Solutions for Homework Assignment #4

Answer to Question 1.

a. $\forall x\, \exists y\, L(x, y)$

This sentence says that for every x there is a larger y. This is true in both $\mathcal{N}$ and $\mathcal{Z}$: For every x, $x + 1$ is an example of a number that is greater than x.

b. $\forall x\, \exists y\, L(y, x)$

This sentence says that for every x there is a smaller y. This is false in $\mathcal{N}$ because if we choose $x = 0$, no natural number is smaller than it. On the other hand, the sentence is true in $\mathcal{Z}$ because for any x, $x - 1$ is an example of a number that is less than x.

c. $\forall x\, \forall y\, \Big(L(x, y) \to \forall u\, \forall v\, \big(P(x, x, u) \wedge P(y, y, v) \to L(u, v)\big)\Big)$

This sentence says that for any numbers x and y, if $x < y$ then $x^2 < y^2$. This is true in $\mathcal{N}$. It can be proved in a variety of ways (including induction on y). Here is a simple proof: $x^2 - y^2 = (x - y)(x + y)$. $x - y$ is negative (since $x < y$) and $x + y$ is positive (since x, y are both natural numbers and $x < y$), so their product is negative. Thus, $x^2 - y^2 < 0$, i.e., $x^2 < y^2$. On the other hand, this sentence is false in $\mathcal{Z}$: If we take $x = -2$ and $y = 0$, we have $x < y$ but $x^2 > y^2$.

d. $\forall x\, \exists y\, \Big(L(x, y) \wedge \forall u\, \forall v \big(P(x, x, u) \wedge P(y, y, v) \to L(u, v)\big)\Big)$

This sentence says that for each x there is a $y > x$ so that $y^2 > x^2$. This is true in both $\mathcal{N}$ and $\mathcal{Z}$: For each x we can choose $y = |x| + 1$. It is clear that $y > x$. Furthermore, $y^2 = |x|^2 + 2|x| + 1 = x^2 + 2|x| + 1 > x^2$.

e. $\exists x\, \forall y\, \big(L(x, y) \vee \approx(x, y)\big) \to \exists y\, \forall x\, \big(L(x, y) \vee \approx(x, y)\big)$

This sentence says that if there is a smallest element of the domain then there is a greatest element of the domain. This is false in $\mathcal{N}$ because there is a smallest element of the domain, namely 0, but there is no largest element of the domain. It is trivially true in $\mathcal{Z}$ because the antecedent is false: there is no smallest element of the domain in this case.

Answer to Question 2.

a. This assertion is false. For example, suppose $A(x)$ is the predicate "x is an odd number that is a multiple of 2", and $B(x)$ is the predicate "x is even". Then $\forall x\, \big(A(x) \to B(x)\big)$ is true (because, for each x $A(x)$ is false and so the conditional $A(x) \to B(x)$ is true), but obviously $\exists x\, \big(A(x) \wedge B(x)\big)$ is false, because no number x can be both odd and even.

b. This assertion is true.

$$
\begin{array}{lll}
 & \exists x\, \big(A(x) \to \neg B(x)\big) & \\
\text{LEQV} & \neg\forall x\, \neg\big(A(x) \to \neg B(x)\big) & \text{[by duality of quantifiers]} \\
\text{LEQV} & \neg\forall x\, \neg\big(\neg A(x) \vee \neg B(x)\big) & \text{[by } \to\text{-rule]} \\
\text{LEQV} & \neg\forall x\, \big(A(x) \wedge B(x)\big) & \text{[by DeMorgan and double negation]}
\end{array}
$$

c. This assertion is false. For example, let $A(x)$ be the predicate "x is male". Then $\exists x\, A(x) \wedge \exists x\, \neg A(x)$ is true since it asserts that there is someone who is male and someone who is not. However, $\exists x\, \big(A(x) \wedge \neg A(x)\big)$ is false, since it asserts that there is someone who is both male and not male.

d. This assertion is true. $\exists x\, A(x) \vee \exists x\, \neg A(x)$ asserts that some x satisfies $A(x)$ or some x satisfies $\neg A(x)$. This is true no matter what predicate the predicate symbol A stands for — i.e., it is a valid formula.

The formula $\exists x\, \big(A(x) \vee \neg A(x)\big)$ asserts that some x satisfies either $A(x)$ or $\neg A(x)$. This is also true no matter what predicate the symbol A stands for. Since both formulas are valid, they are logically equivalent.

e. This assertion is false. For example, let E be the formula $A(x)$ and F be the formula $\forall y A(y)$. In this case, the formula $\forall x\, E \leftrightarrow F$ is

$$G: \qquad \forall x\, A(x) \leftrightarrow \forall y\, A(y).$$

This formula is valid by the renaming logical equivalence and Theorem 5.11(b). On the other hand, the formula $\forall x\, (E \leftrightarrow F)$ is

$$H: \qquad \forall x\, \big(A(x) \leftrightarrow \forall y\, A(y)\big).$$

This formula is not valid: For example, if $A(x)$ stands for the predicate "$x = 0$", H is false. Since G is valid and H is not, the two formulas cannot be logically equivalent.

