Computer Vision: Calibration and Reconstruction

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What did we see in class last week?
Today’s Readings

- Chapter 6 and 11 of Szeliski’s book
Let’s look at camera calibration
Find the quantities internal to the camera that affect the imaging process as well as the position of the camera with respect to the world

- Rotation and translation
Camera Calibration

Find the quantities internal to the camera that affect the imaging process as well as the position of the camera with respect to the world

- Rotation and translation
- Position of image center in the image
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- Rotation and translation
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- Focal length
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Find the quantities internal to the camera that affect the imaging process as well as the position of the camera with respect to the world

- Rotation and translation
- Position of image center in the image
- Focal length
- Different scaling factors for row pixels and column pixels
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- Rotation and translation
- Position of image center in the image
- Focal length
- Different scaling factors for row pixels and column pixels
- Skew factor
Camera Calibration

Find the quantities internal to the camera that affect the imaging process as well as the position of the camera with respect to the world

- Rotation and translation
- Position of image center in the image
- Focal length
- Different scaling factors for row pixels and column pixels
- Skew factor
- Lens distortion (pin-cushion effect)
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- Rotation and translation
- Position of image center in the image
- Focal length
- Different scaling factors for row pixels and column pixels
- Skew factor
- Lens distortion (pin-cushion effect)
Why do we need calibration?

- Have good reconstruction
- Interact with the 3D world

[Source: Ramani]
Camera and Calibration Target

Most methods assume that we have a known 3D target in the scene

[Source: Ramani]
Many algorithms!

- Calibration target: 2 planes at right angle with checkerboard patterns (Tsai grid)
Most common used Procedure

Many algorithms!

- Calibration target: 2 planes at right angle with checkerboard patterns (Tsai grid)
- We know positions of pattern corners only with respect to a coordinate system of the target
Many algorithms!

- **Calibration target**: 2 planes at right angle with checkerboard patterns (Tsai grid)
- **We know positions of pattern corners only with respect to a coordinate system of the target**
- **We position camera in front of target and find images of corners**
Most common used Procedure

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- Calibration target: 2 planes at right angle with checkerboard patterns (Tsai grid)
- We know positions of pattern corners only with respect to a coordinate system of the target
- We position camera in front of target and find images of corners
- We obtain equations that describe imaging and contain internal parameters of camera

[Source: Ramani]

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Most common used Procedure

Many algorithms!

- Calibration target: 2 planes at right angle with checkerboard patterns (Tsai grid)
- We know positions of pattern corners only with respect to a coordinate system of the target
- We position camera in front of target and find images of corners
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- We also find position and orientation of camera with respect to target (camera pose)

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[Source: Ramani]
Obtaining 2D-3D correspondences

- Canny edge detection
- Straight line fitting to detect linked edges. How?
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  - We get pairs (image point)–(world point), \((x_i, y_i) \rightarrow (X_i, Y_i, Z_i)\).

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[Source: Ramani]
Estimate matrix $P$ using 2D-3D correspondences

Estimate $K$ and $(R, t)$ from $P$

$$P = K \cdot R \cdot [I_{3 \times 3} \mid t]$$
Calibration

- Estimate matrix $P$ using 2D-3D correspondences
- Estimate $K$ and $(R, t)$ from $P$

$$P = K \cdot R \cdot [I_{3 \times 3} \mid t]$$

- Left 3x3 submatrix of $P$ is product of upper-triangular matrix and orthogonal matrix
Calibration

- Estimate matrix $\mathbf{P}$ using 2D-3D correspondences
- Estimate $\mathbf{K}$ and $(\mathbf{R}, \mathbf{t})$ from $\mathbf{P}$

\[
\mathbf{P} = \mathbf{K} \cdot \mathbf{R} \cdot \begin{bmatrix} \mathbf{I}_{3 \times 3} & \mathbf{t} \end{bmatrix}
\]

- Left 3x3 submatrix of $\mathbf{P}$ is product of upper-triangular matrix and orthogonal matrix
Inherent Constraints

