Auction-Based Multi-Robot Routing

Abstract— **Recently auction methods have been investigated as effective, decentralized methods for multi-robot coordination. Experimental research has shown great potential, but has not been complemented yet by theoretical analysis. In this paper we contribute a theoretical analysis of the performance of auction methods for multi-robot routing. We suggest a generic framework for auction-based multi-robot routing and analyze a variety of bidding rules for different team objectives. This is the first time that auction methods are shown to offer theoretical guarantees for such a variety of bidding rules and team objectives.**

I. INTRODUCTION

Robot teams are increasingly becoming a popular alternative to single robots for a variety of difficult robotic tasks, such as planetary exploration or planetary base assembly. Robot teams offer many advantages over single robots: robustness (due to redundancy), efficiency (due to parallelism), and flexibility (due to reconfigurability). However, an important factor for the success of a robot team is the ability to coordinate the team members in an effective way. Coordination involves the allocation and execution of individual tasks through an efficient, decentralized mechanism.

In this paper, we focus on multi-robot routing, a class of problems where a team of mobile robots must visit a set of locations for some purpose (e.g., delivery or acquisition) with routes that optimize certain criteria (e.g., minimization of consumed energy, completion time, or average latency). Examples include search-and-rescue in areas hit by disasters, surveillance of a facility, placement of sensors in a sensor network, delivery of parts, and measurements over a wide area. Such routing problems, including Vehicle Routing Problems (VRPs) and several variants of the Traveling Salesman Problem (TSP), have been widely studied from a centralized point of view in the operations research literature, and more recently in robotics with a focus on decentralized approaches.

Even in decentralized multi-robot coordination, some information should be communicated between the robots to facilitate efficient performance; it is desirable to enable good decision making while communicating as little information as possible. One promising approach of this type is the use of market-based mechanisms, in particular, auction-based methods, where the communicated information consists of bids robots place on various tasks and coordination is achieved by a process similar to winner determination in auctions.

The efficiency of auction-based methods has been demonstrated experimentally [1]–[9], but there has been little theoretical study [8]. In this paper we make the following contributions: (1) we suggest a generic framework for auction-based multi-robot routing, and (2) we derive and analyze six bidding rules for three team objectives (minimizing total cost, maximum cost, or average service cost), specifically, we provide lower and upper bounds on their performance relative to optimal performance. This is the first time that auction methods are shown to offer theoretical guarantees for such a variety of bidding rules and team objectives.

II. MULTI-ROBOT ROUTING

A *multi-robot routing* problem is specified by a set of robots, $R = \{r_1, r_2, \ldots, r_n\}$, a set of targets, $T = \{t_1, t_2, \ldots, t_m\}$, their locations, and a non-negative cost function $c(i, j)$, $i, j \in R \cup T$, that denotes the cost of moving between locations i and j. We assume that these costs are symmetric, $c(i, j) = c(j, i)$, are the same for all robots, and satisfy the triangle inequality. The travel distances and travel times between locations often satisfy these assumptions in an environment with or without obstacles. The objective of multi-robot routing is to find an allocation of targets to robots and a path for each robot that visits all targets allocated to it so that a team objective is optimized[1]. In this research, we study three team objectives:

MINISUM: Minimize the sum of the robot path costs over all robots.
MINIMAX: Minimize the maximum robot path cost over all robots.
MINIAVE: Minimize the average target path cost over all targets.

The *robot path cost* for any robot is the sum of the costs along its entire path, from its initial location to the last target on its path. The *target path cost* for any target is the sum of the costs from the initial location of the robot that visits that target up to the target in question.

III. OPTIMAL SOLUTIONS

Optimizing performance for any of the three team objectives is NP-hard, as shown by the following theorem.

Theorem 1: There is no polynomial time algorithm for solving multi-robot routing optimally with the MINISUM, the MINIMAX, or the MINIAVE objective, unless P = NP.

Proof: We show that an algorithm for multi-robot routing with any of the three objectives implies a polynomial time algorithm for Hamiltonian path, a well known NP-complete problem. An instance of the Hamiltonian path problem consists of a graph $G = (V, E)$ and a vertex v and we are asked to decide if there exists a path starting from v that visits all the vertices exactly once. We define an instance of multi-robot routing as follows. Let $G' = (V, c)$ be the complete weighted graph on the vertices V of G with the following weights: if $(u, w) \in E$, then $c(u, w) = 1$, otherwise $c(u, w) = 2$. One robot is placed at vertex v and the remaining $|V| - 1$ nodes are

[1] Although we assume that robots are not required to return to their initial locations, our algorithms and results apply also to the case of closed tours.

designated as targets. It is easy to see that the costs (weights) in G' satisfy the triangle inequality.

We claim that G has a Hamiltonian path if and only if an optimal MINISUM solution in G' has a cost of $|V|-1$. To see this, suppose G had a Hamiltonian path. Then this path is also a solution in G' with cost $|V|-1$, hence an optimal MINISUM solution has a cost of $|V|-1$ (this is the minimum possible cost for visiting $|V|-1$ targets as it is necessary to use $|V|-1$ edges). On the other hand, if G does not have a Hamiltonian path, then any path in G' that starts from v and visits all the vertices has to use some edge of cost 2 in G'. Hence, an optimal solution will be at least $|V|$.

Similarly, it is true that G has a Hamiltonian path if and only if an optimal MINIMAX solution in G' has a cost of $|V|-1$. Finally, it is easy to check that G has a Hamiltonian path if and only if an optimal MINIAVE solution in G' has a cost of at most $\big(1+2+\ldots+(|V|-2)+(|V|-1)\big)/\big(|V|-1\big) = |V|/2$. ■

Given this hardness result, we focus on efficient approximation algorithms for solving large-scale instances of multi-robot routing. However, optimal solutions for small instances can be obtained through a mixed-integer programming formulation.

