]$. And to ensure that all elements of C that remain uncovered by triples of Group I and II can be covered, and we add to T triples $(\widehat{a}_k, \widehat{b}_k, x_{ij}^1)$ and $(\widehat{a}_k, \widehat{b}_k, x_{ij}^0)$ for every $i \in [1..n]$, $j \in [1..m]$, and $k \in [1..m(n-1)]$.

To recap, the instance (A, B, C, T) of 3DM constructed from the 3-CNF formula F with variables $x_1, x_2, \ldots x_n$ and clauses $C_1, C_2, \ldots, C_m$, where $C_j = (\ell_j^1 \vee \ell_j^2 \vee \ell_j^3)$ for literals ℓ_j^1, ℓ_j^2, and ℓ_j^3 is as follows:

$$
\begin{aligned}
A &= \{a_{ij} : \; i \in [1..n] \text{ and } j \in [1..m]\} \cup \{a_j : \; j \in [1..m]\} \cup \{\widehat{a}_k : \; k \in [1..m(n-1)]\}.\\
B &= \{b_{ij} : \; i \in [1..n] \text{ and } j \in [1..m]\} \cup \{b_j : \; j \in [1..m]\} \cup \{\widehat{b}_k : \; k \in [1..m(n-1)]\}.\\
C &= \{x_{ij}^1, x_{ij}^0 : \; i \in [1..n] \text{ and } j \in [1..m]\}.\\
T &= \{(a_{ij}, b_{ij}, x_{ij}^1), (a_{ij\oplus 1}, b_{ij}, x_{ij}^0) : \; i \in [1..n] \text{ and } j \in [1..m]\}\\
&\quad \cup \{(a_j, b_j, x_{ij}^0) : \; j \in [1..m],\, i \in [1..n], \text{ and } \ell_j^t = x_i \text{ for some } t \in \{1,2,3\}\}\\
&\quad \cup \{(a_j, b_j, x_{ij}^1) : \; j \in [1..m],\, i \in [1..n], \text{ and } \ell_j^t = \overline{x}_i \text{ for some } t \in \{1,2,3\}\}\\
&\quad \cup \{(\widehat{a}_k, \widehat{b}_k, x_{ij}^1), (\widehat{a}_k, \widehat{b}_k, x_{ij}^0) : \; i \in [1..n],\, j \in [1..m], \text{ and } k \in [1..m(n-1)]\}.
\end{aligned}
$$

The sets A, B, and C have $2mn$ elements each, and T has $2mn + 3m + mnm(n-1) = O(m^2n^2)$ triples. Therefore the size of (A, B, C, T) is a polynomial of the size of F (m clauses of three variables each, on n variables). So, $\langle A, B, C, T\rangle$ can be computed from $\langle F\rangle$ in polynomial time.

It remains to show that the construction satisfies $(*)$.

[ONLY IF] Suppose F is satisfiable and let τ be a truth assignment that satisfies it. Then collect triples from T into a set M (that will become a matching) as follows:

(1) For all $i \in [1..n]$,

- if $\tau(x_i) = 1$ then add to M the triples $(a_{ij}, b_{ij}, x_{ij}^1)$ for all $j \in [1..m]$;
- if $\tau(x_i) = 0$ then add to M the triples $(a_{ij\oplus 1}, b_{ij}, x_{ij}^0)$ for all $j \in [1..m]$.

These are Group I triples that cover all a_{ij}s and b_{ij}s, and mn of the x_{ij}^1s and x_{ij}^0s.

(2) For each $j \in [1..m]$, let $t \in \{1,2,3\}$ be such that $\tau(\ell_j^t) = 1$. Such a t must exist for every j, since τ satisfies every clause C_j of F. If there are multiple such ts for some j, pick any one of them. Then add to M the following triples from T:

- if $\ell_j^{t_j} = x_i$ then add (a_j, b_j, x_{ij}^0) to M
- if $\ell_j^{t_j} = \overline{x}_i$ then add (a_j, b_j, x_{ij}^1) to M.

These are Group II triples that cover all a_js and b_js, and m of the x_{ij}^0s and x_{ij}^1s.

(3) For each $k \in [1..m(n-1)]$, add to M a triple $(\widehat{a}_k, \widehat{b}_k, x_{ij}^b)$, where $b \in \{0,1\}$, for one of the $m(n-1)$ x_{ij}^bs that are not covered by triples added to M in (1) or (2).

By construction, M has the right number of triples $2mn$, and every element of A, B, and C is included in some triple of M, so M is a matching.

[If] Suppose M is a matching of (A, B, C, T). We will show that there is a truth assignment τ that satisfies F.

First consider the Group I triples associated with variable x_i, i.e., triples of the form $(a_{ij}, b_{ij}, x_{ij}^b)$ for $j \in [1..m]$ and $b \in \{0,1\}$. Since M is a matching, exactly one of the following is the case: either
(1) M contains $(a_{ij}, b_{ij}, x_{ij}^1)$ for <u>all</u> $j \in [1..m]$, or
(2) M contains $(a_{ij\oplus 1}, b_{ij}, x_{ij}^0)$ for <u>all</u> $j \in [1..m]$.
Accordingly define

$$\tau(x_i) = \begin{cases} 1, & \text{if (1) is the case} \\ 0, & \text{if (2) is the case.} \end{cases}$$

Next consider the Group II triples associated with clause C_j, $j \in [1..m]$, i.e., triples of the form (a_j, b_j, x_{ij}^b), for some $i \in [1..n]$ and $b \in \{0,1\}$. There are three such triples and exactly one of them is in M (because they all share a_j and b_j and no other triple has these elements). Let (a_j, b_j, x_{ij}^b) be the triple of this form that is in M. There are two cases:

Case 1. $b = 1$. Then, by definition of T, for some $t \in [1..3]$, $\ell_j^t = \overline{x}_i$. Since M contains (a_j, b_j, x_{ij}^1), it cannot contain $(a_{ij}, b_{ij}, x_{ij}^1)$ (otherwise two triples would have the same third component, contradicting that M is a matching); so M must contain $(a_{ij\oplus 1}, b_{ij}, x_{ij}^0)$ (because these are the only two triples that contain b_{ij}). Then, by the above definition of τ, $\tau(x_i) = 0$ and so $\tau(\overline{x}_i) = 1$. Since $\ell_j^t = \overline{x}_i$, τ satisfies one of the literals of clause C_j and therefore the entire clause.

Case 2. $b = 0$. By similar reasoning, τ satisfies clause C_j.

We have proved that τ satisfies every clause C_j; therefore F is satisfiable. □

The ***two-dimensional*** counterpart of 3DM is known as ***bipartite matching*** and is solvable in polynomial time by reduction to the maximum flow problem.