CSC 2515 (Lecture 3): Linear Classification

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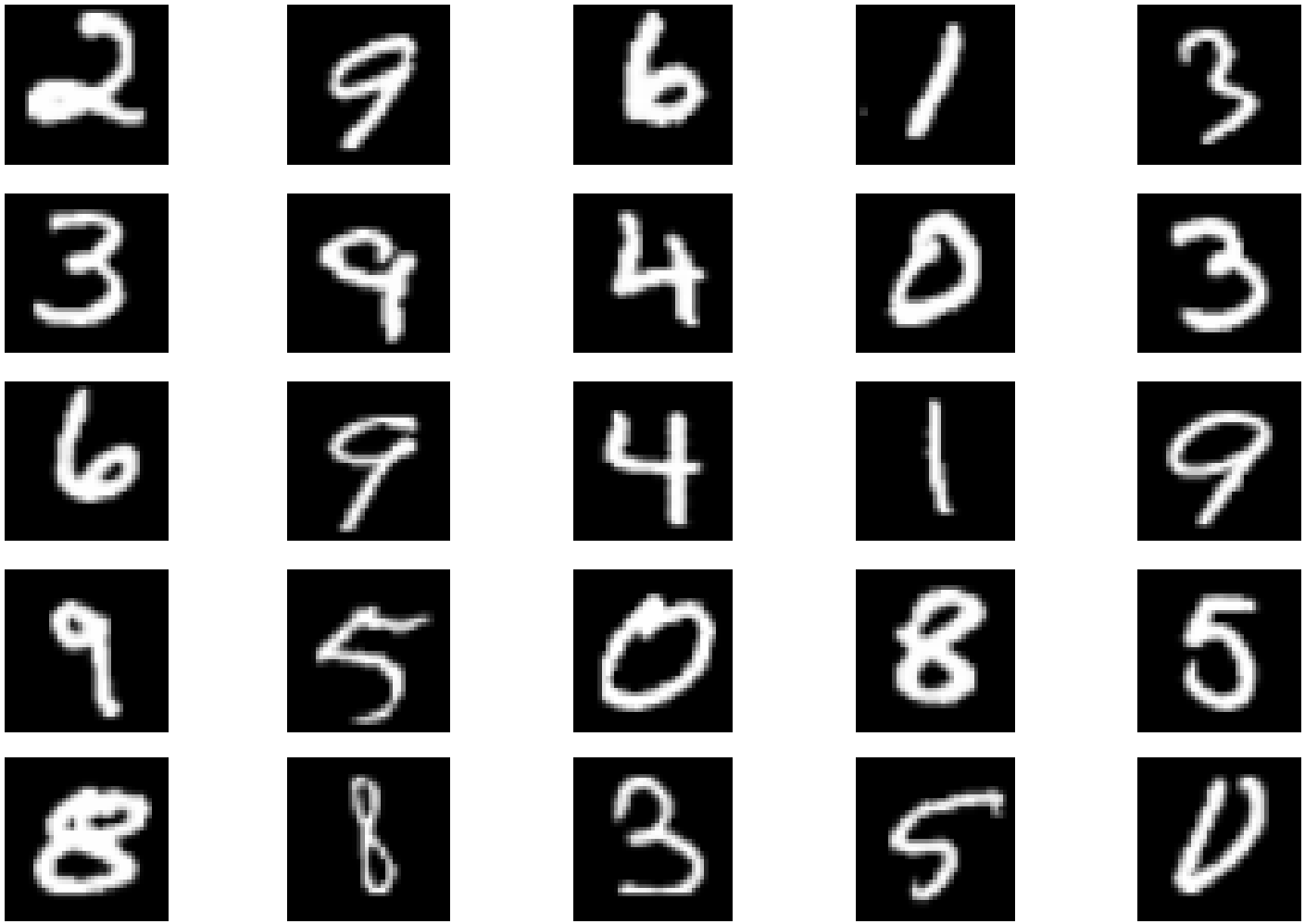
Today

- Classification as regression
- Decision boundary
- Loss functions
- Logistic Regression
- Regularization
- Nearest Neighbors

Classification vs Regression

- We are interested in mapping the input $\mathbf{x} \in \mathcal{X}$ to a *label* $t \in \mathcal{Y}$
- In regression typically $\mathcal{Y} = \Re$
- Now its a category
- Examples?

Examples of Classification



What digit is this?
How can I predict this? What are my input features?

Examples of Classification

Is this a dog?
How can I predict this? What are my input features?

Examples of Classification

what about this one?

Examples of Classification

Am I going to pass the exam?
How can I predict this? What are my input features?

Examples of Classification

Do I have diabetes?
How can I predict this? What are my input features?

Discriminative vs Generative

- **Generative approach**: Model the distribution of inputs characteristic of the class (Bayes classifier)
  - build a model of $p(\mathbf{x}|t = k)$ for every class
  - apply Bayes rule to predict the class
- **Discriminative approach:** estimate parameters of decision boundary/class separator directly from labeled examples
  - learn boundary parameters directly (e.g., logistic regression)
  - learn mappings from inputs to classes $x \rightarrow t$ (e.g., neural nets)

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- How do I compute a label for a new example? Let's see an example

Classification as Regression

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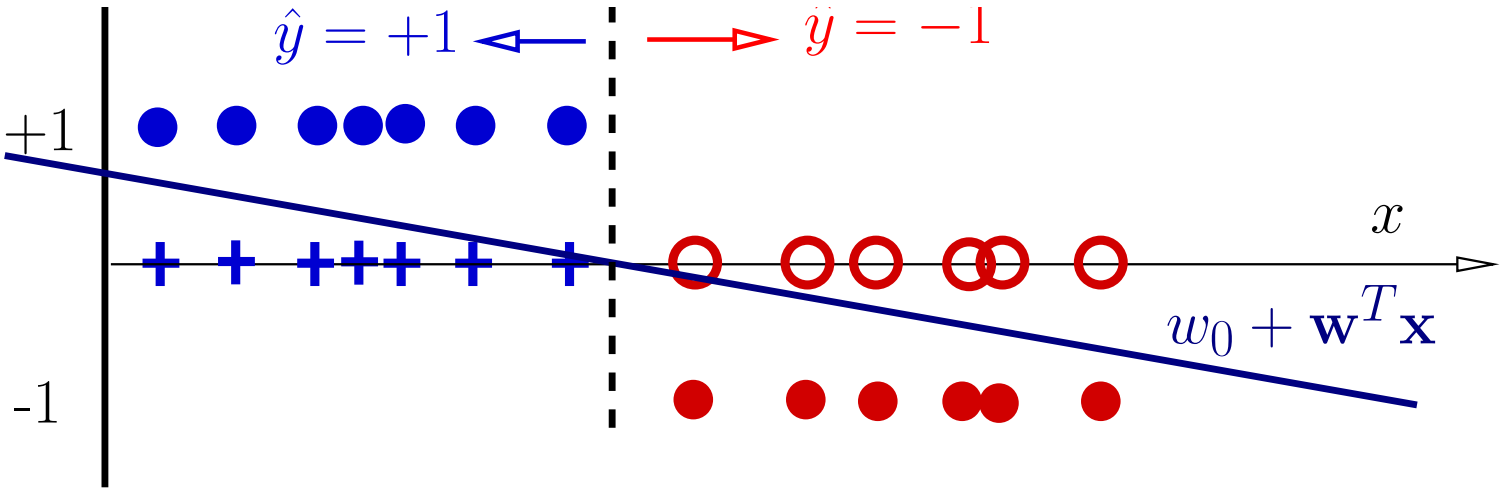
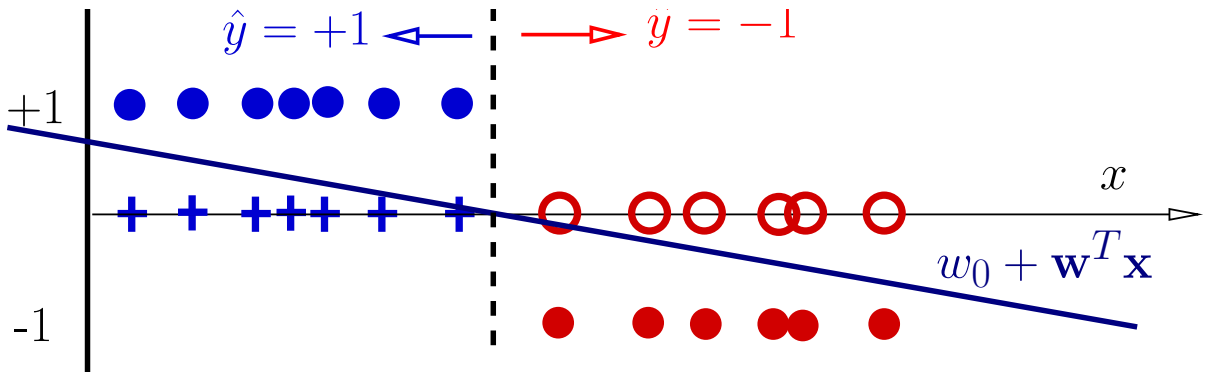