IV. Auction Framework

Our auction-based coordination system for multi-robot routing considers the robots as bidders and the targets as goods and operates as follows. All targets are initially unallocated. During each round of bidding, all robots bid on all unallocated targets. The robot that places the overall lowest bid on any target wins and is allocated that particular target. A new round of bidding starts, and all robots bid again on all unallocated targets, and so on until all targets have been allocated to robots. Note that each robot needs to bid only on a single target at each round, namely on that target for which its bid is the lowest, since all other bids from the same robot have no chance of winning. Upon allocation of all targets, each robot computes a path for visiting the targets allocated to it and then moves along that path. A robot does not move if no targets are allocated to it. Bid selection and path computation are the key factors that affect team performance.

The main advantage of this multi-round auction mechanism is its simplicity and the fact that it allows for a decentralized implementation on real robots. Initially, each robot needs to know its own location, the location of all targets, and the number of robots, but not the locations of the other robots. At each round, each robot computes its single bid locally and in parallel with the other robots, broadcasts the bid to the other robots, listens to the broadcasts of the other robots, collects all bids, and then locally determines the winning bid. Broadcasting can be achieved by means of relaying messages from robot to robot. This procedure is repeated in every round of the auction. Clearly, there is no need for a central auctioneer, and therefore there is no single point of failure in the system.

V. Paths versus Trees

We explore two ways of obtaining approximate solutions to the multi-robot routing within the auction framework described above: paths and trees. In particular, during the auction, one can consider constructing paths that span all targets (one path for each robot) or constructing trees that span all targets (a forest with one tree rooted at each robot). Considering paths is a direct method, whereas considering trees is an indirect method since the trees must be converted to paths to obtain a solution to the original problem. This extra step is relatively easy and does not significantly affect the quality of the solution.

The choice of paths versus trees depends on how efficiency and performance are affected. The rationale behind the idea of constructing trees rather than paths is that trees with certain properties might be readily computable compared to paths with similar properties. For example, given any weighted graph, a minimum-cost spanning tree can be obtained in polynomial time, whereas a minimum-cost path through all nodes poses an NP-hard problem. Therefore, instead of directly seeking paths that achieve a team objective, one may seek to find trees that achieve an equivalent objective, and then convert the trees to paths that approximate the original team objective.

For MINISUM, the equivalent objective is to find a minimum spanning forest (MSF), that is a collection of trees rooted at the robots that span all targets with minimum total cost. Such a tree is computable in polynomial time by a variant of Prim's algorithm [10]. For MINIMAX, the equivalent objective is to find a minimax spanning forest which is a collection of trees rooted at the robots that span all targets and the cost of the most expensive tree is minimized. Computing the minimax tree is an NP-hard problem [11]. Finally, for MINIAVE, the equivalent objective is to find a minimum average-cost spanning forest which is a collection of trees rooted at the robots that span all targets and the average root-target cost over all targets is minimized. Such a forest can be trivially computed by connecting each target to the closest root and consists of stars.

A tree can be easily turned into a path using the minimum spanning tree (MST) heuristic commonly used for TSP [12]. The MST heuristic constructs a path from a tree by performing a depth-first search on the tree to derive the ordering of nodes in the path while skipping previously visited nodes (short-cutting). It is a well-known fact that the cost of the resulting path will be no more than twice the cost of the tree [12]. Alternatively, one could use any sophisticated TSP algorithm on the nodes of each tree to obtain a good path for each robot. The specific method used does not affect the results in this paper as long as the cost of each path is at most twice the cost of the corresponding tree, which can be guaranteed through the MST heuristic. Thus, in our auction framework, we assume that the final step of converting trees into paths incurs an approximation factor of at most 2.

VI. Bidding Rules

In every round of the auction, the robots use a bidding rule to determine the appropriate (according to the team objective) bid for each target. We suggest a generic methodology for deriving such rules for any given team objective and we derive six bidding rules for the three team objectives we consider. We distinguish the bidding rules in two classes depending on whether they aim to build paths or trees.

Suppose that the team objective is expressed as

$$\min_{\mathcal{A}} f\big(g(r_1, A_1), \ldots, g(r_n, A_n)\big)$$

where function g determines the performance of each robot, function f determines the performance of the team, and $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ is a partition of the set of targets, where targets in A_i are allocated to robot r_i. The three team objectives we consider fit this structure. Let $RPC(r_i, A_i)$ denote the minimum robot path cost for robot r_i to visit all targets in A_i from its current location. Similarly, let $CTPC(r_i, A_i)$ denote the minimum cumulative target path cost of all targets in A_i, again, if robot r_i visits all targets in A_i from its current location. Then, the three team objectives can be expressed as

$$\begin{aligned}
\textsc{MiniSum} &: \quad \min_{\mathcal{A}} \sum_j RPC(r_j, A_j),\\
\textsc{MiniMax} &: \quad \min_{\mathcal{A}} \max_j RPC(r_j, A_j),\\
\textsc{MiniAve} &: \quad \min_{\mathcal{A}} \frac{1}{m} \sum_j CTPC(r_j, A_j).
\end{aligned}$$

Let $(S_1, S_2, \ldots, S_n)$ be the current partial allocation of targets to robots at some round of the auction and let t be an unallocated target. We propose the following bidding rule, which is directly derived from the team objective.

> **Bidding Rule** Robot r bids on unallocated target t the difference in performance for the given team objective between the current allocation of targets to robots and the allocation that results from the current one if robot r is additionally allocated target t.

