Solutions for Homework Assignment #1

Answer to Question 1. Let $S(n)$ be the predicate: $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$. We will prove that $S(n)$ holds for every integer $n \geq 1$ by (simple) induction.
BASIS: $\sum_{i=1}^{1} \frac{1}{i^2} = 1 \leq 2 - \frac{1}{1}$, so $S(1)$ is true.

INDUCTION STEP: Let $u \geq 1$, and suppose that $S(u)$ holds. We want to show that $S(u+1)$ holds. We have:

$$\begin{aligned}
\sum_{i=1}^{u+1} \frac{1}{i^2} &= \left(\sum_{i=1}^{u} \frac{1}{i^2}\right) + \frac{1}{(u+1)^2} \\
&\leq \left(2 - \frac{1}{u}\right) + \frac{1}{(u+1)^2} && \text{[by i.h.]} \\
&= 2 - \frac{u^2+u+1}{u(u+1)^2} \\
&\leq 2 - \frac{u^2+u}{u(u+1)^2} \\
&= 2 - \frac{1}{u+1}
\end{aligned}$$

as wanted.

Answer to Question 2. Let $P(n)$ be the predicate

$$P(n): \qquad f(n) \leq 3 \cdot n \cdot 2^{n-2}$$

We will use complete induction to prove that $P(n)$ is true for all integers $n \geq 3$. Using the definition of f we calculate $f(2)$, $f(3)$ and $f(4)$. (As we will see, $f(3)$ and $f(4)$ are needed for the "base" case of the complete induction, i.e., the case that we need to establish without recourse to any induction hypotheses; $f(2)$ is needed to calculate $f(3)$.) We have

$$\begin{aligned}
f(2) &= 4f(0) + 2^2 = 4 \cdot 1 + 4 = 8 \\
f(3) &= 4f(1) + 2^3 = 4 \cdot 2 + 8 = 16 \\
f(4) &= 4f(2) + 2^4 = 4 \cdot 8 + 16 = 48
\end{aligned}$$

Let i be an arbitrary integer such that $i \geq 3$. Assume that $P(j)$ is true for all integers j such that $3 \leq j < i$. That is, assume that for all integers j such that $3 \leq j < i$, $f(j) \leq 3 \cdot j \cdot 2^{j-2}$. We will prove that $P(i)$ is also true. That is, we will prove that $f(i) \leq 3 \cdot i \cdot 2^{i-2}$.

CASE 1. $i = 3$. In this case, $3 \cdot i \cdot 2^{i-2} = 3 \cdot 3 \cdot 2^1 = 18$. Comparing to the value of $f(3)$ calculated above we see that indeed $f(3) \leq 18$, so $P(i)$ is true in this case.

CASE 2. $i = 4$. In this case, $3 \cdot i \cdot 2^{i-2} = 3 \cdot 4 \cdot 2^2 = 48$. Comparing to the value of $f(4)$ calculated above we see that indeed $f(4) \leq 48$, so $P(i)$ is true in this case.

CASE 3. $i \geq 5$. Thus, $i - 2 \geq 3$ and so by induction hypothesis, $P(i-2)$ is true — i.e., $f(i-2) \leq 3(i-2)2^{i-4}$. We have

$$\begin{aligned}
f(i) &= 4f(i-2) + 2^i && \text{[by def of } f\text{, since } i \geq 5 \text{ and so } i \geq 2] \\
&\leq 4\big(3(i-2)2^{i-4}\big) + 2^i && \text{[by induction hypothesis]} \\
&= 3 \cdot i \cdot 2^{i-2} - \frac{3}{2}2^i + 2^i \\
&= 3 \cdot i \cdot 2^{i-2} - \frac{1}{2}2^i \\
&\leq 3 \cdot i \cdot 2^{i-2}
\end{aligned}$$

so $P(i)$ is true in this case.

Answer to Question 3.

a.

n	dyadic representation	n	dyadic representation
0	ϵ	8	112
1	1	9	121
2	2	10	122
3	11	11	211
4	12	12	212
5	21	13	221
6	22	14	222
7	111	15	1111

b. Consider the predicate

$$P(m): \qquad m \text{ has a dyadic representation}$$

We will prove that $P(m)$ is true for all $m \in \mathbb{N}$.