Decision Rules



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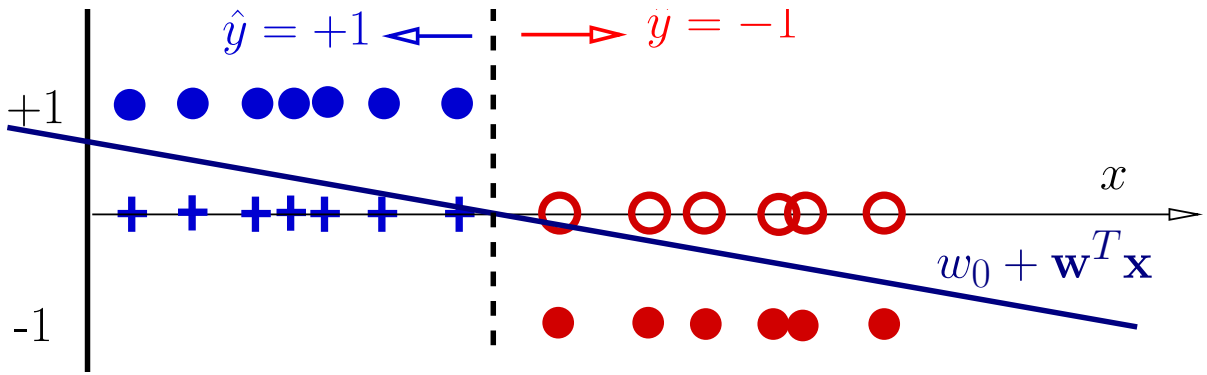

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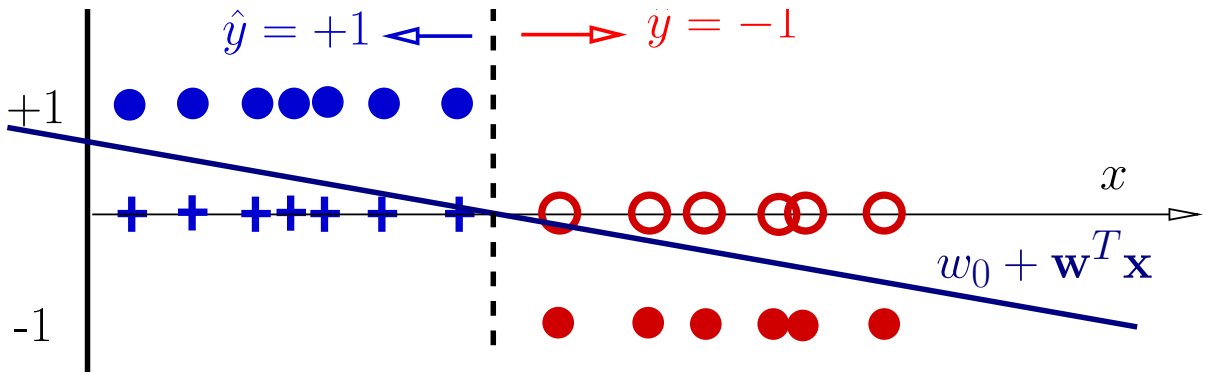

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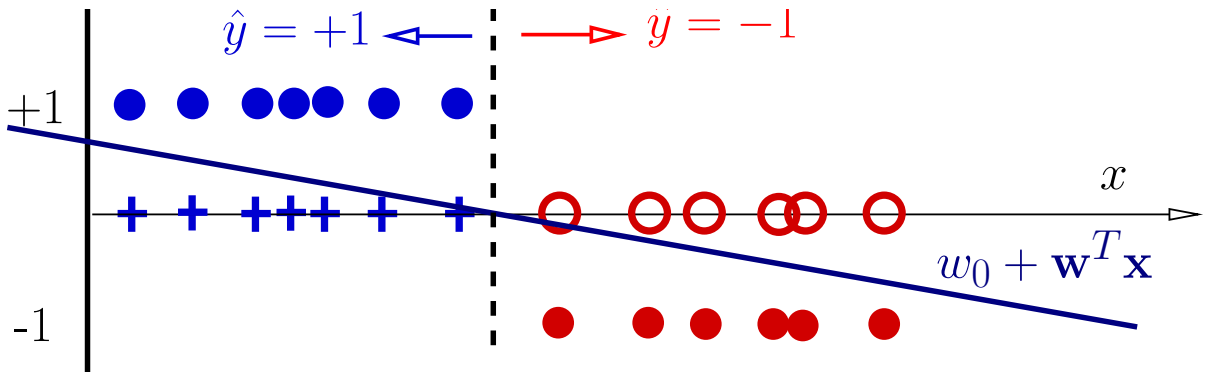

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- How does this function look like?

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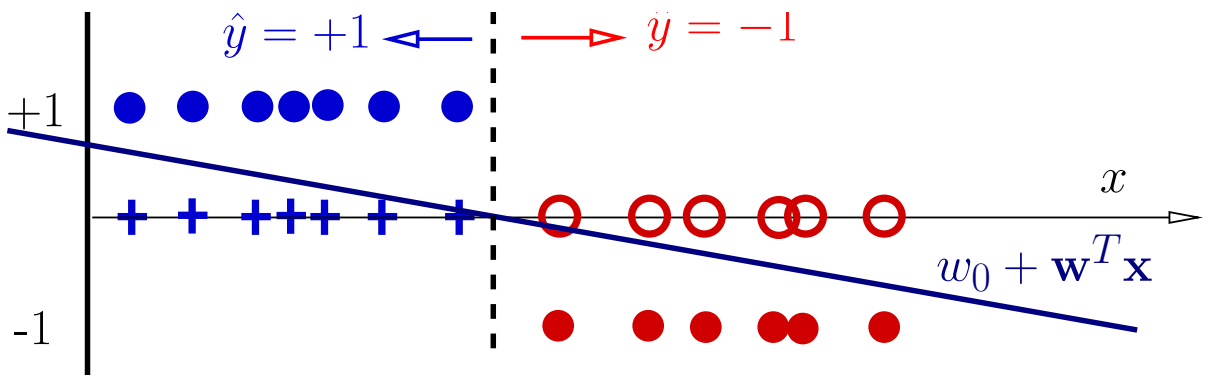


- How can I mathematically write this rule?

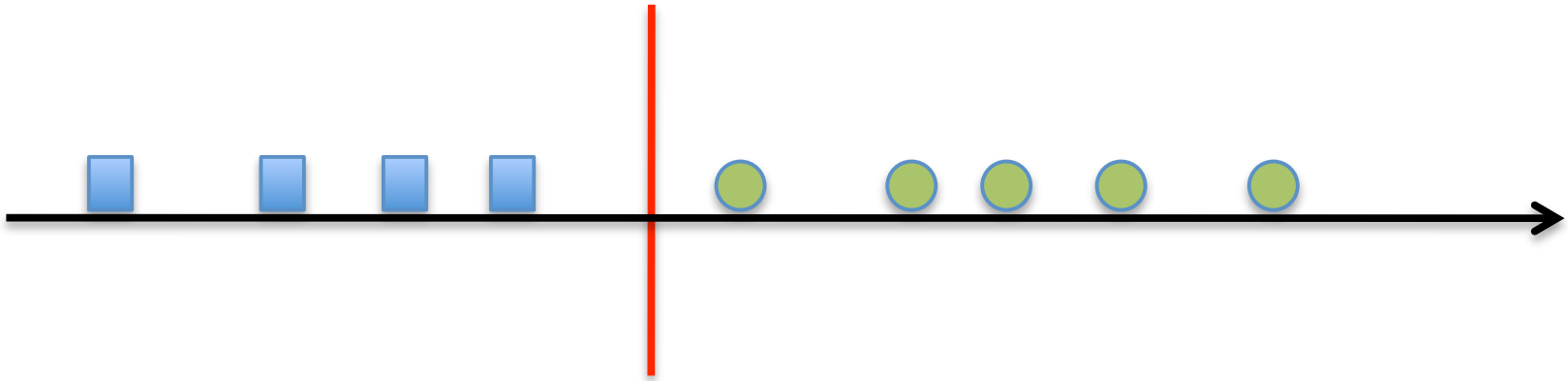
$$y = \text{sign}(w_0 + \mathbf{w}^T\mathbf{x})$$

- This specifies a **linear classifier**: it has a **linear boundary (hyperplane)**

$$w_0 + \mathbf{w}^T\mathbf{x} = 0$$

which separates the space into two "half-spaces"

Example in 1D



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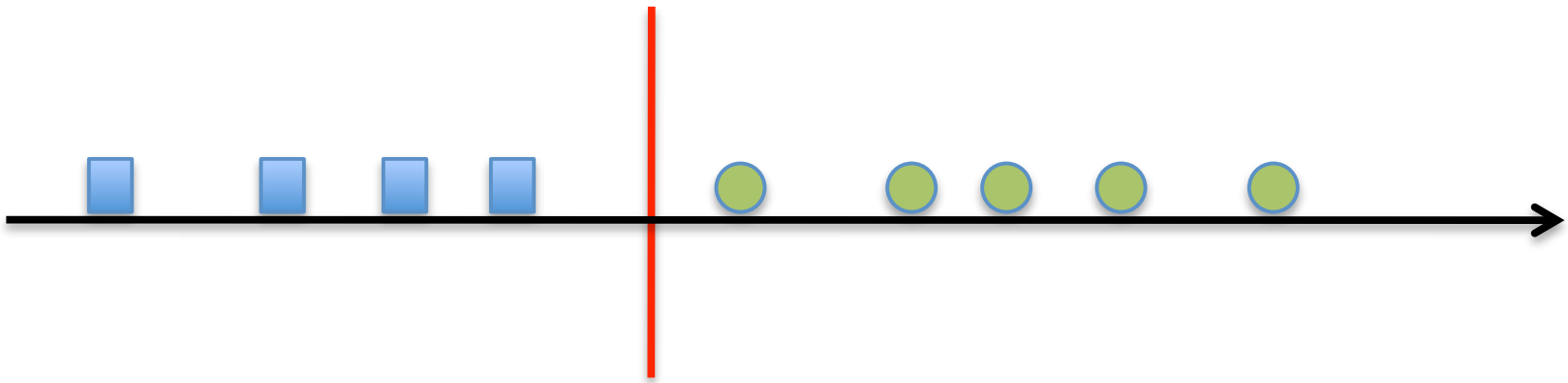