Consequently, robot r_i should bid on target t the difference

$$f\Big(g(r_1, S_1'), \ldots, g(r_n, S_n')\Big) - f\Big(g(r_1, S_1), \ldots, g(r_n, S_n)\Big),$$

where $S_i' = S_i \cup \{t\}$ and $S_j' = S_j$ for $i \neq j$. This generic bidding rule thus performs some sort of hill climbing aiming to find a good, but not necessarily optimal, allocation.

For the MINISUM team objective, robot r_i bids on target t

$$\begin{aligned}
&\sum_j RPC(r_j, S_j') - \sum_j RPC(r_j, S_j)\\
&= RPC(r_i, S_i \cup \{t\}) - RPC(r_i, S_i).
\end{aligned}$$

For the MINIMAX team objective, robot r_i bids on target t

$$\begin{aligned}
&\max_j RPC(r_j, S_j') - \max_j RPC(r_j, S_j)\\
&= RPC(r_i, S_i \cup \{t\}) - \max_j RPC(r_j, S_j).
\end{aligned}$$

This derivation uses the fact that $\max_j RPC(r_j, S_j') = RPC(r_i, S_i')$, otherwise target t would have already been allocated in a previous round of bidding. The term $\max_j RPC(r_j, S_j)$ can be dropped since the outcome of the auction remains unchanged if all bids change by a constant. Thus, robot r_i can bid just $RPC(r_i, S_i \cup \{t\})$ on target t. Last, for the MINIAVE team objective, robot r_i bids on target t

$$\begin{aligned}
&\frac{1}{m}\sum_j CTPC(r_j, S_j') - \frac{1}{m}\sum_j CTPC(r_j, S_j)\\
&= \frac{1}{m}\Big(CTPC(r_i, S_i \cup \{t\}) - CTPC(r_i, S_i)\Big).
\end{aligned}$$

The factor $1/m$ can be dropped since the outcome of the auction remains unchanged if all bids are multiplied by a constant factor. Thus, robot r_i can bid just $CTPC(r_i, S_i \cup \{t\}) - CTPC(r_i, S_i)$ on target t.

Thus, the bidding rules for the three team objectives are

- BIDSUMPATH: $RPC(r_i, S_i \cup \{t\}) - RPC(r_i, S_i)$,
- BIDMAXPATH: $RPC(r_i, S_i \cup \{t\})$, and
- BIDAVEPATH: $CTPC(r_i, S_i \cup \{t\}) - CTPC(r_i, S_i)$.

The robots need to be able to calculate their bids efficiently, but computing $RPC(r_i, S_i \cup \{t\})$ or $CTPC(r_i, S_i \cup \{t\})$ is NP-hard. Therefore, in practice each robot r_i uses a heuristic method to compute bids. In particular, we assume that it makes use of the insertion heuristic for TSP to find a good path that visits the targets in $S_i \cup \{t\}$ for a given team objective. Since it already has a good path that visits the targets in S_i, it inserts target t into all possible positions on the existing path, one after the other, picking the one that minimizes the cost of the new path. The specific method used for computing the bids does not affect the results of this paper as long as the bids are not worse than those computed using the insertion heuristic.

A similar analysis can be used to derive bidding rules for the case of constructing trees. For any robot r_i and any set of targets S_i, let $TC(r_i, S_i)$ denote the minimum tree cost, that is the cost of a minimum spanning tree over the nodes $\{r_i\} \cup S_i$. Similarly, let $CTC(r_i, S_i)$ denote the minimum cumulative tree cost which is the sum of root-target costs for all targets in S_i in a spanning tree over $\{r_i\} \cup S_i$ with root r_i. Without going through details, the bidding rules for the three team objectives in this case are

- BIDSUMTREE: $TC(r_i, S_i \cup \{t\}) - TC(r_i, S_i)$,
- BIDMAXTREE: $TC(r_i, S_i \cup \{t\})$, and
- BIDAVETREE: $CTC(r_i, S_i \cup \{t\}) - CTC(r_i, S_i)$.

Given the sequential nature of the allocation of targets during the different rounds of bidding and the incremental construction of trees, the TREE bidding rules can be further simplified. In particular, a tree over S_i remains unchanged within the tree over $S_i \cup \{t\}$ under any of the three objectives. This is true because for the BIDSUMTREE and BIDMAXTREE rules target t is connected to S_i through the cheapest edge, whereas for the BIDAVETREE rule it is connected directly to the root (r_i) because of the triangle inequality assumption. In other words, $TC(r_i, S_i \cup \{t\}) = TC(r_i, S_i) + c(T_i \cup \{r_i\}, t)$, where $c(S_i \cup \{r_i\}, t)$ is the cost of the cheapest edge between any node in $S_i \cup \{r_i\}$ and t, and $CTC(r_i, S_i \cup \{t\}) = CTC(r_i, S_i) + c(r_i, t)$. Thus, the rules can be expressed as:

- BIDSUMTREE: $c(T_i \cup \{r_i\}, t)$,
- BIDMAXTREE: $TC(r_i, S_i) + c(S_i \cup \{r_i\}, t)$, and

- BIDAVETREE: $c(r_i, t)$.

Bids for the TREE rules are computable in polynomial time.

VII. SUMMARY OF RESULTS

Our goal is to assess the performance of each bidding rule theoretically in comparison to optimal performance and with respect to each of the three team objectives. This is done in terms of upper and lower bounds on the performance ratio.

