University of Toronto
Scarborough Campus
October 25, 2003

CSC B36 Midterm Examination

Aids allowed: One 8.5×11 handwritten, non-photocopied 'cheat sheet'

Duration: One hour and fifty minutes

- There should be 6 pages in this exam booklet, including this cover page.
- Answer all questions.
- Put all answers in this booklet, in the spaces provided.
- For rough work, use the backs of the pages; ***these will not be marked***.
- Good luck!

Family Name ________________________ Given Name ____________________

Student Number ____

Problem	Marks Rec'ved	Marks Worth
1.		10
2.		20
3.		20
4.		10
5.		10
6.		30
TOTAL		100

QUESTION 1. (10 marks)

Use induction to prove that, for every $n \in \mathbb{N}$, $4^n - 1$ is a multiple of 3.

PROOF:

QUESTION 2. (20 marks)

Let $f : \mathbb{N} \to \mathbb{N}$ be the function defined recursively as follows:

$$f(n) = \begin{cases} 2, & \text{if } n = 0 \\ 3, & \text{if } n = 1 \\ 5, & \text{if } n = 2 \\ f(n-1) + 2f(n-3), & \text{if } n > 2 \end{cases}$$

Prove that for all integers $n \geq 5$, $f(n) \geq n^2$.

PROOF:

QUESTION 3. (20 marks)

Let P, Q and R be propositional formulas so that P is a ***tautology***, Q is ***unsatisfiable*** and R is a ***satisfiable formula that is not a tautology***. For each of the statements (a)–(d) below, indicate whether it is true or false. Do not justify your answers.

Do not guess: 4 for each correct answer, 0 for no answer, -1 for wrong answer.

(a) $\neg P$ is logically equivalent to Q.

(b) $R \rightarrow Q$ is a tautology.

(c) $P \rightarrow R$ is a tautology.

(d) $R \rightarrow P$ is a tautology.

(e) $R \leftrightarrow Q$ is satisfiable.

QUESTION 4. (10 marks)

For each of the statements (a)–(d) below, indicate whether it is true or false. Do not justify your answers.

Do not guess: 2 for each correct answer, 0 for no answer, -0.5 for wrong answer.

(a) $x \vee \neg y \vee z$ is in conjunctive normal form.

(b) $(x \rightarrow y) \vee (x \rightarrow z)$ is in disjunctive normal form.

(c) $x \rightarrow y$ is logically equivalent to $\neg x \rightarrow \neg y$.

(d) $x \leftrightarrow y$ is logically equivalent to $\neg x \leftrightarrow \neg y$.

(e) $x \wedge \neg x$ logically implies $y \rightarrow z$.

QUESTION 5. (10 marks)

Prove that the following propositional formula is **not** a tautology:

$$\neg\big((p \wedge q) \rightarrow (r \vee s)\big) \ \rightarrow \ \big((t \wedge \neg s) \rightarrow (u \vee v)\big)$$

(Hint: This formula has seven propositional variables, so its truth table has $2^7 = 128$ rows. This is too big to write down in an exam!)

PROOF:

QUESTION 6. (30 marks)

The program SQUARE(n) below returns n^2. More precisely, it is correct with respect to:

Precondition: n is a natural number.

Postcondition: SQUARE(n) returns n^2.

```
SQUARE(n)
sqr := 0
k := 1
while k ≠ 2n + 1 do
    sqr := sqr + k
    k := k + 2
end while
return sqr
```

a. (10 marks) Prove the following

Loop Invariant Lemma: For all $i \in \mathbb{N}$, if the loop is executed at least i times then $sqr_i = i^2$ and $k_i = 2i + 1$.

PROOF:

[continued on the next page]

b. (10 marks) Prove that SQUARE(n) is partially correct.

PROOF:

c. (10 marks) Prove that SQUARE(n) terminates.

PROOF:

THE END