- The **linear classifier** has a **linear boundary (hyperplane)**

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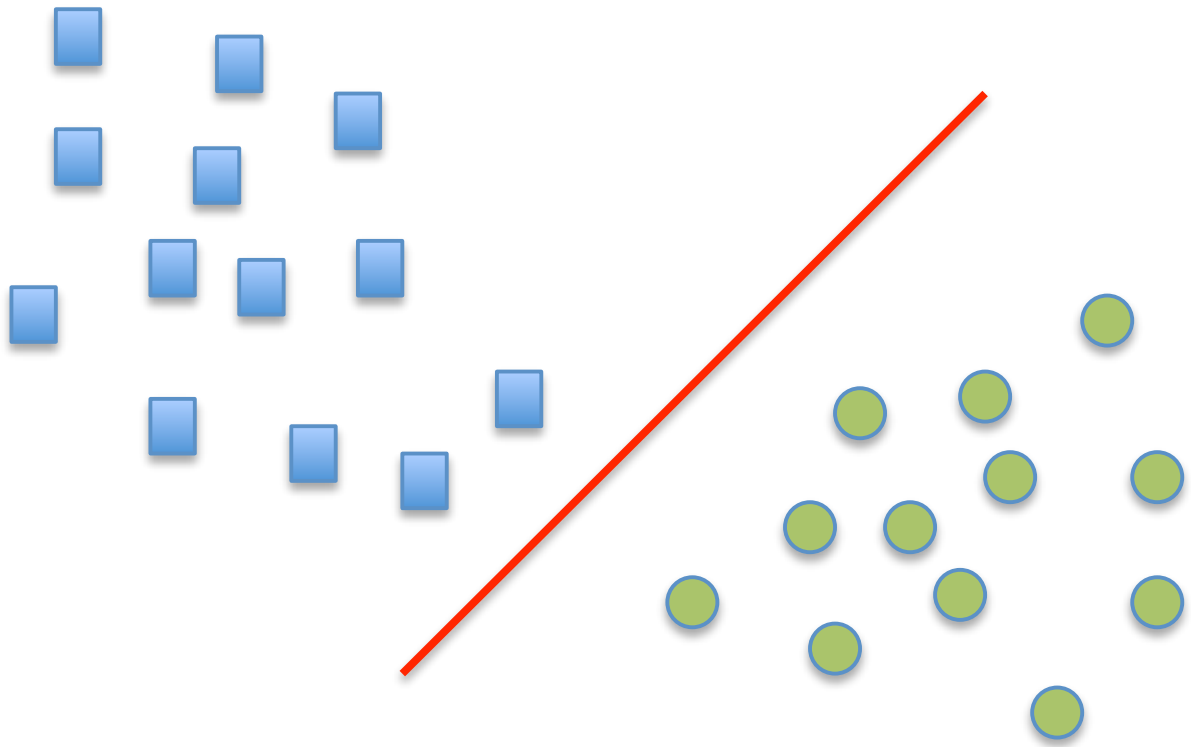
- In 1D this is simply a threshold

Example in 2D



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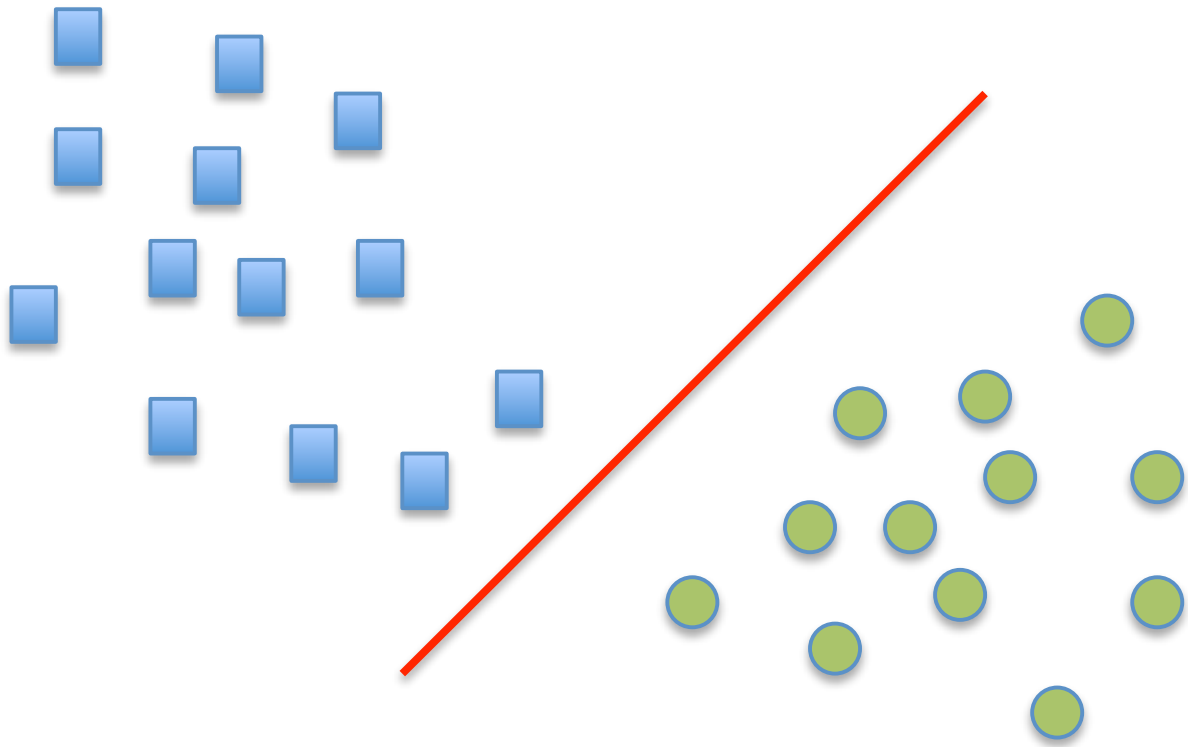

- The **linear classifier** has a **linear boundary (hyperplane)**

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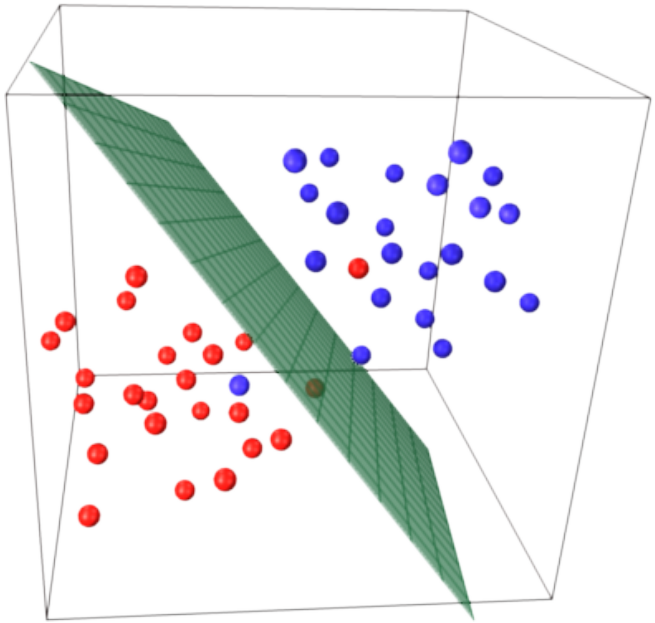
- In 2D this is a line

Example in 3D



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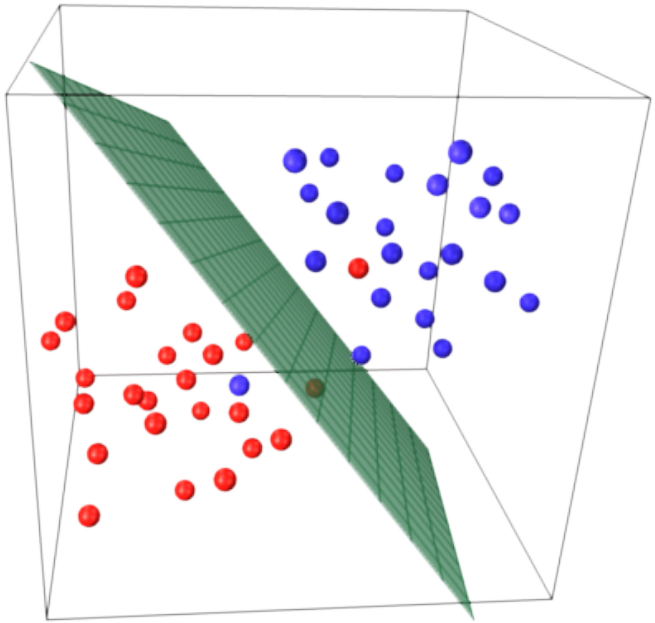

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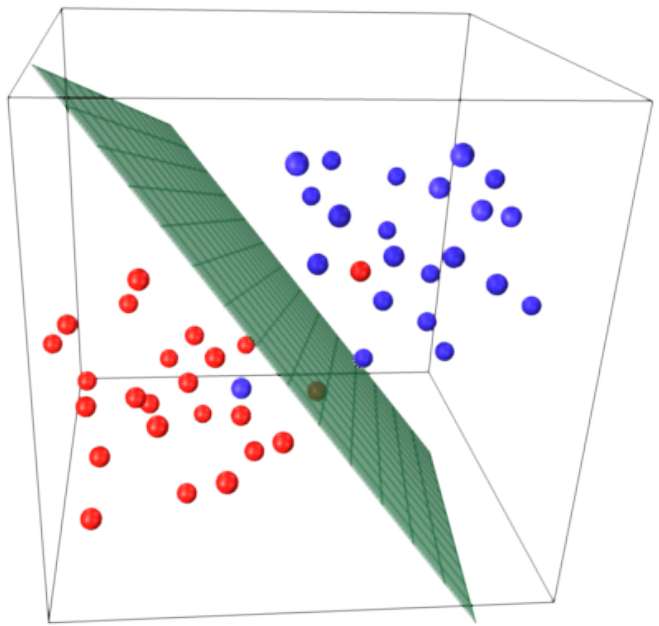

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- In 3D this is a plane
- What about higher-dimensional spaces?