If $I(n, m)$ is the class of all instances of multi-robot routing with n robots and m targets, we seek an upper bound $UB(n, m, R, X)$ for each rule R and for each objective X, such that for any n and m:

$$\max_{I \in I(n,m)} \frac{R(I, X)}{O(I, X)} \leq UB(n, m, R, X)$$

where $R(I, X)$ is the cost of the solution under objective X for instance $I \in I(n, m)$ obtained using rule R and $O(I, X)$ is the optimal cost under objective X for instance I. Similarly, we seek a matching lower bound $LB(n, m, R, X)$ such that for some instance $I \in I(n, m)$:

$$LB(n, m, R, X) \leq \frac{R(I, X)}{O(I, X)}$$

and therefore there is no possible upper bound less than $LB(n, m, R, X)$. An upper bound provides a guarantee on the performance of the corresponding rule for the corresponding team objective; no matter how n or m or $I \in I(n, m)$ are chosen, the performance ratio will not exceed this bound. However, a lower bound usually represents pathological special cases that demonstrate how tight the upper bound is.

It should be pointed out that each bidding rule essentially represents a family of rules. For the PATH rules, we do not specify a particular choice for the computation of the functions RPC and $CTPC$. This choice can be anything between computing them optimaly (NP-hard) and computing them approximately through the insertion heuristic (polynomial). However, we assume that whatever the choice, it will not be worse than the insertion heuristic approximation. Similarly, for the TREE rules we do not specify a particular choice for the conversion of trees to paths, which can range from computing optimal paths (NP-hard) to using the MST heuristic. Once again, we only assume that, whatever the choice, the cost of each path is at most twice the cost of the corresponding tree as guaranteed by the MST heuristic. Our bounds apply to the entire family of rules. To make this possible, for upper bounds we assume that nothing better than the insertion heuristic or the MST heuristic is used, whereas for lower bounds we assume that RPC and $CTPC$, as well as the conversion of trees to paths, can be computed optimally.

Table I summarizes all our results. It is interesting that the PATH and the TREE rules offer almost identical guarantees, which implies that they are not fundamentally different from a theoretical point of view. Although extensive experimentation is required to assess their actual performance, the TREE rules might be preferable from a practical perspective, since their bid computation is much faster compared to the PATH rules. Given that in general $n \ll m$, it is clear that the best guarantees are offered for the MINISUM and the MINIMAX objectives, whereas there are only loose guarantees for the MINIAVE objective. Independently of the objective, the BIDSUMPATH and BIDSUMTREE rules provide uniformly the best guarantees. Overall, our results show that our auction-based methods constitute a principled, viable approach to multi-robot routing.

TABLE I

BOUNDS ON PERFORMANCE RATIO (RULE PERFORMANCE OVER OPTIMAL PERFORMANCE) WITH n ROBOTS AND m TARGETS.

Bidding Rule	Team Objective MINISUM		MINIMAX		MINIAVE	
	Lower	Upper	Lower	Upper	Lower	Upper
BIDSUMPATH	1.5	2	n	$2n$	$\frac{m+1}{2}$	$2m$
BIDMAXPATH	n	$2n$	$\frac{n+1}{2}$	$2n$	$\Omega(m^{1/3})$	$2m$
BIDAVEPATH	m	$2m^2$	$\frac{n+1}{2}$	$2m^2n$	$\Omega(m^{1/3})$	$2m^2$
BIDSUMTREE	1.5	2	n	$2n$	$\frac{m+1}{2}$	$2m$
BIDMAXTREE	n	$2n$	$\frac{n+1}{2}$	$2n$	$\Omega(m^{1/3})$	$2m$
BIDAVETREE	m	$2m$	$\frac{n+1}{2}$	$2mn$	$\Omega(m^{1/3})$	$2m^2$

VIII. ANALYSIS

In this section we prove the bounds in Table I. We make the following notational conventions. The solution found by using any of the bidding rules is marked with the name of the rule, e.g. BIDSUMTREE. An optimal solution for each team objective is denoted by OPTSUM, OPTMAX, and OPTAVE respectively. The cost of a solution S according to each team objective is marked by SUM(S), MAX(S), and AVE(S) respectively. With a slight abuse of notation, if F is a forest, we also use SUM(F) for the total cost of the forest, MAX(F) for the cost of the most expensive tree in the forest, and AVE(F) for the average of all root-target costs in the forest.

The following lemma on the relationship of the various objective functions is used repeatedly.

Lemma 1: Let F be a spanning forest rooted at the robots that spans all targets in an instance of multi-robot routing with n robots and m targets. Then it holds that

$$\text{AVE(F)} \leq \text{MAX(F)} \leq \text{SUM(F)} \leq n \ \text{MAX(F)},$$

$$\text{SUM(F)} \leq m \ \text{AVE(F)}.$$

Proof: The maximum root-target cost of any target can be at most equal to the cost of the most expensive tree in the forest. Therefore, the average of the root-target costs cannot be more than the cost of the most expensive tree in the forest. Furthermore, the cost of the most expensive tree in the forest cannot exceed the total cost of the forest. The total cost of the forest cannot exceed an n-multiple of the cost of the most expensive tree, since there are at most n trees in the forest. Finally, there are m targets in the forest and the contribution of each target to the total cost of the forest is no more than its root-target cost. Therefore, the total cost of the forest cannot exceed the sum of all root-target costs, which can be expressed as an m-multiple of the average root-target cost. ■

A. Upper Bounds for BIDSUMPATH

Theorem 2: The performance ratio of the BIDSUMPATH bidding rule for the MINISUM team objective is at most 2.