- The visual angle between any pair of 2D points must be the same as the angle between their corresponding 3D points.
Simplest to form a set of linear equations (analog to the 2D case)

\[ x_i = \frac{p_{00} X_i + p_{01} Y_i + p_{02} Z_i + p_{03}}{p_{20} X_i + p_{21} Y_i + p_{22} Y_i + p_{23}} \]
\[ y_i = \frac{p_{10} X_i + p_{11} Y_i + p_{12} Z_i + p_{13}}{p_{20} X_i + p_{21} Y_i + p_{22} Y_i + p_{23}} \]

with \((x_i, y_i)\), the measured 2D points, and \((X_i, Y_i, Z_i)\) are known 3D locations

This can be solved in a linear fashion for \( P \). How?
Direct Linear Transform

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  Similar to the homography case.
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- This can be solved in a linear fashion for \(P\). How?
- Similar to the homography case.
- This is called the **Direct Linear Transform** (DLT)
- How many unknowns? how many correspondences?
  - 11 or 12 unknowns, 6 correspondences
Direct Linear Transform

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Once $P$ is recovered, how do I obtain the other matrices?

$t$ is the null vector of matrix $P$ as

$$Pt = 0$$
Finding Camera Translation

- Once $P$ is recovered, how do I obtain the other matrices?

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- $t$ is the unit singular vector of $P$ corresponding to the smallest singular value
Once $\mathbf{P}$ is recovered, how do I obtain the other matrices?

- $\mathbf{t}$ is the null vector of matrix $\mathbf{P}$ as

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- The last column of $\mathbf{V}$, where

$$\mathbf{P} = \mathbf{UDV}^T$$

is the SVD of $\mathbf{P}$
Finding Camera Translation

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- $\mathbf{t}$ is the null vector of matrix $\mathbf{P}$ as

$$\mathbf{P} \mathbf{t} = 0$$

- $\mathbf{t}$ is the unit singular vector of $\mathbf{P}$ corresponding to the smallest singular value.

- The last column of $\mathbf{V}$, where

$$\mathbf{P} = \mathbf{U} \mathbf{D} \mathbf{V}^T$$

is the SVD of $\mathbf{P}$.
Left $3 \times 3$ submatrix $M$ of $P$ is of form

$$M = KR$$

where $K$ is an upper triangular matrix, and $R$ is an orthogonal matrix.

Any non-singular square matrix $M$ can be decomposed into the product of an upper-triangular matrix $K$ and an orthogonal matrix $R$ using the RQ factorization.
Left 3 × 3 submatrix $M$ of $P$ is of form

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Any non-singular square matrix $M$ can be decomposed into the product of an upper-triangular matrix $K$ and an orthogonal matrix $R$ using the RQ factorization

Similar to QR factorization but order of 2 matrices is reversed
• Left $3 \times 3$ submatrix $\mathbf{M}$ of $\mathbf{P}$ is of form

$$\mathbf{M} = \mathbf{K}\mathbf{R}$$

where $\mathbf{K}$ is an upper triangular matrix, and $\mathbf{R}$ is an orthogonal matrix.

• Any non-singular square matrix $\mathbf{M}$ can be decomposed into the product of an upper-triangular matrix $\mathbf{K}$ and an orthogonal matrix $\mathbf{R}$ using the RQ factorization.

• Similar to QR factorization but order of 2 matrices is reversed.
RQ Factorization

Define the matrices

\[ R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}, \quad R_y = \begin{bmatrix} c' & 0 & s' \\ 0 & 1 & 0 \\ -s' & 0 & c' \end{bmatrix}, \quad R_z = \begin{bmatrix} c'' & -s'' & 0 \\ s'' & c'' & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

We can compute

\[ c = -\frac{M_{33}}{(M_{32}^2 + M_{33}^2)^{1/2}} \quad s = \frac{M_{32}}{(M_{32}^2 + M_{33}^2)^{1/2}} \]
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- Multiply \(M\) by \(R_x\). The resulting term at (3, 2) is zero because of the values selected for \(c\) and \(s\)
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Multiply the resulting matrix by \( R_y \), after selecting \( c \) and \( s \) so that the resulting term at position (3, 1) is set to zero.