Geometry

$\mathbf{w}^T\mathbf{x} = 0$ a line passing though the origin and orthogonal to $\mathbf{w}$
$\mathbf{w}^T\mathbf{x} + w_0 = 0$ shifts it by w_0

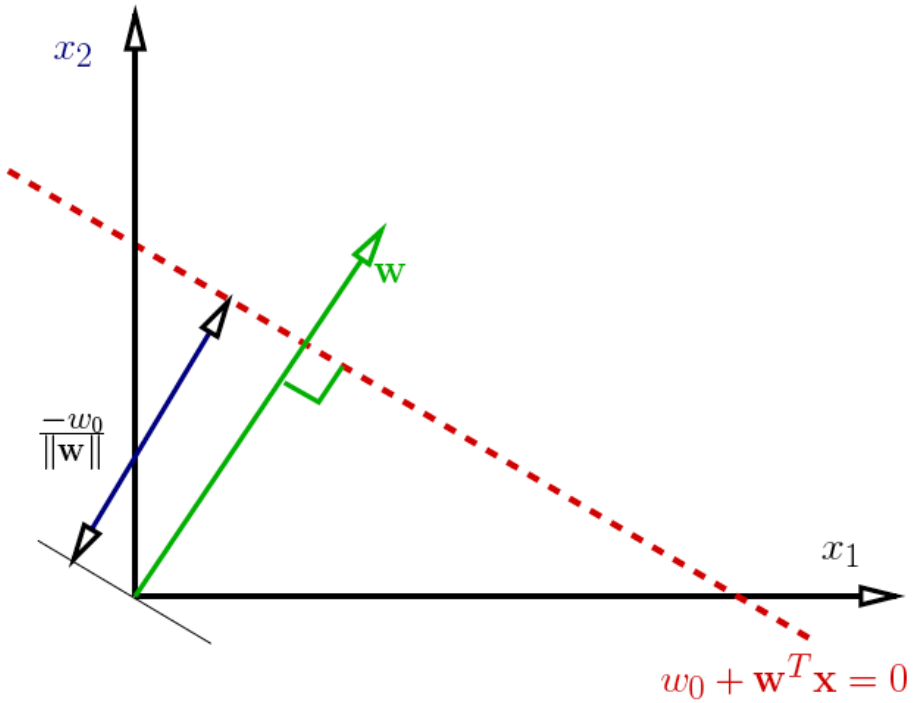

Figure from G. Shakhnarovich

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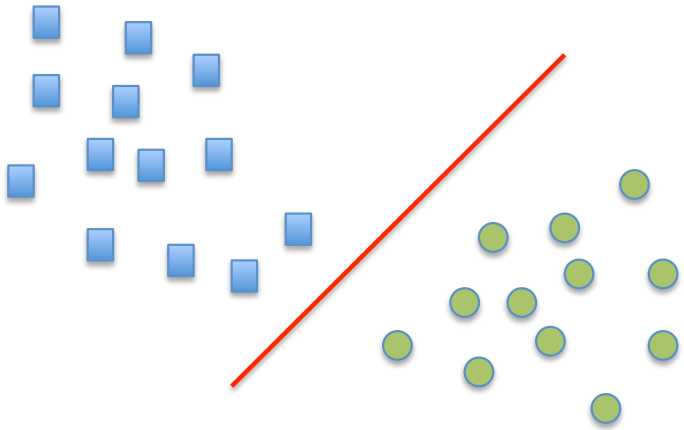

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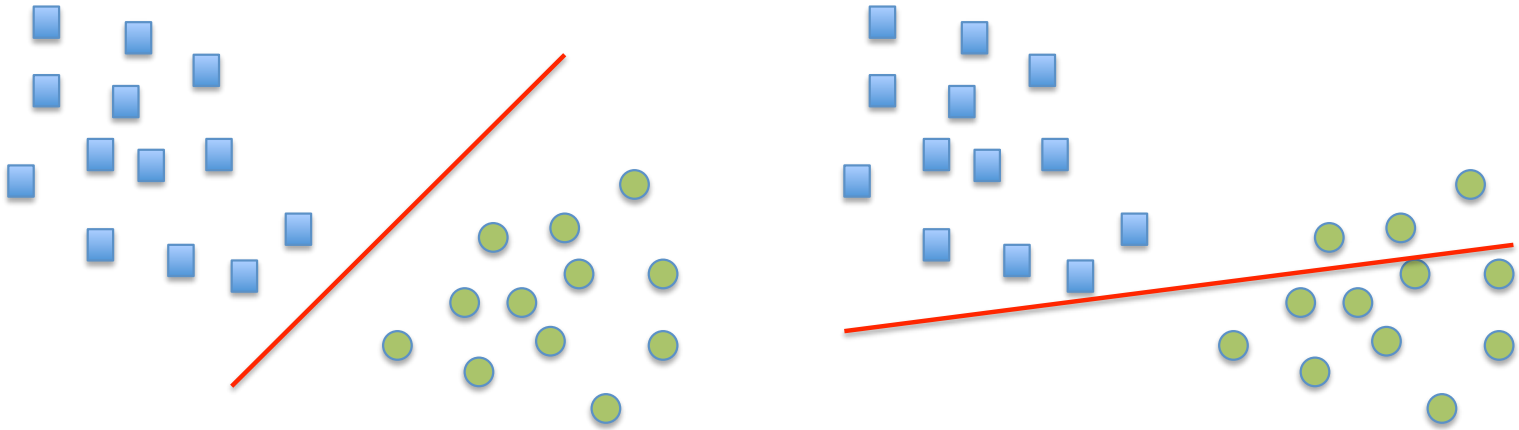

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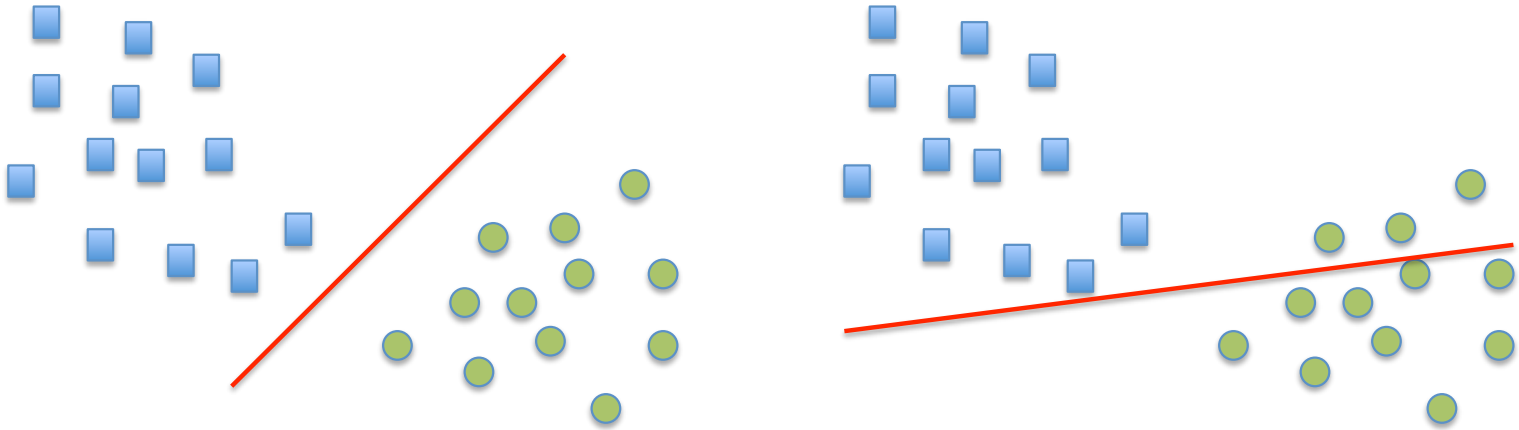

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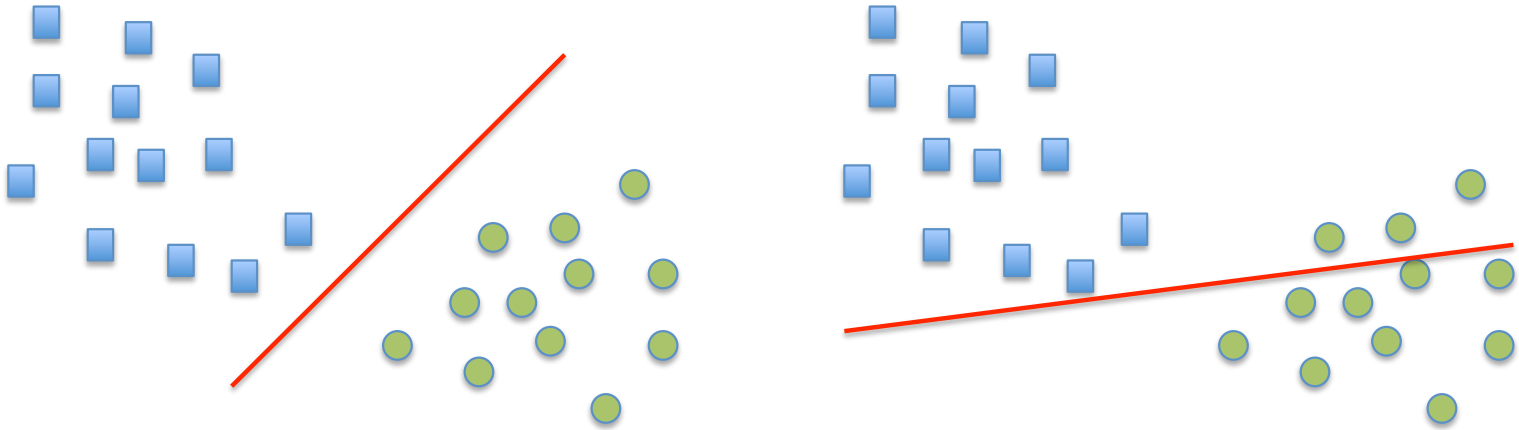

- We need a criteria that tell us how to select the parameters
- Do you know any?