Proof: Let $G = (R \cup T, c)$ be the weighted graph over all robot and target nodes. At each round k of the auction, $k = 0, \ldots, m-1$, let V_k be the set of robot nodes and allocated target nodes and $\bar{V}_k$ the set of unallocated target nodes. The sets V_k and $\bar{V}_k$ define a cut over G and, obviously, $V_0 = R$, $\bar{V}_0 = T$, $V_m = R \cup T$, and $\bar{V}_m = \emptyset$. At each round k, BIDSUMPATH selects a target $t \in \bar{V}_k$ that can be added to one of the paths in V_k with the least additional cost. Let this cost be $b(V_k, \bar{V}_k)$ which is exactly the bid placed by the winning robot. Therefore, the SUM cost of the solution found by BIDSUMPATH at the end of the auction is:

$$\textsc{Sum}(\textsc{BidSumPath}) = \sum_{k=0}^{m-1} b(V_k, \bar{V}_k)$$

Let $c(V_k, \bar{V}_k)$ be the cost of a cheapest edge across the cut $(V_k, \bar{V}_k)$. A target in $\bar{V}_k$ corresponding to a cheapest edge can be inserted to some path in V_k with an increase of at most $2c(V_k, \bar{V}_k)$ in SUM cost (because of the triangle inequality assumption). Since the BIDSUMPATH rule identifies an insertion with minimum increase in SUM cost, it must be the case that $b(V_k, \bar{V}_k) \leq 2c(V_k, \bar{V}_k)$. Hence,

$$\textsc{Sum}(\textsc{BidSumPath}) \leq 2 \sum_{k=0}^{m-1} c(V_k, \bar{V}_k)$$

Consider another graph G' which is identical to G except that exactly m edges have their costs lowered. In particular, for every cut $(V_k, \bar{V}_k)$ the cost of a cheapest edge connecting t (the target selected by BIDSUMPATH) to V_k is lowered to $c(V_k, \bar{V}_k)$ (the cost of a cheapest edge across the cut) in G'. Clearly, those m edges in G' form an MSF in G' (it is equivalent to running Prim's algorithm starting with the robot nodes connected to each other with zero cost). An MSF in G cannot have less SUM cost than any MSF in G', since we have only lowered costs while constructing G' from G. Therefore, we obtain:

$$\sum_{k=0}^{m-1} c(V_k, \bar{V}_k) = \textsc{Sum}\big(\text{MSF}(G')\big) \leq \textsc{Sum}\big(\text{MSF}(G)\big).$$

which implies that

$$\textsc{Sum}(\textsc{BidSumPath}) \leq 2\ \textsc{Sum}\big(\text{MSF}(G)\big). \qquad (1)$$

An optimal solution OPTSUM for the MINISUM team objective is also a spanning forest in G, therefore it is true that

$$\textsc{Sum}\big(\text{MSF(G)}\big) \leq \textsc{Sum}(\textsc{OptSum}).$$

Thus, we conclude that

$$\textsc{Sum}(\textsc{BidSumPath}) \leq 2\ \textsc{Sum}(\textsc{OptSum}).$$

■

Using Equation 1, Lemma 1, and the fact that both OPTMAX and OPTAVE are spanning forests, we also conclude:

Corollary 1: The performance ratio of the BIDSUMPATH bidding rule for the MINIMAX team objective is at most $2n$.

Corollary 2: The performance ratio of the BIDSUMPATH bidding rule for the MINIAVE team objective is at most $2m$.

B. Upper Bounds for BIDMAXPATH

Theorem 3: The performance ratio of the BIDMAXPATH bidding rule for the MINISUM team objective is at most $2n$.

Proof: As in Theorem 2, consider the cuts $(V_k, \bar{V}_k)$ at each round k of the auction. Let $c(V_k, \bar{V}_k)$ be the cost of a cheapest edge across the cut $(V_k, \bar{V}_k)$. We establish by induction that at any round of the auction, the SUM cost of any path P_i^k, $i = 1, \ldots, n$, in V_k is bounded as follows:

$$\textsc{Sum}(P_i^k) \leq 2 \sum_{j=0}^{k-1} c(V_j, \bar{V}_j)$$

The base case is certainly true as $P_i^0 = \{r_i\}$ (a single node), $\textsc{Sum}(P_i^0) = 0$. Assume that the assertion holds for k. At the next round, BIDMAXPATH allocates a target $t \in \bar{V}_k$ that minimizes the cost of the most expensive path in V_{k+1}. The path P_r^{k+1} where t was added must be the most expensive path in V_{k+1}, otherwise t would have been allocated in some previous round. Therefore, for any path P_i^{k+1} in V_{k+1} it is true that

$$\textsc{Sum}(P_i^{k+1}) \leq \textsc{Sum}(P_r^{k+1}).$$

Let $t' \in \bar{V}_k$ be the target and $P_{r'}^k \in V_k$ be the path corresponding to a cheapest edge across the cut $(V_k, \bar{V}_k)$. Target t' can be inserted in path $P_{r'}^k$ with an increase of at most $2c(V_k, \bar{V}_k)$ in the SUM cost of $P_{r'}^k$ (because of the triangle inequality assumption). Since BIDMAXPATH selects to insert t in P_r^k at round k, it must be the case that

$$\textsc{Sum}(P_r^{k+1}) \leq \textsc{Sum}(P_{r'}^k) + 2\ c(V_k, \bar{V}_k)$$

Finally, by the inductive hypothesis we have

$$\textsc{Sum}(P_i^{k+1}) \leq 2 \sum_{j=0}^{k-1} c(V_j, \bar{V}_j) + 2\ c(V_k, \bar{V}_k) \leq 2 \sum_{j=0}^{k} c(V_j, \bar{V}_j).$$

Since the MAX cost of the BIDMAXPATH solution is the SUM cost of the most expensive path, we conclude that

$$\textsc{Max}(\textsc{BidMaxPath}) \leq 2 \sum_{j=0}^{m-1} c(V_j, \bar{V}_j).$$

Using the construction for graph G' as in Theorem 2, we have

$$\textsc{Max}(\textsc{BidMaxPath}) \leq 2\ \textsc{Sum}(\text{MSF}). \qquad (2)$$

An optimal solution OPTSUM for the MINISUM team objective is also a spanning forest, so by Lemma 1 we have

$$\textsc{Sum}(\textsc{BidMaxPath}) \leq 2n\ \textsc{Sum}(\textsc{OptSum}).$$

■

Using Equation 2, Lemma 1, and the fact that both OPTMAX and OPTAVE are spanning forests, we also conclude:

Corollary 3: The performance ratio of the BIDMAXPATH bidding rule for the MINIMAX team objective is at most $2n$.