Multiply the resulting matrix by \( R_z \), after selecting \( c \) and \( s \) so that the resulting term at position (2, 1) is set to zero.
RQ Factorization

- Define the matrices
  
  \[ R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}, \quad R_y = \begin{bmatrix} c' & 0 & s' \\ 0 & 1 & 0 \\ -s' & 0 & c' \end{bmatrix}, \quad R_z = \begin{bmatrix} c'' & -s'' & 0 \\ s'' & c'' & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

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- Multiply the resulting matrix by \( R_z \), after selecting \( c \) and \( s \) so that the resulting term at position (2, 1) is set to zero
Why does this algorithm work?

\[ MR_x R_y R_z = K \]

Thus we have

\[ M = KR_z^T R_y^T R_x^T = KR \]
Why does this algorithm work?

\[ MR_x R_y R_z = K \]

Thus we have

\[ M = K R_z^T R_y^T R_x^T = KR \]
Improved computation of $P$

- We have equations involving homogeneous coordinates
  
  $x_i = PX_i$

- Thus $x_i$ and $X_i$ just have to be proportional
  
  $x_i \times PX_i = 0$
**Improved computation of \( P \)**

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  \[ x_i \times PX_i = 0 \]

- Let \( p_1^T, p_2^T, p_3^T \) be the row vectors of \( P \)

\[
PX_i = \begin{bmatrix}
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p_2^T X_i \\
p_3^T X_i 
\end{bmatrix}
\]
Improved computation of $\mathbf{P}$

- We have equations involving homogeneous coordinates
  \[ x_i = \mathbf{P} \mathbf{X}_i \]

- Thus $x_i$ and $\mathbf{X}_i$ just have to be proportional
  \[ x_i \times \mathbf{P} \mathbf{X}_i = 0 \]

- Let $\mathbf{p}_1^T, \mathbf{p}_2^T, \mathbf{p}_3^T$ be the row vectors of $\mathbf{P}$
  \[ \mathbf{P} \mathbf{X}_i = \begin{bmatrix} \mathbf{p}_1^T \mathbf{X}_i \\ \mathbf{p}_2^T \mathbf{X}_i \\ \mathbf{p}_3^T \mathbf{X}_i \end{bmatrix} \]

- Therefore
  \[ x_i \times \mathbf{P} \mathbf{X}_i = \begin{bmatrix} v_i \mathbf{p}_3^T \mathbf{X}_i - w_i \mathbf{p}_2^T \mathbf{X}_i \\ w_i \mathbf{p}_3^T \mathbf{X}_i - u_i \mathbf{p}_2^T \mathbf{X}_i \\ u_i \mathbf{p}_3^T \mathbf{X}_i - v_i \mathbf{p}_2^T \mathbf{X}_i \end{bmatrix} \]
Improved computation of $\mathbf{P}$

- We have equations involving homogeneous coordinates
  \[ \mathbf{x}_i = \mathbf{P}\mathbf{X}_i \]

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  \[ \mathbf{x}_i \times \mathbf{P}\mathbf{X}_i = \begin{bmatrix} v_i \mathbf{p}_3^T \mathbf{X}_i - w_i \mathbf{p}_2^T \mathbf{X}_i \\ w_i \mathbf{p}_3^T \mathbf{X}_i - u_i \mathbf{p}_2^T \mathbf{X}_i \\ u_i \mathbf{p}_3^T \mathbf{X}_i - v_i \mathbf{p}_2^T \mathbf{X}_i \end{bmatrix} \]
Improved computation of $P$

- We have
  \[
  x_i \times PX_i = \begin{bmatrix}
  v_i p_3^T X_i - w_i p_2^T X_i \\
  w_i p_3^T X_i - u_i p_2^T X_i \\
  u_i p_3^T X_i - v_i p_2^T X_i
  \end{bmatrix} = 0
  \]

