Computer Science B36 | November 1, 2006
Scarborough Campus | University of Toronto

Homework Assignment #4
Due: November 15, 2006, by 12 noon
(in the course drop box)

Appended to this document is a cover page for your assignment. Fill it out, staple your answers to it, and deposit the resulting document into the course drop box. Please do ***not*** *enclose your assignment in an envelope.*

Question 1. (10 marks) Consider the first-order language of arithmetic described in Section 6.2.1 of the notes. Let $\mathcal{N}$ and $\mathcal{Z}$ be structures for this language, with domains $\mathbb{N}$ and $\mathbb{Z}$, respectively, and the standard meaning for the predicate symbols. More formally:

$$\begin{aligned} S^{\mathcal{N}} &= \{(a,b,c) \in \mathbb{N}^3 : a+b=c\} & S^{\mathcal{Z}} &= \{(a,b,c) \in \mathbb{Z}^3 : a+b=c\} \\ P^{\mathcal{N}} &= \{(a,b,c) \in \mathbb{N}^3 : a\cdot b=c\} & P^{\mathcal{Z}} &= \{(a,b,c) \in \mathbb{Z}^3 : a\cdot b=c\} \\ L^{\mathcal{N}} &= \{(a,b) \in \mathbb{N}^2 : a<b\} & L^{\mathcal{Z}} &= \{(a,b) \in \mathbb{Z}^2 : a<b\} \\ \approx^{\mathcal{N}} &= \{(a,b) \in \mathbb{N}^2 : a=b\} & \approx^{\mathcal{Z}} &= \{(a,b) \in \mathbb{Z}^2 : a=b\} \\ \mathit{0}^{\mathcal{N}} &= 0 & \mathit{0}^{\mathcal{Z}} &= 0 \\ \mathit{1}^{\mathcal{N}} &= 1 & \mathit{1}^{\mathcal{Z}} &= 1 \end{aligned}$$

For each of the sentences below, state whether it is true or false in each of $\mathcal{N}$ and $\mathcal{Z}$. Justify your answer by translating the formula into a statement (in precise English) about numbers, and then explain why that statement is true or false for natural numbers and for integers.

a. $\forall x\, \exists y\, L(x,y)$

b. $\forall x\, \exists y\, L(y,x)$

c. $\forall x\, \forall y\, \Big(L(x,y) \to \forall u\, \forall v\, \big(P(x,x,u) \land P(y,y,v) \to L(u,v)\big)\Big)$

d. $\forall x\, \exists y\, \Big(L(x,y) \land \forall u\, \forall v \big(P(x,x,u) \land P(y,y,v) \to L(u,v)\big)\Big)$

e. $\exists x\, \forall y\, \big(L(x,y) \lor \approx(x,y)\big) \to \exists y\, \forall x\, \big(L(x,y) \lor \approx(x,y)\big)$

Question 2. (10 marks) For each of the following assertions, state whether it is true or false, and justify your answer. In (a)–(d) A and B are unary predicates in the first-order language.

a. $\forall x\, \big(A(x) \to B(x)\big)$ logically implies $\exists x\, \big(A(x) \land B(x)\big)$.

b. $\exists x\, \big(A(x) \to \neg B(x)\big)$ is logically equivalent to $\neg \forall x\, \big(A(x) \land B(x)\big)$.

c. $\exists x\, A(x) \land \exists x\, \neg A(x)$ is logically equivalent to $\exists x\, \big(A(x) \land \neg A(x)\big)$.

d. $\exists x\, A(x) \lor \exists x\, \neg A(x)$ is logically equivalent to $\exists x\, \big(A(x) \lor \neg A(x)\big)$.

e. For any first-order formulas E and F such that x does not appear free in F, $\forall x\, E \leftrightarrow F$ is logically equivalent to $\forall x\, (E \leftrightarrow F)$

Question 3. (10 marks) Consider the same first-order language of arithmetic as in Question 1, enriched with a new constant symbol $\boldsymbol{2}$ which is intended to represent the integer 2. In this question we will be working with this language in the structure $\mathcal{N}$ also defined in Question 1 (with the additional stipulation that $\boldsymbol{2}^{\mathcal{N}} = 2$. The following formula $Prime(x)$ expresses the predicate "x is prime":

$$L(\boldsymbol{1}, x) \wedge \forall y \forall z \left(P(y, z, x) \rightarrow (\approx(y, \boldsymbol{1}) \vee \approx(z, \boldsymbol{1}))\right)$$

a. ***Goldbach's conjecture*** asserts that every even integer greater than 2 can be expressed as the sum of two prime numbers. Nobody knows whether this is true or false. Write a formula to express Goldbach's conjecture. In your answer you may use the predicate $Prime(x)$ for which a formula was given above.