Proof following first hint. The key observation is that a dyadic representation of $m+1$ is obtained from that of m by changing the rightmost 1 of the dyadic representation of m to 2, and changing all subsequent 2s to 1s. In the special case where *all* symbols of the dyadic representation of m are 2s, we change them all to 1s and add one leading 1.

We use simple induction to prove that $P(m)$ is true for all $m \in \mathbb{N}$.

BASIS: $m = 0$. Then m has a dyadic representation, namely the empty string. So, $P(0)$ holds.

INDUCTION STEP: Let n be an arbitrary natural number. Assume that $P(n)$ is true. That is, assume that n has a dyadic representation, say $d_{k-1}d_{k-2}\ldots d_0$. We will prove that $n+1$ also has a dyadic representation.

CASE 1. One of the symbols of the dyadic representation of n is 1. That is, there is some integer i s.t. $0 \le i < k$ and $d_i = 1$. Without loss of generality, let i be the smallest such integer. That is, $d_i = 1$ and for every j s.t. $0 \le j < i$, $d_j = 2$. (In other words, we are looking at the rightmost 1 of $d_{k-1}d_{k-2}\ldots d_0$.) Let $e_{k-1}e_{k-2}\ldots e_0$ be defined as follows:

$$e_0 = e_1 = \ldots = e_{i-1} = 1, \quad e_i = 2, \quad \text{and } e_j = d_j \text{ for all } j \text{ s.t. } i < j < k$$

(That is, we change the rightmost 1 of $d_{k-1}d_{k-2}\ldots d_0$ to 2, we change all the succeeding 2s to 1s, and we leave all the preceding symbols unchanged.) We will now show that $e_{k-1}e_{k-2}\ldots e_0$ is a dyadic representation of $n+1$. Indeed we have

$$\begin{aligned}
n+1 &= (d_{k-1}\cdot 2^{k-1} + \cdots + d_{i+1}\cdot 2^{i+1} + d_i \cdot 2^i + d_{i-1}\cdot 2^{i-1} + \cdots + d_0 \cdot 2^0) + 1 && \text{[by i.h.]}\\
&= (e_{k-1}\cdot 2^{k-1} + \cdots + e_{i+1}\cdot 2^{i+1} + 1\cdot 2^i + 2\cdot 2^{i-1} + \cdots + 2\cdot 2^0) + 1\\
&= e_{k-1}\cdot 2^{k-1} + \cdots + e_{i+1}\cdot 2^{i+1} + 1\cdot 2^i + 1\cdot 2^i + \cdots + 1\cdot 2^1 + 1\cdot 2^0\\
&= e_{k-1}\cdot 2^{k-1} + \cdots + e_{i+1}\cdot 2^{i+1} + 2\cdot 2^i + 1\cdot 2^{i-1} + \cdots + 1\cdot 2^1 + 1\cdot 2^0\\
&= e_{k-1}\cdot 2^{k-1} + \cdots + e_{i+1}\cdot 2^{i+1} + e_i\cdot 2^i + e_{i-1}\cdot 2^{i-1} + \cdots + e_1\cdot 2^1 + e_0\cdot 2^0
\end{aligned}$$

as wanted.

CASE 2. There is no 1 in the dyadic representation of n. That is, for all i s.t. $0 \le i < k$, $d_i = 2$. Let $e_k e_{k-1} \ldots e_0$ be defined by

$$e_k = e_{k-1} = \ldots = e_0 = 1.$$

(That is, we change the string consisting of k 2s to the string consisting of $k+1$ 1s.) We will now show that $e_k e_{k-1} \ldots e_0$ is a dyadic representation of $n+1$. Indeed we have

$$\begin{aligned} n+1 &= (d_{k-1} \cdot 2^{k-1} + d_{k-2} \cdot 2^{k-2} + \cdots + d_0 \cdot 2^0) + 1 \qquad \text{[by i.h.]} \\ &= (2 \cdot 2^{k-1} + 2 \cdot 2^{k-2} + \cdots + 2 \cdot 2^0) + 1 \\ &= 2^k + 2^{k-1} + \cdots + 2^0 \\ &= 1 \cdot 2^k + 1 \cdot 2^{k-1} + \cdots + 1 \cdot 2^0 \\ &= e_k \cdot 2^k + e_{k-1} \cdot 2^{k-1} + \cdots + e_0 \cdot 2^0 \end{aligned}$$

as wanted.

