Question 1. (12 marks) Consider the following family of ternary strings. Let \( S \) be the smallest set such that:

**Basis:** \( 0 \in S \)

**Induction Step:** if \( x, y \in S \), then so are \( x0y, 1x2, 0x \).

a. (7 marks) Prove that if \( k \in \mathbb{N} \), there is no string in \( S \) with exactly \( 5^k + 1 \) zeros. **(HINT:** first use structural induction to prove that every string in \( S \) has an odd number of zeros).

b. (5 marks) Prove with a similar approach that if \( k \in \mathbb{N} \), there is no string in \( S \) with exactly \( 2^{k+1} \) digits.

Question 2. (14 marks) For each statement below, state whether it is true or false and justify your answer. No credit will be given without proper justification.

(a) \( n^5 + 2n^2 + 3n + 6 \in O(n^5) \)

(b) \( \log_5 n \in \Omega(\log_7 n^4) \)

(c) \( n^3 + 1000n^2 + 2000n \in O(n^2) \)

(d) \( 2^{n+1} \in O(2^n) \)

(e) \( 2^{2n} \in O(2^n) \)

(f) \( f(n) \in \Theta(n^2) \), where

\[
f(n) = \begin{cases} 
1 & n = 1 \\
3f(\lceil \frac{n}{3} \rceil) + 2f(\lfloor \frac{n}{3} \rfloor) + 4n & n \geq 2 
\end{cases}
\]

in the case that \( l = 1 \). What about when \( l = 2 \)?

Question 3. (18 marks)

a. (12 marks) Consider the following recursive definition:

\[
f(n) = \begin{cases} 
0 & n = 1 \\
3f(\lceil \frac{n}{2} \rceil) + (n - 3)^2 & n \geq 2 
\end{cases}
\]

Present positive constants \( c \) and \( d \) and show that for every \( n \geq 2 \), \( f(n) \leq cn^2 - 2n - d \). You are not allowed to use the master theorem for this problem. You may use the fact that \( \frac{n-1}{2} \leq \lfloor \frac{n}{2} \rfloor \leq \frac{n}{2} \).

b. (6 marks) Consider the following function:

\[
f(n) = \begin{cases} 
10 & n = 1 \\
3f(\lceil \frac{n}{2} \rceil) + 5n^2 & n \geq 2 
\end{cases}
\]
Show that for every \( n \geq 1 \), \( f(n) \geq 8n^2 + n^{\log_2 3} \). You may use the fact that \( \lceil \frac{n}{2} \rceil \geq \frac{n}{2} \).

**Question 4.** (16 marks)

\[
f(n) = \begin{cases} 
4 & n = 1 \\
3f(\lceil \frac{n}{3} \rceil) + 2n & n \geq 2 
\end{cases}
\]

a. (10 marks) Find a closed-form formula for \( f(n) \) when \( n \) is a power of 3, i.e., \( n = 3^k \) for some \( k \in \mathbb{N} \). Use repeated substitution to guess the formula and then induction to prove it.

b. (6 marks) Note that the master theorem tells us that for large enough \( n \), \( f(n) \in \Theta(n \log_3 n) \). Without using the master theorem, find explicit positive constants \( c_1 \) and \( c_2 \) and show that for all \( n \geq 2 \), \( c_1 n \log_3 n \leq f(n) \leq c_2 n \log_3 n \).

**HINT:** one way to do this is to use (a) and the fact that \( f \) is nondecreasing; you do not need to prove that \( f \) is nondecreasing.

**Question 5.** (10 marks) Prove that the recursive program below is correct with respect to the following precondition/postcondition pair.

**Precondition:** \( x \in \mathbb{N} \) and \( x \geq 1 \).

**Postcondition:** The program returns \((x + 1)^2\).

```c
int foo(int x)
if (x = 1) then
    return 4
else
    a = foo(x div 2) // x div 2 = \lfloor x/2 \rfloor
    if (x mod 2 = 0) then
        return 4 * a - 2 * x - 3
    else
        return 4 * a
end if
end if
```

```c
```