- We can thus write
  \[
  \begin{bmatrix}
  0_4 & -w_i X_i^T & v_i X_i^T \\
  w_i X_i^T & 0_4 & -u_i X_i^T \\
  -v_i X_i^T & u_i X_i^T & 0_4
  \end{bmatrix}
  \begin{bmatrix}
p_1 \\
p_2 \\
p_3
  \end{bmatrix} = 0
  \]
Improved computation of $P$

- We have
  \[
  \mathbf{x}_i \times P\mathbf{x}_i = \begin{bmatrix}
  v_i \mathbf{p}_3^T \mathbf{x}_i - w_i \mathbf{p}_2^T \mathbf{x}_i \\
  w_i \mathbf{p}_3^T \mathbf{x}_i - u_i \mathbf{p}_2^T \mathbf{x}_i \\
  u_i \mathbf{p}_3^T \mathbf{x}_i - v_i \mathbf{p}_2^T \mathbf{x}_i
  \end{bmatrix} = 0
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  \[
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  0_4 \\
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  \end{bmatrix}
  \begin{bmatrix}
  v_i \mathbf{X}_i^T \\
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  -u_i \mathbf{X}_i^T \\
  \end{bmatrix}
  \begin{bmatrix}
  p_1 \\
  p_2 \\
  p_3
  \end{bmatrix} = 0
  \]

- Third row can be obtained from sum of $u_i$ times first row $-v_i$ times second row
Improved computation of $P$

- We have

$$x_i \times P X_i = \begin{bmatrix} v_i p_3^T X_i - w_i p_2^T X_i \\ w_i p_3^T X_i - u_i p_2^T X_i \\ u_i p_3^T X_i - v_i p_2^T X_i \end{bmatrix} = 0$$

- We can thus write

$$\begin{bmatrix} 0 & -w_i X_i^T & v_i X_i^T \\ w_i X_i^T & 0 & -u_i X_i^T \\ -v_i X_i^T & u_i X_i^T & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = 0$$

- Third row can be obtained from sum of $u_i$ times first row $-v_i$ times second row

- 2 independent equations in 11 unknowns (ignoring scale)
Improved computation of $P$

- We have
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  v_i X_i^T \\
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- 2 independent equations in 11 unknowns (ignoring scale)

- With 6 correspondences, we get enough equations to compute matrix $P$

  \[
  Ap = 0
  \]
Improved computation of $P$

- We have
  \[
  x_i \times PX_i = \begin{bmatrix}
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  \end{bmatrix}
  \begin{bmatrix}
    p_1 \\
    p_2 \\
    p_3
  \end{bmatrix} = 0
  \]

- Third row can be obtained from sum of $u_i$ times first row $-v_i$ times second row

- 2 independent equations in 11 unknowns (ignoring scale)

- With 6 correspondences, we get enough equations to compute matrix $P$

  \[
  Ap = 0
  \]
Solving the system

\[ \mathbf{A} \mathbf{p} = 0 \]

- Minimize \( \| \mathbf{A} \mathbf{p} \| \) with the constraint \( \| \mathbf{p} \| = 1 \)
Solving the system

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- Minimize \( \| \mathbf{A} \mathbf{p} \| \) with the constraint \( \| \mathbf{p} \| = 1 \)
- \( \mathbf{p} \) is the unit singular vector of \( \mathbf{A} \) corresponding to the smallest singular value
Solving the system

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- The last column of \( \mathbf{V} \), where

\[ \mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T \]

is the SVD of \( \mathbf{A} \)
Solving the system

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\[ A = UDV^T \]

is the SVD of \( A \)

- Called **Direct Linear Transformation (DLT)**
Solving the system

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- Called **Direct Linear Transformation (DLT)**
Improving the $P$ estimation

- In most applications we have prior knowledge about some of the parameters of $K$, e.g., pixels are squared, skew is small, optical center near the center of the image

- Use this constraints and frame the problem as a minimization

\[
\sum_i \rho(x_i, PX_i), \text{ with } \rho \text{ a robust estimator}
\]

Use Levenberg-Marquardt iterative minimization

See 6.2.2. in Szeliski's book
Improving the $P$ estimation

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  - Find $P$ using DLT

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  - Use Levenberg-Marquardt iterative minimization
- See 6.2.2. in Szeliski’s book
We have assumed that lines are imaged as lines.