Proof following second hint. The key observation is that, for $m > 0$, a dyadic representation of m is obtained inductively from that of $\lfloor (m-1)/2 \rfloor$ by appending to the latter a 1 if m is odd, and a 2 if m is even.

We use complete induction to prove that $P(m)$ holds for every $m \in \mathbb{N}$. Let i be an arbitrary natural number. Assume that $P(j)$ holds for every natural number j s.t. $0 \le j < i$. We will prove that $P(i)$ holds as well.

CASE 1. $i = 0$. Then i has a dyadic representation, namely the empty string. So, $P(i)$ holds in this case.

CASE 2. $i > 0$. Then $0 \le \lfloor (i-1)/2 \rfloor < i$ and so $P(\lfloor (i-1)/2 \rfloor)$ holds by the induction hypothesis. Therefore, there is a string $d_{k-1} d_{k-2} \ldots d_0$ of 1s and 2s that is a dyadic representation of $\lfloor (i-1)/2 \rfloor$. We will now prove that $d_{k-1} d_{k-2} \ldots d_0 d$ is a dyadic representation of i, where $d = 1$ if i is odd and $d = 2$ if i is even. (That is, a dyadic representation of i is obtained from that of $\lfloor (i-1)/2 \rfloor$ by appending to it a 1 if i is odd, or a 2 if i is even.)

If i is odd we have,

$$\begin{aligned} i &= 2\lfloor (i-1)/2 \rfloor + 1 \\ &= 2(d_{k-1} \cdot 2^{k-1} + d_{k-2} \cdot 2^{k-2} + \cdots + d_0 \cdot 2^0) + 1 \qquad \text{[by i.h.]} \\ &= d_{k-1} \cdot 2^k + d_{k-2} \cdot 2^{k-1} + \cdots + d_0 \cdot 2^1 + 1 \cdot 2^0 \end{aligned}$$

and so $d_{k-1} d_{k-2} \ldots d_0 1$ is a dyadic representation of i.

If i is even we have,

$$\begin{aligned} i &= 2\lfloor (i-1)/2 \rfloor + 2 \\ &= 2(d_{k-1} \cdot 2^{k-1} + d_{k-2} \cdot 2^{k-2} + \cdots + d_0 \cdot 2^0) + 2 \qquad \text{[by i.h.]} \\ &= d_{k-1} \cdot 2^k + d_{k-2} \cdot 2^{k-1} + \cdots + d_0 \cdot 2^1 + 2 \cdot 2^0 \end{aligned}$$

and so $d_{k-1} d_{k-2} \ldots d_0 2$ is a dyadic representation of i.

In either case, i has a dyadic representation and so $P(i)$ holds, as wanted.

c. Let $d_{k-1} d_{k-2} \ldots d_0$ and $e_{\ell-1} e_{\ell-2} \ldots e_0$ be two different strings of 1s and 2s representing, respectively, m and n. We will prove that $m \ne n$. Since the two strings are different there are two cases:

CASE 1. The two strings have different lengths. Without loss of generality, assume that $k > \ell$. We will prove that $m > n$.

$$\begin{aligned} m &= d_{k-1} \cdot 2^{k-1} + d_{k-2} \cdot 2^{k-2} + \cdots + d_0 \cdot 2^0 \\ &\geq d_\ell \cdot 2^\ell + d_{\ell-1} \cdot 2^{\ell-1} + \cdots + d_0 \cdot 2^0 \\ &\geq 1 \cdot 2^\ell + 1 \cdot 2^{\ell-1} + \cdots + 1 \cdot 2^0 \\ &= 2 \cdot 2^{\ell-1} + 2 \cdot 2^{\ell-2} + \cdots + 2 \cdot 2^0 + 1 \\ &\geq e_{\ell-1} \cdot 2^{\ell-1} + e_{\ell-2} \cdot 2^{\ell-2} + \cdots + e_0 \cdot 2^0 + 1 \\ &= n + 1 \end{aligned}$$

CASE 2. The two strings have the same length but differ in some position i. Let i be the largest index in which the two strings differ. That is, for some i s.t. $0 \leq i < k$, $d_i \neq e_i$; and for all j s.t. $i < j < k$, $d_j = e_j$. Without loss of generality, assume that $d_i = 2$ and $e_i = 1$. We will prove that $m > n$.

