Homework Assignment #4
Due: December 15

On the cover page of your assignment, you must write and sign the following statement: “I have read and understood the homework collaboration policy described in the Course Information handout.” Without such a signed statement your homework will not be marked.

**Question 1.** (12 marks) Consider the first order language of arithmetic described in Section 6.2 and the structures \( \mathcal{N} \) and \( \mathcal{Z} \) described on page 152 in the notes. For each sentence below state whether it is true in \( \mathcal{N} \), \( \mathcal{Z} \), both or neither. Justify your answer by translating each sentence into a statement in precise English about numbers and then explain why that statement is true or false for the natural numbers or for the integers.

1. \( \exists x \forall y (\neg \approx (x, y) \rightarrow L(x, y)) \)
2. \( \forall x \forall y ((L(0, y) \land S(x, x, y)) \rightarrow L(x, 0)) \)
3. \( \forall x \forall y \forall z ((L(x, 0) \land P(x, y, z) \land L(z, 0)) \rightarrow L(0, y)) \)
4. \( \forall x \exists y \exists z \exists w (S(y, z, w) \land \neg L(w, x)) \)

**Question 2.** (8 marks) Consider the database schema for a library as defined in the lecture notes with some small additions. There are five predicates:

1. \( \text{Book}(b, t, n) \): This is true if a book with id \( b \), has title \( t \) and is written by author \( n \).
2. \( \text{Subscriber}(s, n, a) \): This is true if a subscriber with SIN \( s \) has name \( n \) and lives in address \( a \).
3. \( \text{Borrowed}(s, b, d) \): This is satisfied when a subscriber with SIN \( s \) has borrowed a book with id \( b \) and it is due on date \( d \).
4. \( \text{Likes_donuts}(s) \): This is true if a person with SIN \( s \) likes to eat donuts.
5. \( \text{Contains}(b, w) \): This is true if the word \( w \) is contained in the title of the book with id \( b \).

**a.** (4 marks) Give a first-order formula that expresses the query: Find the names of all subscribers who like donuts and who have borrowed at least one of the following books:
   
   (i) Donut recipes for dummies
   
   (ii) The role of donuts in modern politics.

**b.** (4 marks) Give a first-order formula that expresses the query: Find the title and due date of every book which is borrowed by Mr. Homer Simpson and which has the word donut in its title.

**Question 3.** (10 marks)

**a.** (5 marks) Write a formula in Prenex Normal Form that is logically equivalent to:

\[ (\forall x P(x) \rightarrow \forall x Q(x)) \lor (\neg \forall x P(x) \rightarrow \neg \forall x Q(x)) \]
b. (5 marks) For each formula below say whether it is valid, satisfiable or unsatisfiable. Justify your answer either with a formal proof using logical equivalences and/or by using appropriate structures.

1. \(\forall x \forall y \forall z (\neg L(x, z) \to (\neg L(y, z) \lor \neg L(x, y)))\)
2. \(\forall x \ A(x) \to \neg \exists x A(x)\)

Question 4. (30 marks)

a. (8 marks) An electrician is designing two-way switches so that the stairway lights of a building can be turned on or off from the first, second and third floor. She selects the alphabet \(\Sigma = \{F, f, S, s, T, t\}\) with the meaning:

1. \(F\) means the first-floor switch is flipped up, \(f\) means it is flipped down.
2. \(S\) means the second-floor switch is flipped up, \(s\) means it is flipped down.
3. \(T\) means the third-floor switch is flipped up, \(t\) means it is flipped down.

A string over \(\Sigma\) defines a sequence of switch flips. For example \(SfTs\) means that the second-floor switch is flipped up, then the first-floor switch is flipped down, then the third-floor switch is flipped up and then the second-floor switch is flipped down. We can even have sequences where the same action repeats, like \(TT\), which corresponds to a frustrated individual in the third floor who flips the third-floor switch up twice in a row, (not noticing in the second time that it is already flipped up).

Assume that initially all switches are down, i.e., the empty sequence \(\epsilon\) leaves 0 switches up. Draw a DFSA for the language
\[
L = \{x \in \Sigma^* : x\ \text{leaves an even number of switches up}\}
\]

For example \(FSfT \in L\) whereas \(FSf \notin L\). You do not need to prove that your automaton works but you should provide a brief justification of its correctness.

b. (12 marks) The electrician above decides that before moving on to her next assignment, it would be beneficial to practice more on regular expressions and finite automata. For each language below, provide a regular expression and draw a DFSA that accepts the language. You do not need to justify your answer.

1. \(L = \{x \in \{0, 1\}^* : x\ \text{does not contain the substring 110}\}\).
2. \(L = \{x \in \{0, 1\}^* : \text{every odd position of } x \text{ is } 1\}\).
3. \(L = \{x \in \{0, 1\}^* : x\ \text{has length at least 3 and its third symbol is a } 0\}\).

c. (10 marks) The purpose of this question is to convince you (and the electrician) that in certain cases, a NFSA may be more convenient to represent a language than a DFSA. Draw a NFSA with at most 4 states for the language
\[
L = \{x \in \{0, 1\}^* : x\ \text{has a } 1\ \text{on the third position from the end}\}
\]
For example \(00100 \in L\) but \(01000 \notin L\). Now draw a DFSA for the same language.

Question 5. (20 marks)

a. (10 marks) Let \(\Sigma = \{0, 1\}\). Construct a DFSA that accepts a string \(s \in \Sigma^*\) if and only if the decimal value of \(s\) is divisible by 3. For example, your DFSA should accept 0 or 11 or 1001 but should not accept 10 or 101. Give a formal proof that your DFSA is correct along the lines of Example 7.12 of your textbook.

b. (10 marks) State whether the following statements are true or false. Briefly justify your answer. No credit will be given without justification.
1. The string 001101 belongs to the language denoted by \((0 + 1)(0^*10^*1)^*\).

2. Consider an infinite sequence of languages \(L_0, L_1, L_2, \ldots\), such that each \(L_i\) is regular. Then the infinite union of these languages \(\bigcup_{i \in \mathbb{N}} L_i\) is also regular.

3. If \(L\) is regular so is \(L - \{x\}\), for any \(x \in L\).

4. If \(L\) is not regular then \(\overline{L}\) is also not regular.

**Question 6.** (10 marks) For each of the following languages show whether they are regular or not. If they are you should provide either a regular expression or a DFSA or a NFSA and briefly justify its correctness. If the language is not regular you should provide a proof by contradiction, using the pumping lemma.

1. \(L = \{xx : x \in \{0, 1\}^*\}\).

2. \(L = \{0^n1^m : m + n \text{ is odd}\}\).

**Question 7.** (10 marks) **Extra credit:** Prove that the following unary language is not regular:

\[ L = \{1^{n^2} : n \in \mathbb{N}\} \]