Significant error for cheap optics and for short focal lengths.
Radial Distortion Correction

- We can write the correction as

\[
x_c - x_0 = L(r)(x - x_0)
\]

\[
y_c - y_0 = L(r)(y - y_0)
\]

with \( L(r) = 1 + \kappa_1 r^2 + \kappa_2 r^4 \) and \( r^2 = (x - x_0)^2 + (y - y_0)^2 \)

- We thus minimize the following function

\[
f(\kappa_1, \kappa_2) = \sum_i (x'_i - x_{ci})^2 + (y'_i - y_{ci})^2
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using lines known to be straight, with \((x', y')\) the radial projection of \((x, y)\) on straight line.
Radial Distortion Correction

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Let’s look at 3D alignment
Motivation

[Source: W. Burgard]
Before we were looking at aligning 2D points, or 2D to 3D points.

Now let’s do it in 3D
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Now let’s do it in 3D

Given two corresponding point sets \( \{x_i, x'_i\} \), in the case of rigid (Euclidean) motion we want \( R \) and \( t \) that minimizes

\[
E_{R3D} = \sum_i ||x'_i - Rx_i - t||^2_2
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\]

This is called the **absolute orientation problem**

Is this easy to do?
If the correct correspondences are known, the correct relative rotation/translation can be calculated in closed form.

[Source: W. Burgard]
The centroids of the two point clouds $c$ and $c'$ can be used to estimate the translation

$$
\mu = \frac{1}{N} \sum_{i=1}^{N} x_i \quad \mu' = \frac{1}{N} \sum_{i=1}^{N} x'_i
$$

Subtract the corresponding center of mass from every point in the two sets,

$$
\bar{x}_i = x_i - \mu \\
\bar{x}'_i = x'_i - \mu'
$$
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$$\bar{x}_i = x_i - \mu$$
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Solving for the Rotation

- Let \( W = \sum_{i=1}^{N} \bar{x}_i' \bar{x}_i^T \)
- We can compute the SVD of \( W \)

\[
W = USV^T
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- If the rank is 3 we have a unique solution

  $$R = UV^T$$
  $$t = \mu - R\mu'$$
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What if we don’t have correspondences?
Solving for the Rotation

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What if we don't have correspondences?
If the correspondences are not known, it is not possible to obtain the global solution.

- **Iterated Closest Points (ICP)** [Besl & McKay 92]
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**Iterated Closest Points (ICP)** [Besl & McKay 92]

Iterative algorithm that alternates between solving for correspondences and solving for \((R, t)\).
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**Iterated Closest Points (ICP) [Besl & McKay 92]**

Iterative algorithm that alternates between solving for correspondences and solving for \((R, t)\).

Does it resemble any algorithm you have seen before?
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- Iterative algorithm that alternates between solving for correspondences and solving for \((R, t)\)
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ICP Variants

- Point subsets (from one or both point sets)
- Weighting the correspondences
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Performance of Variants

- Speed
- Stability (local minima)
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ICP Variants

- Use all points
- Uniform sub-sampling
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- Random sampling
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ICP Variants

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- Feature based Sampling

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ICP Variants

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[Source: W. Burgard]
Normal-based sampling

- Ensure that samples have normals distributed as uniformly as possible

![Uniform sampling vs. normal-space sampling](image)

- Better for mostly-smooth areas with sparse features

[Source: W. Burgard]
Feature-based sampling

- try to find important points
- decrease the number of correspondences
- higher efficiency and higher accuracy
- requires preprocessing

3D Scan (~200,000 Points)  Extracted Features (~5,000 Points)

[Source: W. Burgard]
ICP Variants

- Point subsets (from one or both point sets)
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Selection vs. Weighting

- Could achieve same effect with weighting
- Hard to guarantee that enough samples of important features except at high sampling rates
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Data association

Has greatest effect on convergence and speed

- **Closest point**: stable, but slow and requires preprocessing

- **Normal shooting**: slightly better than closest point for smooth structures, worse for noisy or complex structures

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[Source: W. Burgard]
ICP Variants

- Point subsets (from one or both point sets)
- Weighting the correspondences
- Data association
- Rejecting certain (outlier) point pairs, e.g., Trimmed ICP rejects a %

Where do these 3D points come from?
Let’s look into stereo reconstruction
Public Library, Stereoscopic Looking Room, Chicago, by Phillips, 1923

[Source: N. Snavely]
Stereo matching is the process of taking two or more images and estimating a 3D model of the scene by finding matching pixels in the images and converting their 2D positions into 3D depths.