Corollary 4: The performance ratio of the BIDMAXPATH bidding rule for the MINIAVE team objective is at most $2m$.

C. Upper Bounds for BIDAVEPATH

Theorem 4: The performance ratio of the BIDAVEPATH bidding rule for the MINIAVE team objective is at most $2m^2$.

Proof: As in Theorem 2, consider the cuts $(V_k, \bar{V}_k)$ at each round k of the auction. At each round k, BIDAVEPATH selects a target $t \in \bar{V}_k$ that is added to one of the paths in V_k with the least increase in the AVE team objective. Let this increase be $b(V_k, \bar{V}_k)$ which corresponds to the bid placed by the winning robot. Therefore, the AVE cost of the solution found by BIDAVEPATH at the end of the auction is:

$$\text{AVE}(\text{BIDAVEPATH}) = \sum_{k=0}^{m-1} b(V_k, \bar{V}_k)$$

Let $c(R, \bar{V}_k)$ be the cost of a cheapest edge across the sets R and $\bar{V}_k$, that is the cost of a cheapest edge between some unallocated target t' and some robot r'. Target t' can always be inserted as the first target in the path of r' in V_k. Such an insertion inflicts an increase of at most $2c(R, \bar{V}_k)$ to the robot-target cost of each target in the path because of the triangle inequality assumption, while the robot-target cost of t' is $c(R, \bar{V}_k)$. In the worst case, this insertion occurs at a path that contains all other targets. Since the increase of the robot-target cost for each of the m targets is at most $2c(R, \bar{V}_k)$, so is the increase in AVE cost. Since the BIDAVEPATH rule identifies the insertion with the least increase at each round, it must be the case that $b(V_k, \bar{V}_k) \le 2c(R, \bar{V}_k)$, and therefore

$$\text{AVE}(\text{BIDAVEPATH}) \le 2 \sum_{k=0}^{m-1} c(R, \bar{V}_k)$$

It holds that $c(R, \bar{V}_k) \le \text{SUM}(\text{MSF})$, since in an MSF no robot can reach a target with cost less than the cheapest direct edge from any robot to that target. Therefore,

$$\text{AVE}(\text{BIDAVEPATH}) \le 2m\ \text{SUM}(\text{MSF}). \quad (3)$$

An optimal solution OPTSUM for the MINISUM team objective is also a spanning forest, so by Lemma 1 we have that

$$\text{SUM}(\text{BIDAVEPATH}) \le 2m^2\ \text{SUM}(\text{OPTSUM}).$$

■

Using Equation 3, Lemma 1, and the fact that both OPTMAX and OPTAVE are spanning forests, we also conclude:

Corollary 5: The performance ratio of the BIDAVEPATH bidding rule for the MINIMAX team objective is at most $2m^2n$.

Corollary 6: The performance ratio of the BIDAVEPATH bidding rule for the MINIAVE team objective is at most $2m^2$.

D. Upper Bounds for BIDSUMTREE

Theorem 5: The performance ratio of the BIDSUMTREE bidding rule for the MINISUM team objective is at most 2 [8].

Proof: The bid placed by each robot at each round is equal to the cost of adding the closest unallocated target to its own subtree. Considering all robot nodes as connected with each other with zero cost, the auction with the BIDSUMTREE rule is identical to Prim's algorithm for MST [10]. Therefore, the tree found by this rule is indeed an MSF. Converting the trees of an MSF to paths incurs a factor of 2, therefore:

$$\text{SUM}(\text{BIDSUMTREE}) \le 2\ \text{SUM}(\text{MSF}). \quad (4)$$

An optimal solution OPTSUM for the MINISUM team objective is also a spanning forest, therefore we conclude that

$$\text{SUM}(\text{BIDSUMTREE}) \le 2\ \text{SUM}(\text{OPTSUM}).$$

■

Using Equation 4, Lemma 1, and the fact that both OPTMAX and OPTAVE are spanning forests, we also conclude:

Corollary 7: The performance ratio of the BIDSUMTREE bidding rule for the MINIMAX team objective is at most $2n$.

Corollary 8: The performance ratio of the BIDSUMTREE bidding rule for the MINIAVE team objective is at most $2m$.

E. Upper Bounds for BIDMAXTREE

Theorem 6: The performance ratio of the BIDMAXTREE bidding rule for the MINISUM team objective is at most $2n$.

Proof: As in Theorem 2, consider the cuts $(V_k, \bar{V}_k)$ at each round k of the auction. Let $c(V_k, \bar{V}_k)$ be the cost of the cheapest edge across the cut $(V_k, \bar{V}_k)$. We establish by induction that at any round of the auction, the SUM cost of any tree T_i^k, $i = 1, \ldots, n$, in V_k is bounded as follows:

$$\text{SUM}(T_i^k) \le \sum_{j=0}^{k-1} c(V_j, \bar{V}_j)$$

The base case is certainly true as $T_i^0 = \{r_i\}$ (a single node), $\text{SUM}(T_i^0) = 0$. Assume that the assertion holds for k. At the next round, BIDMAXTREE allocates a target $t \in \bar{V}_k$ that minimizes the cost of the most expensive tree in V_{k+1}. The tree T_r^{k+1} where t was added must be the most expensive tree in V_{k+1}, otherwise t would have been allocated in some previous round. Therefore, for any tree T_i^{k+1} in V_{k+1} it is true that