$$\begin{aligned} m &= d_{k-1} \cdot 2^{k-1} + \cdots + d_{i+1} \cdot 2^{i+1} + d_i \cdot 2^i + d_{i-1} \cdot 2^{i-1} + \cdots + d_1 \cdot 2^1 + d_0 \cdot 2^0 \\ &\geq e_{k-1} \cdot 2^{k-1} + \cdots + e_{i+1} \cdot 2^{i+1} + 2 \cdot 2^i + 1 \cdot 2^{i-1} + \cdots + 1 \cdot 2^1 + 1 \cdot 2^0 \\ &= e_{k-1} \cdot 2^{k-1} + \cdots + e_{i+1} \cdot 2^{i+1} + 1 \cdot 2^i + 2 \cdot 2^{i-1} + 2 \cdot 2^{i-2} + \cdots + 2 \cdot 2^0 + 1 \\ &\geq e_{k-1} \cdot 2^{k-1} + \cdots + e_{i+1} \cdot 2^{i+1} + e_i \cdot 2^i + e_{i-1} \cdot 2^{i-1} + e_{i-2} \cdot 2^{i-2} + \cdots + e_1 \cdot 2^1 + e_0 \cdot 2^0 + 1 \\ &= n + 1 \end{aligned}$$

In both cases, $m \geq n + 1$. Therefore $m \neq n$, as wanted.

Answer to Question 4. a. Let $P(n)$ be the predicate:

$P(n)$: For every set S of n-bit strings no two of which differ in *exactly* one position, $|S| \leq 2^{n-1}$.

We use (simple) induction to prove that $P(n)$ holds for every integer $n \geq 1$.

BASIS: We wish to show $P(i)$ holds for $i = 1$. Let S be a set of 1-bit strings no two of which differ in exactly one position. There are only two 1-bit strings, namely 0 and 1. Since these strings differ in exactly one position, S cannot contain both of them. So $|S| \leq 1 = 2^{i-1}$.

INDUCTION STEP: Let $i \geq 1$ be an arbitrary integer such that $P(i)$ holds. We wish to show that $P(i+1)$ holds as well.

So let S be a set of $(i+1)$-bit strings, such that no two strings in S differ in exactly one position. We wish to show that $|S| \leq 2^i$. Let S_0 be the set of strings in S that begin with 0, and let S_1 be the set of strings in S that begin with 1. Since every string from S is in one of S_0 or S_1 but not both, we have $|S| = |S_0| + |S_1|$.

We now claim that $|S_0| \leq 2^{i-1}$. To see this, define the set of i-bit strings $S'_0 = \{x \mid 0x \text{ is in } S_0\}$; that is, S'_0 is the set of strings obtained by chopping off the first 0 from all strings in S_0. No two strings in S'_0 differ in exactly one position (since if $x, y \in S'_0$ differed in exactly one position, then $0x$ and $0y$ would be two strings in S that differ in exactly one position). So by the induction hypothesis $P(i)$, $|S'_0| \leq 2^{i-1}$. Since $|S_0| = |S'_0|$, we have $|S_0| \leq 2^{i-1}$.

In a similar manner we can prove that $|S_1| \leq 2^{i-1}$. So $|S| = |S_0| + |S_1| \leq 2^{i-1} + 2^{i-1} = 2^i$.

b. Let n be a positive integer. Let S be the set of n-bit strings x that contain an *even* number of 0s. To see that no two strings in S differ in exactly one position, consider two strings $x, y \in S$. Say that x and y

differ in exactly one position. Then x and y are exactly the same, except that in one position x has a 0 and y has a 1. So 0 occurs in x exactly one more time than in y, contradicting the fact that both x and y are in S.

Now we claim that $|S| = 2^{n-1}$. This can be proved using induction, but here is a different proof. Let T be the set of n-bit strings y with an *odd* number of 0s, Let $f : S \to T$ be the function defined as follows: $f(x)$ is the string obtained by complementing the first bit of x. (It is obvious that this is a function from S to T because, if x has an even number of 0s then $f(x)$ has an odd number of 0s.) Furthermore, it is easy to see that f is one-to-one and onto, so that $|S| = |T|$. Let B be the entire set of n-bit strings. Clearly, $S \cup T = B$ and $S \cap T = \emptyset$. Therefore $|B| = |S| + |T|$. We know that $|B| = 2^n$ and we just proved that $|S| = |T|$. Thus $2 \cdot |S| = 2^n$ and therefore $|S| = 2^{n-1}$.