We perceived depth based on the difference in appearance of the right and left eye.
Stereo matching is the process of taking two or more images and estimating a **3D model** of the scene by **finding matching pixels** in the images and converting their 2D positions into 3D depths.

We perceived depth based on the difference in appearance of the right and left eye.
Given two images from different viewpoints

- The depth is proportional to the inverse of the disparity
- How can we compute the depth of each point in the image?
Given two images from different viewpoints

- The depth is proportional to the inverse of the **disparity**
- How can we compute the depth of each point in the image?
  - Based on how much each pixel moves between the two images

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Given two images from different viewpoints

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[Source: N. Snavely]
Pixel in one image $x_0$ projects to an **epipolar line** segment in the other image.

The segment is bounded at one end by the projection of the original viewing ray at infinity $p_\infty$ and at the other end by the projection of the original camera center $c_0$ into the second camera, which is known as the **epipole** $e_1$. 
Epipolar Geometry

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Epipolar Geometry

- If we project the epipolar line in the second image back into the first, we get another line (segment), this time bounded by the other corresponding epipole $e_0$.

- Extending both line segments to infinity, we get a pair of corresponding epipolar lines, which are the intersection of the two image planes with the epipolar plane that passes through both camera centers $c_0$ and $c_1$ as well as the point of interest $p$. 
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Rectification

- The epipolar geometry depends on the relative pose and calibration of the cameras.
- This can be computed using the fundamental matrix.
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Rectification, the process of transforming the image so that the search is along horizontal line.
The epipolar geometry depends on the relative pose and calibration of the cameras. This can be computed using the fundamental matrix. Once this is computed, we can use the epipolar lines to restrict the search space to a 1D search. **Rectification**, the process of transforming the image so that the search is along horizontal line.
The disparity for pixel \((x_1, y_1)\) is \((x_2 - x_1)\) if the images are rectified.

This is a one dimensional search for each pixel.
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- Very challenging to estimate the correspondences.

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The disparity for pixel \((x_1, y_1)\) is \((x_2 - x_1)\) if the images are rectified.

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Projective geometry depends only on the cameras internal parameters and relative pose of cameras.

Fundamental matrix $F$ encapsulates this geometry.
**Fundamental Matrix**

- Projective geometry depends only on the cameras internal parameters and relative pose of cameras.
- Fundamental matrix $\mathbf{F}$ encapsulates this geometry.
- For any pair of points corresponding in both images:
  \[ x_0^T \mathbf{F} x_1 = 0 \]
Fundamental Matrix

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- Fundamental matrix $F$ encapsulates this geometry.
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  $$x_0^T F x_1 = 0$$
[Source: Ramani]
Pencils of Epipolar Lines

[Source: Ramani]
Computation of Fundamental Matrix

- $F$ can be computed from correspondences between image points alone
- No knowledge of camera internal parameters required
Computation of Fundamental Matrix

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Computation of Fundamental Matrix

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Take $x$ in camera $P$ and find scene point $X$ on ray of $x$ in camera $P$

Find the image $x'$ of $X$ in camera $P'$
Take $x$ in camera $P$ and find scene point $X$ on ray of $x$ in camera $P$

Find the image $x'$ of $X$ in camera $P'$

Find epipole $e'$ as image of $C$ in camera $P'$, $e' = P'C$
Take \( x \) in camera \( P \) and find scene point \( X \) on ray of \( x \) in camera \( P \).

Find the image \( x' \) of \( X \) in camera \( P' \).

Find epipole \( e' \) as image of \( C \) in camera \( P' \), \( e' = P'C \).