Loss functions

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- For movie ratings: The "truth" is that Alice thinks E.T. is worthy of a 4. How bad is it to predict a 5? How about a 2?

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- A possible loss to minimize is the zero/one loss

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- Is this minimization easy to do? why?

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- Absolute Error

$$L_{quadratic}(y(\mathbf{x}), t) = |t - y(\mathbf{x})|$$

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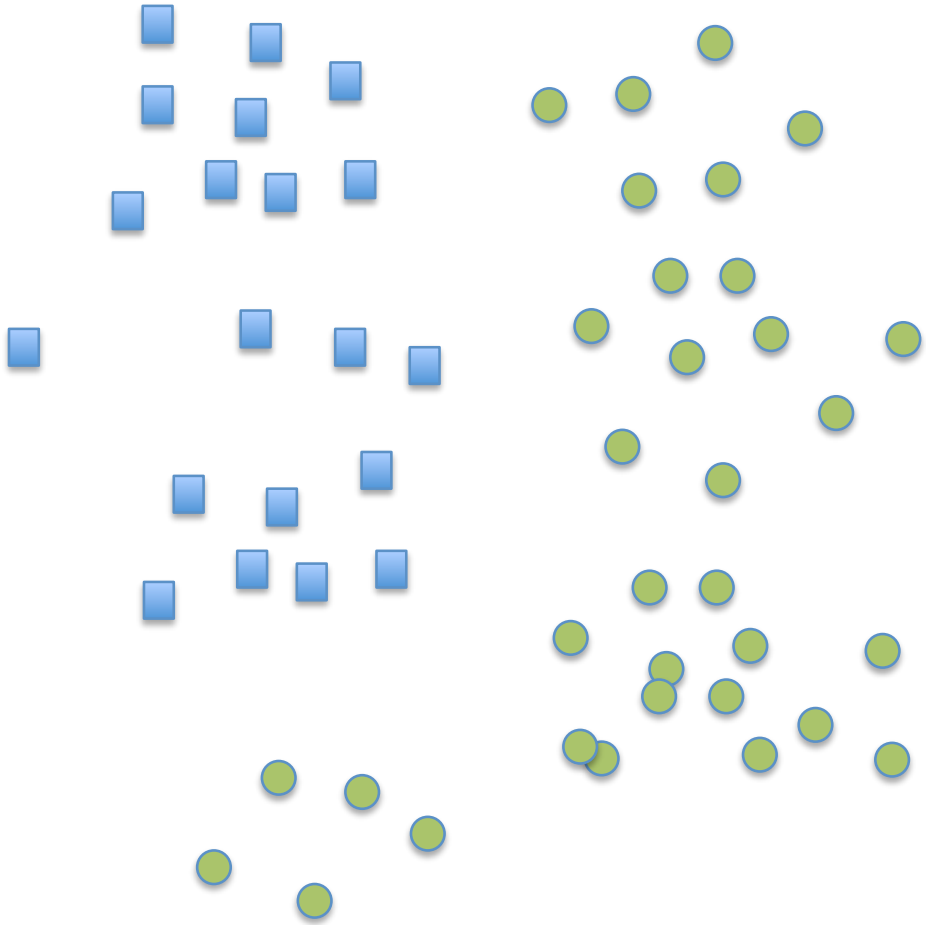
- In what order does Alice prefer E.T., Amelie and Titanic?
- Possibilities:
  - 0-1 loss on the winner
  - Permutation distance
  - Accuracy of top K movies.

Can we always separate the classes?



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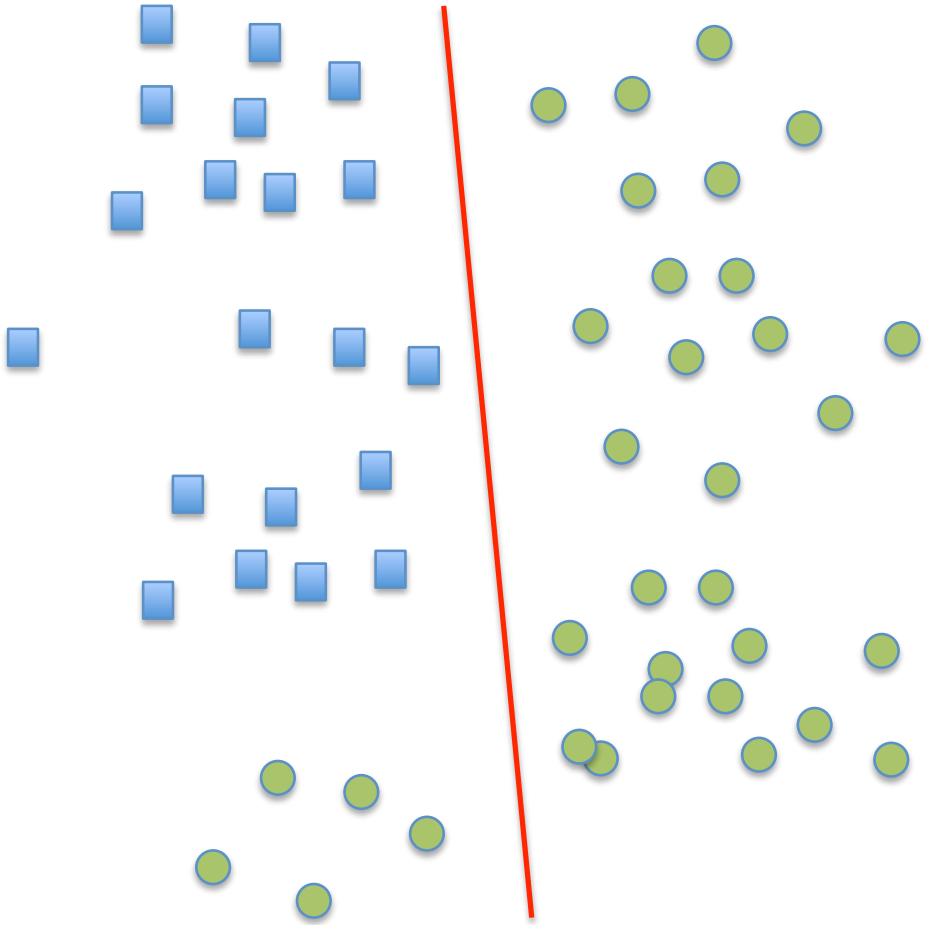

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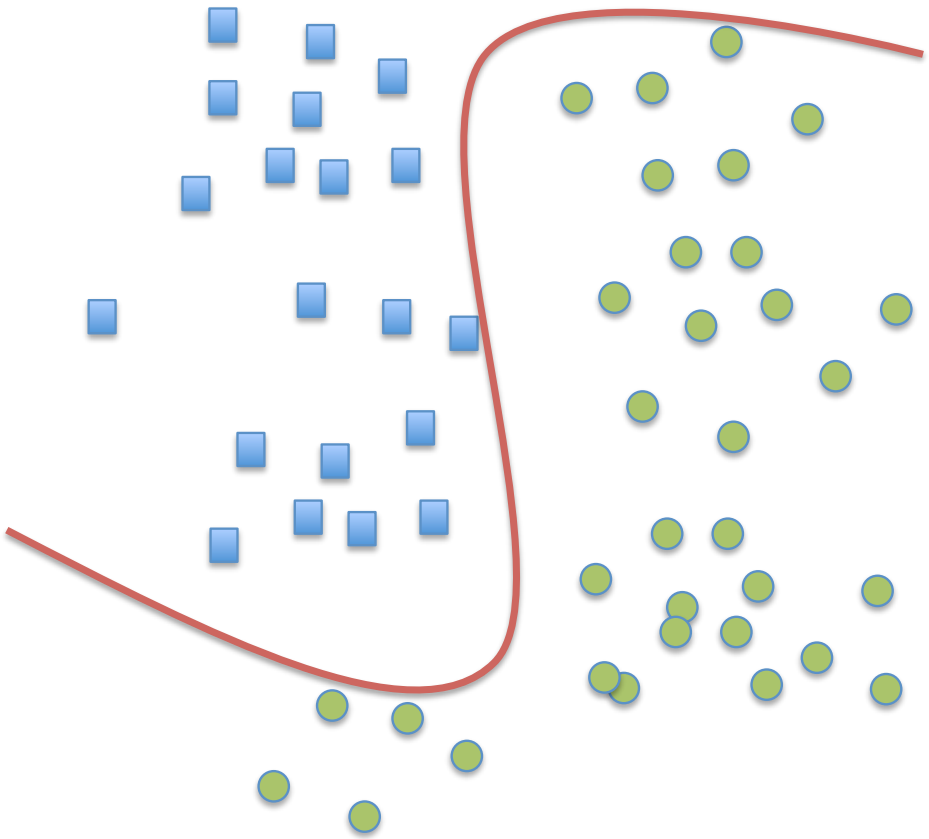

How can we obtain a non-linear decision boundary?

Can we always separate the classes?

Causes of non perfect separation

- Model is too simple
- Noise in the inputs (i.e., data attributes)
- Simple features that do not account for all variations
- Errors in data targets (miss labelings)

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Should we make the model complex enough to have perfect separation in the training data?

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- Can we have a smoother function such that things become easier to optimize?