b. ***Twin primes*** are prime numbers that are two apart; e.g., 3 and 5 are twin primes, as are 17 and 19. The ***twin-prime conjecture*** asserts that there are infinitely many twin primes. Nobody knows whether this is true or false. Write a formula to express the twin-prime conjecture. In your answer you may use the predicate $Prime(x)$ for which a formula was given above. (**Hint:** You can say that there are infinitely many numbers with property P by saying that for each number there is a larger number with property P.)

c. Write a formula to express Proposition 1.7 (page 27) in the notes.

Question 4. (10 marks) Transform the following first-order formula into an equivalent PNF formula in which the quantifier-free part uses only the connectives $\wedge$, $\vee$ and $\neg$, and every occurrence of $\neg$ is applied to an *atomic* formula. Demonstrate the steps through which you obtain your final formula.

$$\left(\forall x \neg \exists y\, A(x, y, z) \rightarrow \neg \forall x\, \exists y\, B(x, y, z)\right) \rightarrow \forall x \left(C(x) \wedge \exists z\, D(x, z)\right)$$

Question 5. (10 marks) Consider the relational database for a library involving the relations $Subscriber(s, n, a)$, $Borrowed(s, b, d)$, and $Book(b, t, n)$, as defined in Section 6.7 of the notes. Write formulas to express the following queries. Your formulas may use the equality predicate symbol. Note the distinction between a copy of a book (which we assume is uniquely identified by a book id), and a book (which we assume is uniquely identified by the title and author's name). There may be several copies of a book in the library.

a. Find the name and address of each subscriber who has borrowed more than one copy of some book.

b. Find the titles of all books written by Chaucer.

c. Find the names of all subscribers who borrowed a copy of a book by Chaucer that is due on 2001/11/22.

d. Find the title of every book by Joyce so that no book by another author has the same title.

e. Find the title of every book borrowed only by subscribers who have never borrowed another book.

[continued]

Question 6 — EXTRA CREDIT. (16 marks) Consider a binary relation $L(x, y)$, loosely interpreted as "x precedes y", where the exact nature of "precedes" is left unspecified. Such a relation is said to be a ***strict partial order*** if it satisfies two properties:

- Anti-reflexivity: Nothing "precedes" itself.
- Transitivity: If x "precedes" y, and y "precedes" z, then x "precedes" z.

These two properties can be stated as the following two first-order formulas:

$$\begin{aligned} F &: \quad \forall x\, \neg L(x, x) \\ G &: \quad \forall x \forall y \forall z\, ((L(x, y) \wedge L(y, z)) \rightarrow L(x, z)) \end{aligned}$$

There are many natural relations that are strict partial orders: The $<$ relation between numbers (as well as $>$), the "is a descendent of" relation between persons (as well as "is an ancestor of"), the "lies above" relation between points in three-dimensional space, the "logically implies but is not logically equivalent to" relation between formulas, the lexicographic ordering relation between strings, and so forth. In this question you will explore what are, and what are not, logical consequences of the properties of strict partial orders.

Consider the following three formulas, expressing properties of the binary relation $L(x, y)$.

$$\begin{aligned} H &: \quad \forall x \forall y\, (L(x, y) \rightarrow \neg L(y, x)) \\ I &: \quad \forall x \forall y\, (L(x, y) \rightarrow \exists z\, (L(x, z) \wedge L(z, y))) \\ J &: \quad \forall x \forall y\, (\approx(x, y) \vee L(x, y) \vee L(y, x)) \end{aligned}$$

Prove or disprove each of the following statements:

a. $(F \wedge G)$ logically implies H.

b. $(F \wedge G)$ logically implies I.

c. $(F \wedge G)$ logically implies $\neg I$.

d. $(F \wedge G)$ logically implies J.

Cover page for
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Submitted by

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By virtue of submitting this homework I/we acknowledge that I am/we are aware of the policy on homework collaboration for this course.