Outline of alternative proof: Take any Gray code of the n-bit strings, say $\langle x_1, x_2, \ldots, x_{2^n} \rangle$, and let $S = \{x_1, x_3, \ldots, x_{2^n-1}\}$. (That is, S is formed by skipping every other string in the Gray code sequence.) It is easy to check from the definition of the Gray sequence that no two strings in S differ in exactly one place, so S has the desired property. Finally, S contains half of the n-bit strings, i.e., 2^{n-1} strings.

Outline of yet another proof: We can define inductively a set S_n of 2^{n-1} binary strings of length n no two of which differ only in one position. In the basis of the induction we start with the set $S_1 = \{0\}$ of 1-bit strings which obviously contains no strings that differ in exactly one position, and has 2^0 elements. In the induction step, we form the set S_{k+1} of $(k+1)$-bit strings by taking one copy of S_k, and adding a 0 at the beginning of each string in S_k, and also taking one copy of S_k, complementing the first bit of each string in S_k and adding a 1 at the beginning of the resulting strings. It is easy to check that the resulting set S_{k+1} contain no strings that differ in exactly one position, and that it has 2^k elements. Thus, for each n we can construct a set of n-bit strings no two of which differ in exactly one position that has 2^{n-1} elements.

Answer to Question 5. The high-level idea is as follows: If $a/b = 1/d$ for some positive integer d then we are done. Otherwise, we let $1/d$ be the largest fraction (with integer denominator) that is less than a/b. So, a/b can be written as $1/d + a'/b'$. If the fraction a'/b' is simplified, it turns out that $a' < a$. So, assuming by induction that we have already handled fractions with numerator less than a, we can assume that a'/b' can be written as the sum of strictly decreasing fractions $1/d_1, 1/d_2, \ldots, 1/d_k$. Then a/b can be written as the sum of strictly decreasing fractions $1/d, 1/d_1, 1/d_2, \ldots, 1/d_k$. We now make this argument more precise.

Consider the predicate

$P(n)$: For every integer $b \geq n$, there exists a finite sequence of positive integers $d_1 < d_2 < \cdots < d_k$ such that $n/b = 1/d_1 + 1/d_2 + \cdots + 1/d_k$

We will use complete induction to prove that $P(n)$ holds for every integer $n \geq 1$. Let i be an arbitrary integer ≥ 1. Assume that $P(j)$ is true for all integers j such that $1 \leq j < i$. We will prove that $P(i)$ is also true. Let b be an integer such that $b \geq i$.

CASE 1. For some positive integer d, $i/b = 1/d$. In this case we are done.

CASE 2. For every positive integer d, $i/b \neq 1/d$. Let d be the smallest positive integer such that $1/d < i/b$. Therefore

$$1/d < i/b < 1/(d-1) \tag{1}$$

(Since $1/d < i/b \leq 1$, it follows that $d > 1$ and so $d - 1 > 0$, so $1/(d-1)$ is a rational number.) By the second inequality in (1), $id - i < b$, and so $id - b < i$. By the first inequality in (1), $id > b$, so $id - b > 0$, and therefore $id - b \geq 1$. So $1 \leq id - b < i$ and so, by the induction hypothesis, $P(id-b)$ holds. Furthermore, $1 > i/b > i/b - 1/d$ and so $(id-b)/bd < 1$.

Thus, by $P(id-b)$, there is a finite sequence of positive integers $d_1 < d_2 < \ldots < d_k$ such that

$$(id-b)/db = 1/d_1 + 1/d_2 + \ldots + 1/d_k \tag{2}$$

It follows that $i/d = 1/d + 1/d_1 + 1/d_2 + \ldots + 1/d_k$.

To complete the proof that $P(i)$ holds, it remains to show that $d < d_1$. As shown above, $id - b < i$ and since $i < b$, we get:

$$id - b < b \Rightarrow id < 2b \Rightarrow i/b < 2/d \Rightarrow i/b - 1/d < 1/d \Rightarrow (id-b)/bd < 1/d.$$

So, by (2),

$$1/d_1 + 1/d_2 + \ldots + 1/d_k < 1/d \Rightarrow 1/d_1 < 1/d \Rightarrow d < d_1$$

as wanted.