Answer to Question 3.

a. Goldbach's conjecture: $\forall x \Big(\big(L(\mathit{2}, x) \wedge \exists y P(\mathit{2}, y, x)\big) \rightarrow \exists y\, \exists z\, \big(\mathit{Prime}(y) \wedge \mathit{Prime}(z) \wedge S(y, z, x)\big)\Big)$

b. Twin prime conjecture: $\forall x\, \exists y\, \Big(L(x, y) \wedge \mathit{Prime}(y) \wedge \exists z\, \big(S(y, \mathit{2}, z) \wedge \mathit{Prime}(z)\big)\Big)$

c. Division theorem:

$$\begin{aligned}\forall x\, \forall y\, \Bigg(\neg{\approx}(y, 0) \rightarrow \exists q\, \exists r\, \exists u\, \Bigg(&\big(P(q, y, u) \wedge S(u, r, x) \wedge ({\approx}(\mathit{0}, r) \vee L(\mathit{0}, r)) \wedge L(r, y)\big) \wedge \\ &\forall q'\, \forall r'\, \forall u'\, \Big(\big(P(q', y', u') \wedge S(u', r', x) \wedge ({\approx}(\mathit{0}, r') \vee L(\mathit{0}, r')) \wedge L(r', y)\big) \rightarrow \\ &\qquad \big({\approx}(q, q') \wedge {\approx}(r, r')\big)\Big)\Bigg)\Bigg)\end{aligned}$$

Roughly speaking, the part of the first line after the implication symbol asserts the existence of a quotient q and a remainder r such that $x = q \cdot y + r$ and $0 \leq r < y$, and the second and third lines assert that the quotient and remainder are unique — in the sense that any two numbers q' and r' with the same property must be equal, respectively, to q and r. (The clause $({\approx}(\mathit{0}, r) \vee L(\mathit{0}, r))$ on the first line asserts that $r \geq 0$; since the domain of $\mathcal{N}$ is $\mathbb{N}$, this clause, and the corresponding clause on the second line, is not necessary and could be left out. The resulting formula would be a correct formula for the Division theorem in $\mathcal{N}$, but not in $\mathcal{Z}$. The formula given above is correct for both structures.)

Answer to Question 4. In the first step, we will rename variables so that each quantifier has a distinct variable.

	$(\forall x\, \neg\exists y\, A(x, y, z) \rightarrow \neg\forall x\, \exists y\, B(x, y, z)) \rightarrow \forall x\, (C(x) \wedge \exists z\, D(x, z))$	
LEQV	$(\forall x\, \neg\exists y\, A(x, y, z) \rightarrow \neg\forall t\, \exists u\, B(t, u, z)) \rightarrow \forall v\, (C(v) \wedge \exists w\, D(v, w))$	[Renaming, four times]
LEQV	$\neg(\neg\forall x\, \neg\exists y\, A(x, y, z) \vee \neg\forall t\, \exists u\, B(t, u, z)) \vee \forall v\, (C(v) \wedge \exists w\, D(v, w))$	[$\rightarrow$ rule, twice]
LEQV	$(\neg\neg\forall x\, \neg\exists y\, A(x, y, z) \wedge \neg\neg\forall t\, \exists u\, B(t, u, z)) \vee \forall v\, (C(v) \wedge \exists w\, D(v, w))$	[DeMorgan]
LEQV	$(\forall x\, \neg\exists y\, A(x, y, z) \wedge \forall t\, \exists u\, B(t, u, z)) \vee \forall v\, (C(v) \wedge \exists w\, D(v, w))$	[double negation, twice]
LEQV	$(\forall x\, \forall y\, \neg A(x, y, z) \wedge \forall t\, \exists u\, B(t, u, z)) \vee \forall v\, (C(v) \wedge \exists w\, D(v, w))$	[Duality]
LEQV	$\forall x\, \forall y\, \forall t\, \exists u\, (\neg A(x, y, z) \wedge B(t, u, z)) \vee \forall v\, \exists w\, (C(v) \wedge D(v, w))$	[Factoring, five times]
LEQV	$\forall x\, \forall y\, \forall t\, \exists u\, \forall v\, \exists w\, ((\neg A(x, y, z) \wedge B(t, u, z)) \vee (C(v) \wedge D(v, w)))$	[Factoring, six times]

Answer to Question 5.

a.

$$\exists s\, \exists b_1\, \exists b_2\, \exists t\, \exists r\, \exists d_1\, \exists d_2\, \big(\mathit{Subscriber}(s,n,a) \wedge \neg \approx (b_1, b_2) \wedge \mathit{Book}(b_1,t,r) \wedge \mathit{Book}(b_2,t,r) \wedge \\ \mathit{Borrowed}(s,b_1,d_1) \wedge \mathit{Borrowed}(s,b_2,d_2)\big).$$

b. $\exists b\, \mathit{Book}(b,t,\text{Chaucer})$.