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where the sigmoid is defined as

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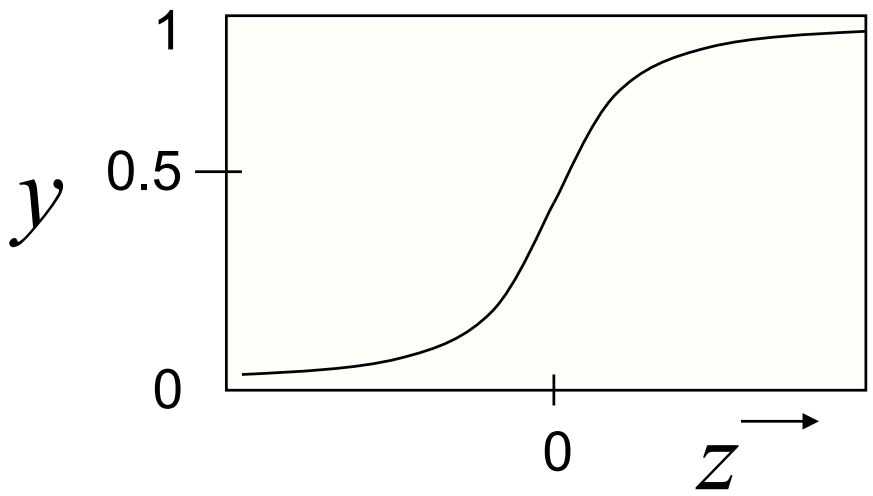

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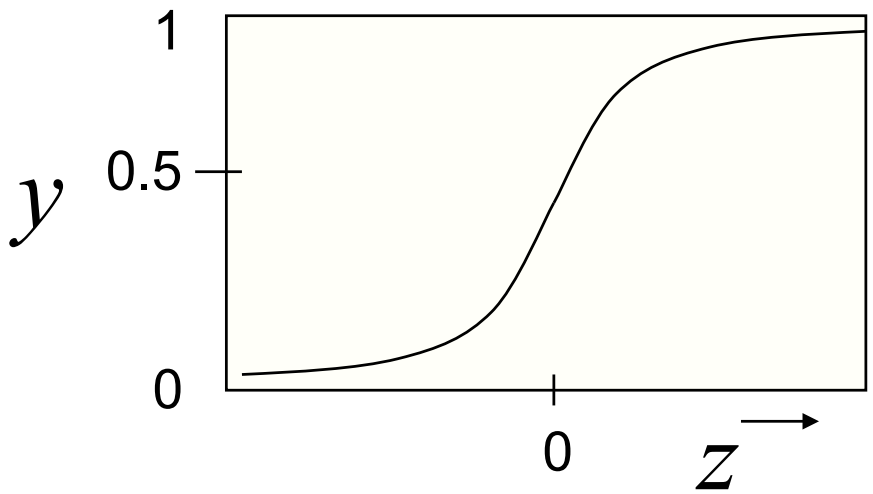

- The output is a smooth function of the inputs and the weights

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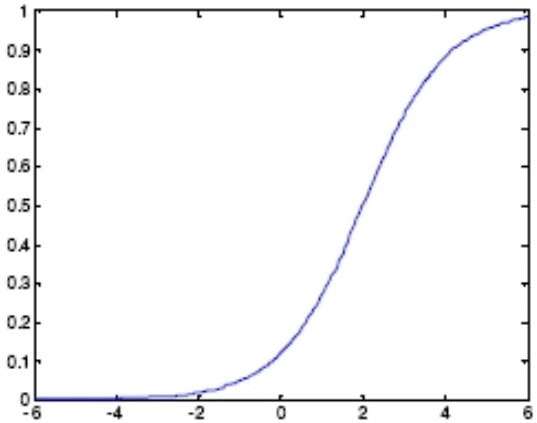
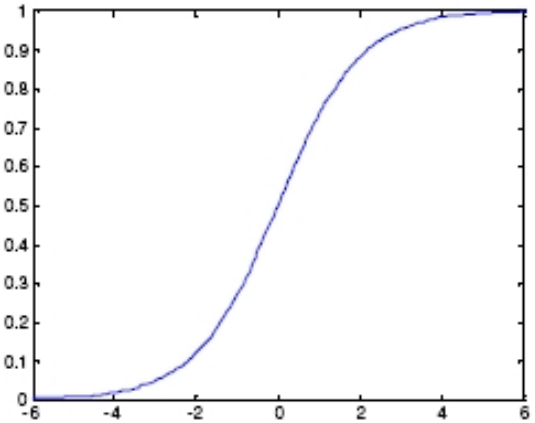
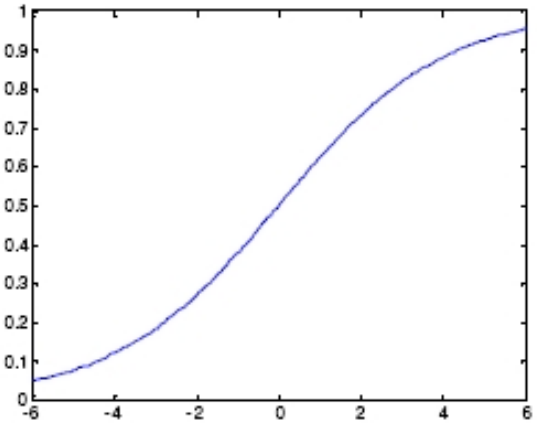


$w_0 = 0, w_1 = 1$



$w_0 = 0, w_1 = 0.5$



Show Matlab demo

Probabilistic Interpretation

- If we have a value between 0 and 1, let's use it to model the posterior

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- Thus (show matlab)

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- Convert this into a minimization so that we can write the **loss function**

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- It's a convex function of $\mathbf{w}$. Can we get the global optimum?

Gradient Descent

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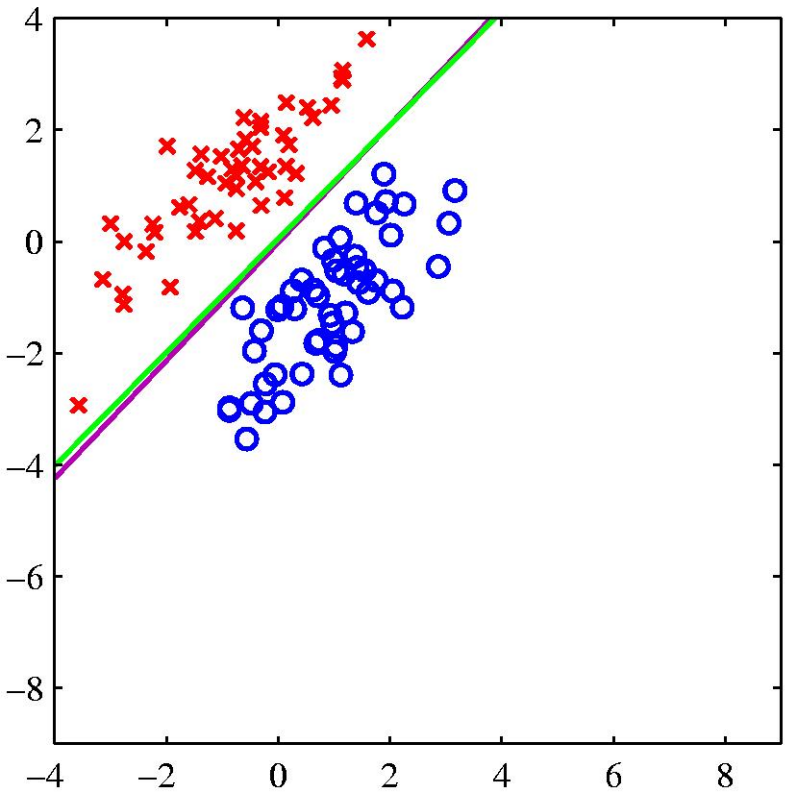
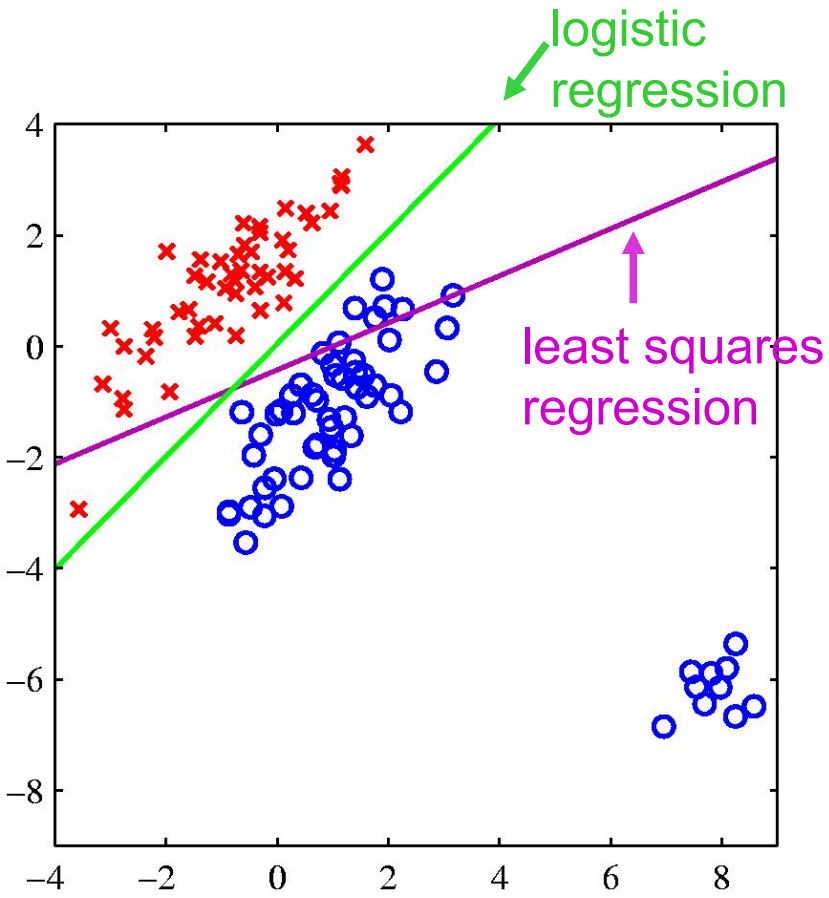
- Logistic regression has linear decision boundary