Find epipolar line \( l' \) from \( e' \) to \( x' \) in \( P' \) as function of \( x \).
Take $x$ in camera $P$ and find scene point $X$ on ray of $x$ in camera $P$.

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The fundamental matrix $F$ is defined $l' = Fx$. 
- Take x in camera P and find scene point X on ray of x in camera P
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- Find epipole e' as image of C in camera P', e' = P'C
- Find epipolar line l' from e' to x' in P' as function of x
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- x' belongs to l', so x'^T l' = 0, so

\[ x'^T F x = 0 \]
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- \( x' \) belongs to \( l' \), so \( x'^T l' = 0 \), so
  \[
  x'^T Fx = 0
  \]
Finding the Fundamental Matrix from known Projections

- Take $x$ in camera $P$ and find one scene point on ray from $C$ to $x$
- Point $X = P^+x$ satisfies $x = PX$ with $P^+ = P^T(PP^T)^{-1}$ so $PX = PP^T(PP^T)^{-1}x = x$
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- Image of this point in camera $P'$ is $x' = P'X = P'P^+x$
Finding the Fundamental Matrix from known Projections

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- Image of this point in camera $P'$ is $x' = P'X = P'P^+x$.
- Image of $C$ in camera $P'$ is epipole $e' = P'C$. 

\[
x_i \\ \pi \\ x'_{i}\
C \\ e \\ e' \\ C' 
\]
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- Image of \( C \) in camera \( P' \) is epipole \( e' = P'C \)
- Epipolar line of \( x \) in \( P' \) is
  \[
  l' = (e') \times (P'P^+x)
  \]
Finding the Fundamental Matrix from known Projections

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Epipolar line of $x$ in $P'$ is

$$l' = (e') \times (P'P^+x)$$

$l' = Fx$ defines the fundamental matrix

$$F = (P'C) \times (P'P^+)$$
Finding the Fundamental Matrix from known Projections

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Epipolar line of $x$ in $P'$ is

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$I' = Fx$ defines the fundamental matrix

$$F = (P'C) \times (P'P^+)$$
Properties of the fundamental matrix

- Matrix $3 \times 3$ since $x'^T F x = 0$

- Let $F$ be the fundamental matrix of camera pair $(P, P')$, the fundamental matrix of camera pair $(P', P)$ is $F' = F^T$
Properties of the fundamental matrix

- Matrix 3 × 3 since $x'^T F x = 0$
- Let $F$ be the fundamental matrix of camera pair $(P, P')$, the fundamental matrix of camera pair $(P', P)$ is $F' = F^T$
- This is true since $x^T F' x' = 0$ implies $x'^T F'^T x = 0$, so $F' = F^T$
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- This is true since $\mathbf{x}^T \mathbf{F}' \mathbf{x}' = 0$ implies $\mathbf{x}'^T \mathbf{F}'^T \mathbf{x} = 0$, so $\mathbf{F}' = \mathbf{F}^T$
- Epipolar line of $\mathbf{x}$ is $\mathbf{l}' = \mathbf{F} \mathbf{x}$
- Epipolar line of $\mathbf{x}'$ is $\mathbf{l} = \mathbf{F}^T \mathbf{x}'$
- Epipole $\mathbf{e}'$ is left null space of $\mathbf{F}$, and $\mathbf{e}$ is right null space.
- All epipolar lines $\mathbf{l}'$ contains epipole $\mathbf{e}'$, so $\mathbf{e}'^T \mathbf{l}' = 0$, i.e. $\mathbf{e}'^T \mathbf{F} \mathbf{x} = 0$ for all $\mathbf{x}$, therefore $\mathbf{e}'^T \mathbf{F} = 0$
- $\mathbf{F}$ is of rank 2 because $\mathbf{F} = \mathbf{e}' \times (\mathbf{P}' \mathbf{P}^+)$ and $\mathbf{e}' \times$ is of rank 2
- $\mathbf{F}$ has 7 degrees of freedom, there are 9 elements, but scaling is not important and $\det(\mathbf{F}) = 0$ removes one degree of freedom
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