$$\text{SUM}(T_i^{k+1}) \le \text{SUM}(T_r^k) + c(T_r^k, t)$$

Let $t' \in \bar{V}_k$ be the target and $T_{r'}^k \in V_k$ be the tree corresponding to the cheapest edge across the cut $(V_k, \bar{V}_k)$. Since BIDMAXTREE selects to attach t to T_r^k in round k, it must be the case that

$$\text{SUM}(T_r^k) + c(T_r^k, t) \le \text{SUM}(T_{r'}^k) + c(T_{r'}^k, t')$$

or, using also that fact that $c(T_{r'}^k, t') = c(V_k, \bar{V}_k)$,

$$\text{SUM}(T_i^{k+1}) \le \text{SUM}(T_{r'}^k) + c(V_k, \bar{V}_k).$$

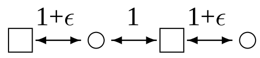

Fig. 1. A simple instance with 2 robots (squares) and two targets (circles).

Finally, by the inductive hypothesis

$$\text{Sum}(T_i^{k+1}) \leq \sum_{j=0}^{k-1} c(V_j, \bar{V}_j) + c(V_k, \bar{V}_k) \leq \sum_{j=0}^{k} c(V_j, \bar{V}_j).$$

Since the Max cost of the BidMaxTree solution is at most twice the Sum cost of the most expensive tree (taking into account the conversion of trees to paths), we conclude that

$$\text{Max}(\text{BidMaxTree}) \leq 2 \sum_{j=0}^{m-1} c(V_j, \bar{V}_j).$$

Using the construction for graph G' as in Theorem 2, we have

$$\text{Max}(\text{BidMaxTree}) \leq 2\ \text{Sum}(\text{MSF}), \tag{5}$$

An optimal solution OptSum for the MiniSum team objective is also a spanning forest, so by Lemma 1 we have

$$\text{Sum}(\text{BidMaxTree}) \leq 2n\ \text{Sum}(\text{OptSum}).$$

■

Using Equation 5, Lemma 1, and the fact that both OptMax and OptAve are spanning forests, we also conclude:

Corollary 9: The performance ratio of the BidMaxTree bidding rule for the MiniMax team objective is at most $2n$.

Corollary 10: The performance ratio of the BidMaxTree bidding rule for the MiniAve team objective is at most $2m$.

F. Upper Bounds for BidAveTree

Theorem 7: The performance ratio of the BidAveTree bidding rule for the MiniSum team objective is at most $2m$.

Proof: Under the BidAveTree rule, the intermediate spanning forest at the end of the auction will consist of stars, one for each robot, where each target is connected directly to the closest (in terms of cost) robot. Because of the triangle inequality assumption, direct connections minimize the robot-target costs, and therefore the average.

Let c^* be the cost of the most expensive robot-target edge in the forest. It holds that $c^* \leq \text{Sum}(\text{MSF})$, since in an MSF no robot can reach a target with cost less than the cheapest direct edge from any robot to that target. Since there are m targets in total, the Sum cost of the forest is at most mc^*, and thus the Sum cost of BidAveTree can be at most $2mc^*$,

$$\text{Sum}(\text{BidAveTree}) \leq 2mc^* \leq 2m\ \text{Sum}(\text{MSF}), \tag{6}$$

where the factor of 2 comes from the conversion of stars to paths. Thus, given that any optimal solution OptSum is also a spanning forest, we conclude that

$$\text{Sum}(\text{BidAveTree}) \leq 2m\ \text{Sum}(\text{OptSum}).$$

■

Using Equation 6, Lemma 1, and the fact that both OptMax and OptAve are spanning forests, we also conclude:

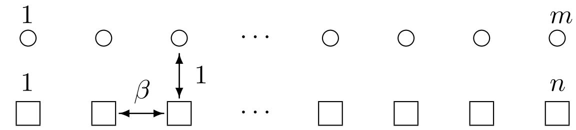

Fig. 2. Parallel lines construction: n robots and $m = n$ targets spaced evenly (distance of β) on two parallel lines (one for robots, one for targets). The distance between the two lines is 1, except for the left-most robot which is a little closer to its corresponding target.

Corollary 11: The performance ratio of the BidAveTree bidding rule for the MiniMax team objective is at most $2mn$.

Corollary 12: The performance ratio of the BidAveTree bidding rule for the MiniAve team objective is at most $2m^2$.

G. Lower bounds

In all example instances, robots are shown as squares, single targets as open circles, clusters of targets as solid circles. and ϵ represents an arbitrarily small positive number. The examples that yield lower bounds for the Tree rules are identical to those for the Path rules, and therefore they are omitted.

Applying BidSumPath to the instance in Figure 1 results in a solution that allocates both targets to the robot on the right with a Sum cost of $3+\epsilon \approx 3$ as opposed to the OptSum cost of $2+2\epsilon \approx 2$ (one target to each robot), so a lower bound to the performance ratio of BidSumPath for MiniSum is 1.5.

Applying the BidSumPath rule to the example in Figure 2 with $\beta = 1-\epsilon$ yields a solution that allocates all targets to the left-most robot and a path that runs through all targets left to right. Obviously, Max(BidSumPath) $\approx n$, whereas Max(OptMax) $= 1$ (each robot visits its corresponding target). Therefore, a lower bound to the performance ratio of BidSumPath for MiniMax is $n = m$, or more precisely $\max(n,m)$. Similarly, Ave(BidSumPath) $\approx (1+2+\ldots+n)/n = (n+1)/2$, whereas Ave(OptAve) $= 1$ (each robot visits its corresponding target). Therefore, a lower bound to the performance ratio of BidSumPath for MiniAve is $(n+1)/2 = (m+1)/2$, or $(\max(n,m)+1)/2$.