c. $\exists t\, \exists b\, \exists s\, \exists a\, \big(\mathit{Subscriber}(s,n,a) \wedge \mathit{Borrowed}(s,b,2001/11/22) \wedge \mathit{Book}(b,t,\text{Chaucer})\big)$.

d. $\exists b\, \big(\mathit{Book}(b,t,\text{Joyce}) \ \wedge\ \forall r\, \forall b'\, (\mathit{Book}(b',t,r) \rightarrow r \approx \text{Joyce})\big)$.

e.

$$\exists b\, \exists r \Bigg(\mathit{Book}(b,t,r) \wedge \exists s\, \exists d\, \mathit{Borrowed}(s,b,d) \wedge \\ \forall s \Bigg(\exists b'\, \exists d\, \big(\mathit{Book}(b',t,r) \wedge \mathit{Borrowed}(s,b',d)\big) \rightarrow \\ \neg \exists b'\, \exists t'\, \exists r'\, \exists d\, \Big(\mathit{Book}(b',t',r') \wedge \mathit{Borrowed}(s,b',d) \wedge \neg\big(\approx(t,t') \wedge \approx(r,r')\big)\Big)\Bigg)\Bigg).$$

Also accepted: Above formula with the second conjunct, $\exists s\, \exists d\, \mathit{Borrowed}(s,b,d)$, removed. This is *not* logically equivalent to the above, as it will return a different answer. In particular, it will return titles of books not borrowed by *any* subscriber. There is an ambiguity in the use of "only" in English. For example, when we say "x has only sisters", we may mean that x has no brothers (which is true of people who have no siblings at all, as well as of people who have at least one sister but no brothers), or we may mean that x has at least one sister but no brothers.

Answer to Question 6.

a. We will prove that $(F \wedge G)$ logically implies H. Suppose, for contradiction, that it does not. Thus, there is an interpretation where both F and G are true, but H is false.

The fact that H is false means that $\neg H$ is true in this interpretation. In other words, $\neg \forall x \forall y\, (L(x,y) \rightarrow \neg L(y,x))$ holds. This is logically equivalent to $\exists x \exists y\, (L(x,y) \wedge L(y,x))$, and therefore this formula is true.

Let c and d be values for x and y (in the domain of the interpretation under consideration) which satisfy the formula $(L(x,y) \wedge L(y,x))$. (Such values must exist, since the formula $\exists x \exists y\, (L(x,y) \wedge L(y,x))$ is satisfied by the interpretation.)

Thus, we have that $L(c,d)$ and $L(d,c)$ both hold in this interpretation. By F (transitivity, which we assume is true), we get that $L(d,d)$ holds. This, however, contradicts the assumption that G (anti-reflexivity) holds.

b. We will prove that $(F \wedge G)$ does not logically imply I. It suffices to show an interpretation in which F and G are both true, but I is false.

Consider the following interpretation. The domain is the set of integers, and $L(x,y)$ is interpreted as the ordinary less-than ($<$) predicate between numbers. It is straightforward to verify that this relation satisfies anti-reflexivity (F), and transitivity (G), but does not satisfy I.

(Intuitively, I states that given two elements in the domain, one of which is "less than" the other, there is an element "between" them. Clearly, there is no integer between, say, 1 and 2.)

c. We will prove that $(F \wedge G)$ does not logically imply $\neg I$. It suffices to show an interpretation in which F and G are both true, but $\neg I$ is false, i.e., I is true.

Consider the following interpretation. The domain is the set of rational number, and $L(x, y)$ is interpreted as the ordinary less-than ($<$) predicate between numbers. It is straightforward to verify that this relation satisfies anti-reflexivity (F), transitivity (G), and I.

Aside: *A strict partial order that also satisfies I is called **dense**. The rationals or the reals with the ordinary less-than relation are examples of dense strict partial orders.*

d. We will prove that $(F \wedge G)$ does not logically imply J.

Intuitively, J states that any two elements in the domain of a strict partial order are either equal, or comparable by the "less than" predicate. Some of the examples of strict partial orders given in the question do not have this property.

For example, if the domain is the set of all persons, and $L(x, y)$ is interpreted as meaning that person x is an ancestor of person y, then there are two distinct persons neither of which is an ancestor of the other. For example, two siblings (persons who share both parents) are not ancestors of each other. Similarly, the "logically implies but is not logically equivalent to" relation between formulas does not have this property, as there are pairs of formulas such that neither formula logically implies the other. Since there are interpretations that satisfy F and G but falsify J, it follows that $(F \wedge G)$ does not logically imply J.

Aside: *A strict partial order that satisfies J is called a **total order**. The integers or rationals or reals with the ordinary less-than relation are examples of total orders.*