Logistic regression vs least squares





If the right answer is 1 and the model says 1.5, it loses, so it changes the boundary to avoid being "too correct" (tilts away from outliers)

33

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- The parameter λ is the importance of the regularization, and it's a **hyper parameter**

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  - The k results from the folds can then be averaged (or otherwise combined) to produce a single estimation

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- **F1 score:** harmonic mean of precision and recall

$$F1 = 2\frac{P \cdot R}{P + R}$$

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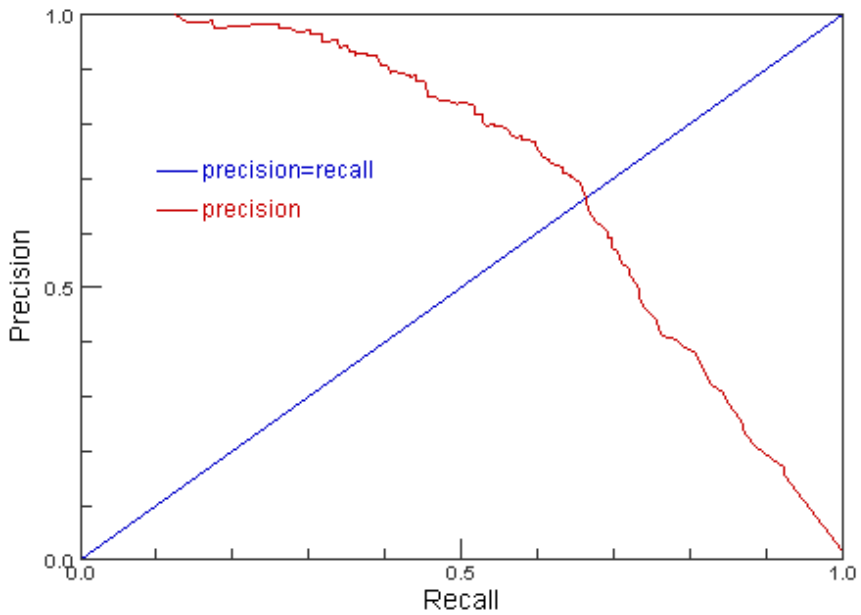

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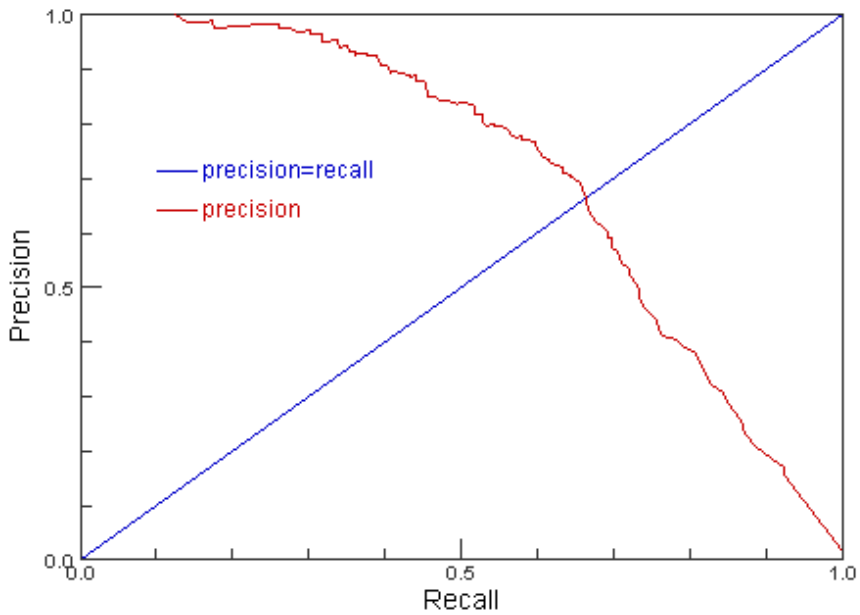
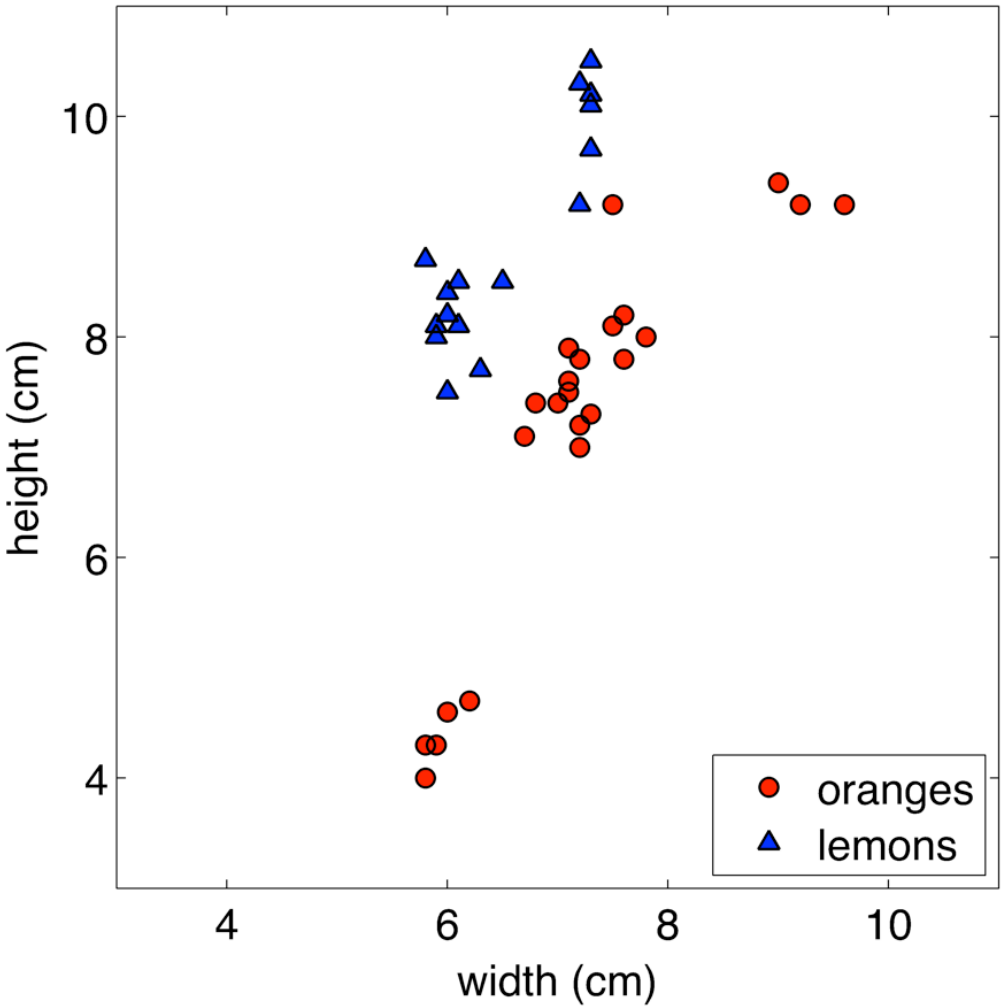
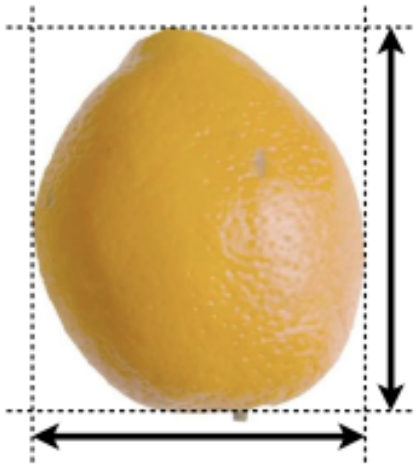