Applying BidMaxPath to the instance in Figure 2 with $\beta = \epsilon$ yields a solution that allocates one target to each robot. Apparently, Sum(BidMaxPath) $= n$, whereas Sum(OptSum) $= 1+n\epsilon \approx 1$ (the left most robot takes all targets) for $\epsilon = o(1/n)$. Thus, a lower bound to the performance ratio of BidMaxPath for MiniSum is $n = m$, or more precisely $\max(n,m)$. Similarly, Sum(BidAvePath) $= n$, whereas Sum(OptSum) ≈ 1 (the left-most robot takes all targets). Thus, a lower bound to the performance ratio of BidAvePath for MiniSum is $n = m$, or $\max(n,m)$.

Applying BidMaxPath (or BidAvePath) to the instance in Figure 3 yields a solution that allocates one target per cluster to each robot. The path of each robot traverses the grid alternating left-to-right and right-to-left, and also from bottom to top. Figure 4 shows the necessary adjustments to make this possible. Apparently, Max(BidMaxPath) $= n(n-1)\beta + (\beta+1)(n-1) = (n\beta+\beta+1)(n-1)$, whereas Max(OptMax) $\leq (n-1)\beta + (\beta+1)(n-1) = (2\beta+1)(n-1)$ (each robot takes one row of the grid). Thus,

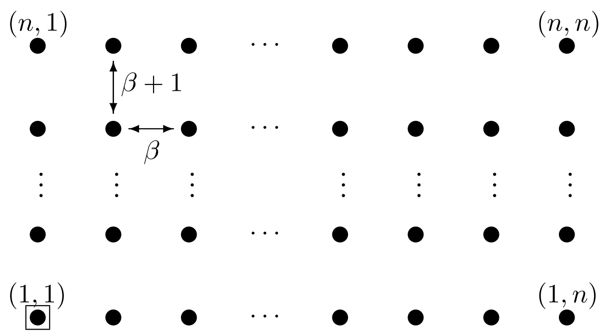

Fig. 3. Grid construction: n robots at $(1,1)$ and $m = n^3$ targets on a $(n \times n)$ rectangular grid; each of the n^2 gridpoints is a cluster of n targets. The intra-row distance is β, whereas the intra-column distance is $\beta + 1$.

a lower bound to the performance ratio of BidMaxPath for MiniMax is $(n + 1)/2 = (m^{1/3} + 1)/2$ for large β, or $(\max(n, m^{1/3}) + 1)/2$. In addition, without going into a detailed analysis, it is obvious that Ave(BidMaxPath) $= \Omega(n^2)$, whereas Ave(OptAve) $= O(n)$ (each robot takes one row of the grid). Thus, a lower bound to the performance ratio of BidMaxPath for MiniAve is $\Omega(n)$ or $\Omega(m^{1/3})$ since $m = n^3$, or more precisely $\Omega(\max(n, m^{1/3}))$. The same bounds hold for BidAvePath.

IX. Related Work

Multi-robot routing falls in the class of Location Routing problems [13]. There has been a tremendous amount of work on centralized algorithms for solving such problems optimally or approximately. The MiniSum objective has been studied in the context of k-TSP problems and can be approximated to a constant factor [14]. The MiniMax objective appears also in the Nurse Location Problem for which there exists an 8-approximation [11]. This objective has also been studied in the context of job scheduling in unrelated machines (makespan) [15]. Finally, the MiniAve objective is also known as the Traveling Repairman Problem (or Minimum Latency Problem) and is approximable to a constant factor [16]. Robotics researchers have studied the MiniSum [2], [7]–[9] objective extensively, but only occasionally the MiniMax [1], [9] and MiniAve [9] objectives.

A variety of auction methods have been used for multi-robot routing. Berhault et. al. [7] have used combinatorial auctions, however their complexity makes them impractical for large problems. Dias and Stentz [2] have proposed a single-item auction similar to BidSumPath which has been implemented on real robots for exploration tasks. The Path rules have been tested experimentally [9] and have been shown to perform best for the corresponding team objective they are designed for. Their actual performance is well below the upper bounds shown in this paper making them practical for large problems.

X. Conclusion

We presented a theoretical analysis of a variety of auction-based methods for multi-robot routing and showed for the first time that they all offer reasonable guarantees. Our future work includes extensive experimentation to assess the actual performance of our algorithms and deeper analysis to tighten our theoretical bounds.

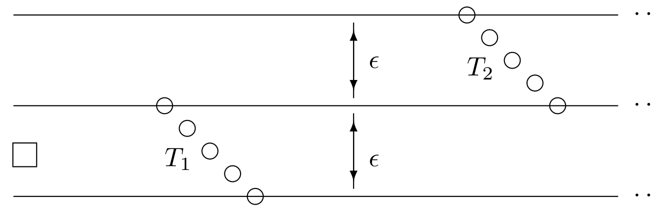

Fig. 4. Cluster arrangement in grid construction: example with 2 clusters of 5 targets each and 5 robots clustered on the left. Targets in cluster T_1 are arranged evenly on a line of slope -45 degrees within a small cost from each other. Each of the 5 robots will visit one target in T_1, since the straight line is shorter than any other path. Targets in T_2 have the same arrangement, but they are shifted up by ϵ to ensure that the cost between corresponding targets in T_1 and T_2 is less than any other inter-cluster cost. Clearly, the robot that visits the first target in T_1 will also visit the first target in T_2, and so on. This pattern continues along the horizontal axis, but it can also be used for clusters arranged vertically with shifting to the right.

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