University of Toronto
Scarborough Campus
October 29, 2004

# CSC B36 Midterm Examination

**Aids allowed:** One $8.5 \times 11$ 'cheat sheet' (may be written on both sides)

**Duration:** One hour and fifty minutes

- There should be 6 pages in this exam booklet, including this cover page.
- Answer all questions.
- Put all answers in this booklet, in the spaces provided.
- For rough work, use the backs of the pages; ***these will not be marked***.
- Good luck!

Family Name ______________________ Given Name ____________________

Student Number ________________________________________

| Problem | Marks Rec'ved | Marks Worth |
|---|---|---|
| 1. | | 15 |
| 2. | | 15 |
| 3. | | 25 |
| 4. | | 15 |
| 5. | | 30 |
| TOTAL | | 100 |

**QUESTION 1.** (15 marks)

Recall that the ***height*** of a binary tree is the maximum number of ***edges*** on a path from the root to a leaf. Thus, the height of a binary tree consisting of a single node is 0.

**a.** (3 marks) In the space below, draw a binary tree that has height 2 and has the largest possible number of nodes among all binary trees of height 2.

**b.** (6 marks) Let $f : \mathbb{N} \to \mathbb{N}$ be the function defined by

$$f(h) = \text{maximum number of nodes of any binary tree of height } h$$

In the space below, write a recursive definition for $f$.

**c.** (6 marks) Use induction to prove that the function $f$ you defined recursively in part (b) satisfies the following: For all $h \in \mathbb{N}$, $f(h) = 2^{h+1} - 1$.

PROOF:

**QUESTION 2.** (15 marks)

Let $f : \mathbb{N} \to \mathbb{N}$ be the function defined recursively as follows:

$$f(n) = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ 2, & \text{if } n = 2 \\ f(n-3) + f(n-2), & \text{if } n \geq 3 \end{cases}$$

Prove that for all integers $n \geq 5$, $f(n) < f(n+1)$.

PROOF:

**QUESTION 3.** (25 marks)

Let $F$ be the formula $(x \vee y) \to (u \wedge v)$. ***Without using truth tables:***

**a.** (15 marks) Prove that $F$ is logically equivalent to $(x \to u) \wedge (y \to u) \wedge (x \to v) \wedge (y \to v)$.

PROOF:

**b.** (10 marks) Prove that $F$ is ***not*** logically equivalent to $\big((x \to u) \vee (y \to u)\big) \wedge \big((x \to v) \vee (y \to v)\big)$. (*Hint:* Compare this formula to the one in part (a).)

PROOF:

**QUESTION 4.** (15 marks)

For each of the statements (a)–(d) below, indicate whether it is true or false. Do not justify your answers. ***Do not guess:*** 3 for each correct answer, 0 for no answer, −1.5 for wrong answer.

(a) $\big((x \wedge \neg y) \vee z\big) \wedge (\neg x \vee y)$ is in conjunctive normal form.

(b) For all sets of Boolean connectives $C$ and $C'$, if $C$ is complete and $C \subseteq C'$ then $C'$ is complete.

(c) The negation of every satisfiable formula is an unsatisfiable formula.

(d) For all propositional formulas $P$, $Q$ and $R$, $(P \wedge Q) \to R$ logically implies $P \to R$.

(e) For all propositional formulas $P$, $Q$ and $R$, $P \to R$ logically implies $(P \wedge Q) \to R$.

**QUESTION 5.** (30 marks)

The program SQRT($k$) below is an iterative version of the recursive program for computing the integer square root that we saw in Assignment #2. More precisely, the program satisfies the following specification:

**Precondition:** $k$ is a natural number.

**Postcondition:** SQRT($k$) returns $\lfloor\sqrt{k}\rfloor$ (i.e., the unique $u \in \mathbb{N}$ such that $u^2 \leq k < (u+1)^2$).

```
SQRT(k)
f := 0;   ℓ := k + 1
while ℓ > f + 1 do
    m := (f + ℓ) div 2
    if m² ≤ k then f := m
    else ℓ := m end if
end while
return f
```

Prove that this program is correct with respect to its specification. You may use, without proof, the following fact: For all integers $f, \ell$, if $f + 1 < \ell$ then $f < (f + \ell)$ **div** $2 < \ell$.

PROOF OF CORRECTNESS:

**[more space, if needed, on the following page]**

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**THE END**