- **Average Precision (AP)**: mean under the curve

Classification: Oranges and Lemons





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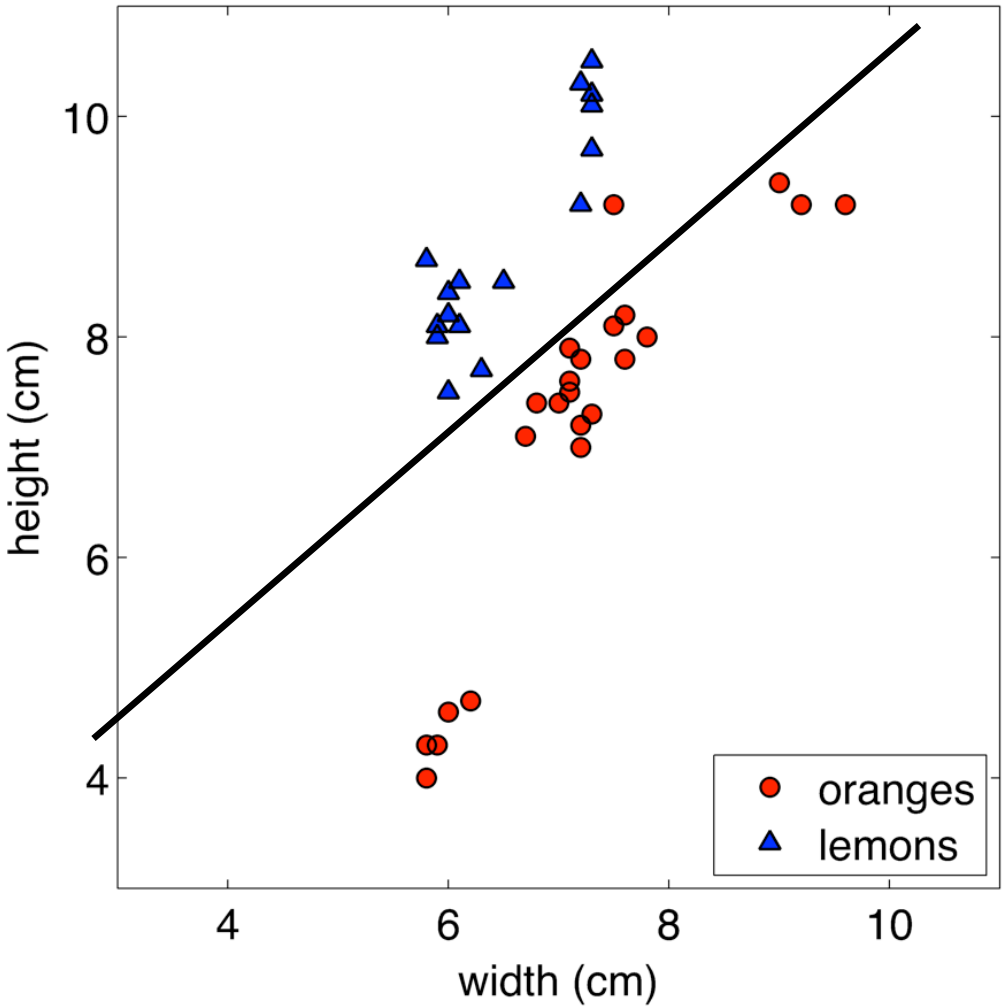
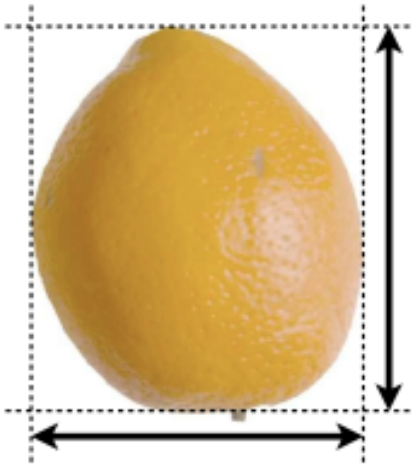


Can construct simple linear decision boundary:

$$y = \text{sign}(w_0 + w_1 x_1 + w_2 x_2)$$



What is the meaning of "linear" classification

- Linear classification means that the part that adapts is linear (just like linear regression)

$$z(x) = \mathbf{w}^T \mathbf{x} + w_0$$

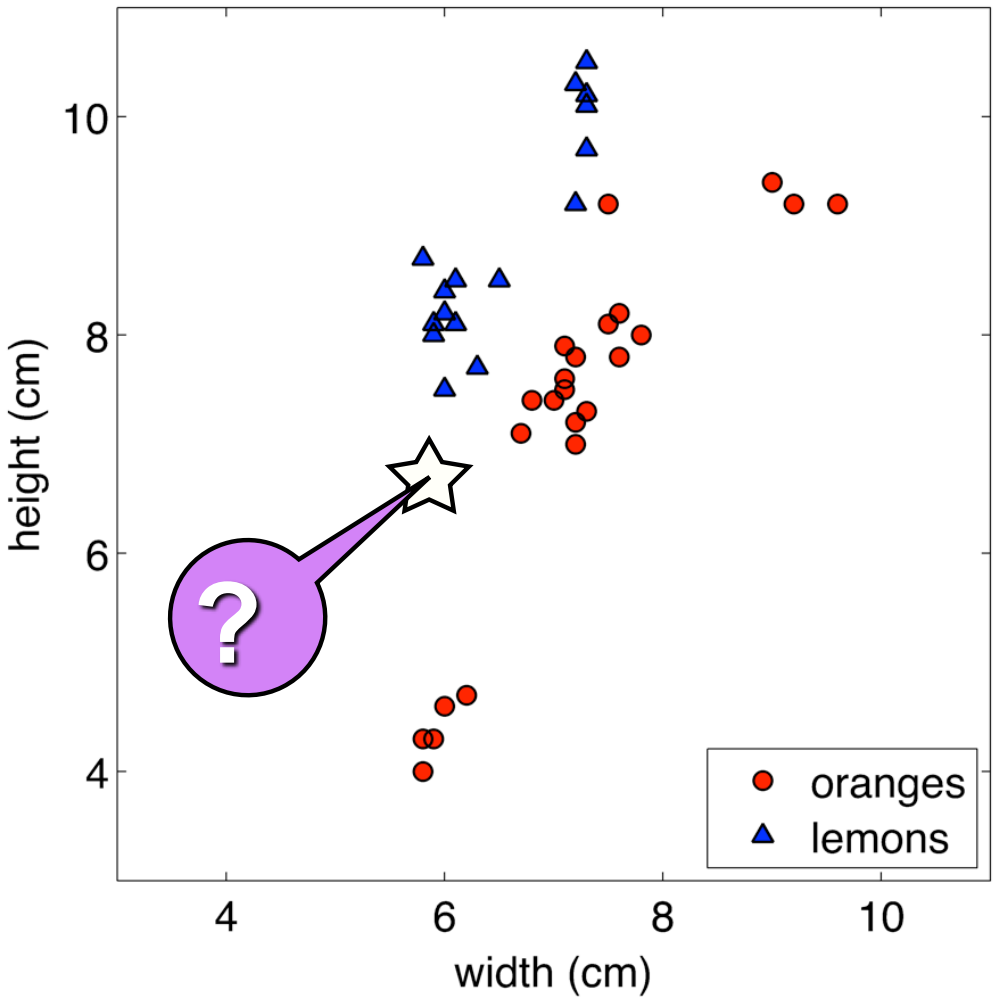
  with adaptive $\mathbf{w}, w_0$

- The adaptive part is follow by a non-linearity

$$y(\mathbf{x}) = f(z(\mathbf{x}))$$

- What f have we seen?

Classification as Induction



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- Test instances classified using similar training instances
- Embodies often sensible underlying assumptions:
  - Output varies smoothly with input
  - Data occupies sub-space of high-dimensional input space

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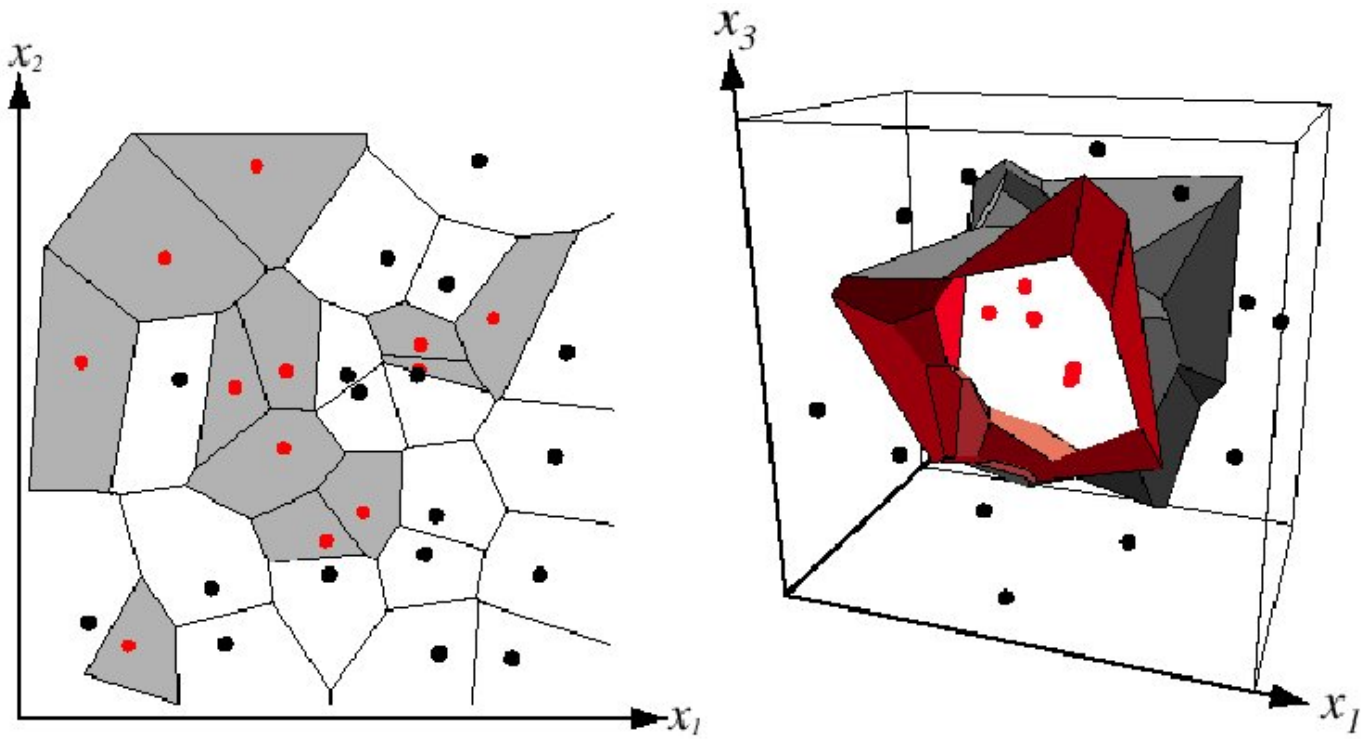
- Note: we don't need to compute the square root. Why?

Nearest Neighbors Decision Boundaries

- Nearest neighbor algorithm does not explicitly compute decision boundaries, but these can be inferred
- Decision boundaries: Voronoi diagram visualization
  - show how input space divided into classes
  - each line segment is equidistant between two points of opposite classes



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- Algorithm:
  1. find k examples $\{\mathbf{x}^{(i)}, t^{(i)}\}$ closest to the test instance $\mathbf{x}$
  2. classification output is majority class

$$y = arg \max_{t^{(z)}} \sum_{r=1}^{k} \delta(t^{(z)}, t^{(r)})$$

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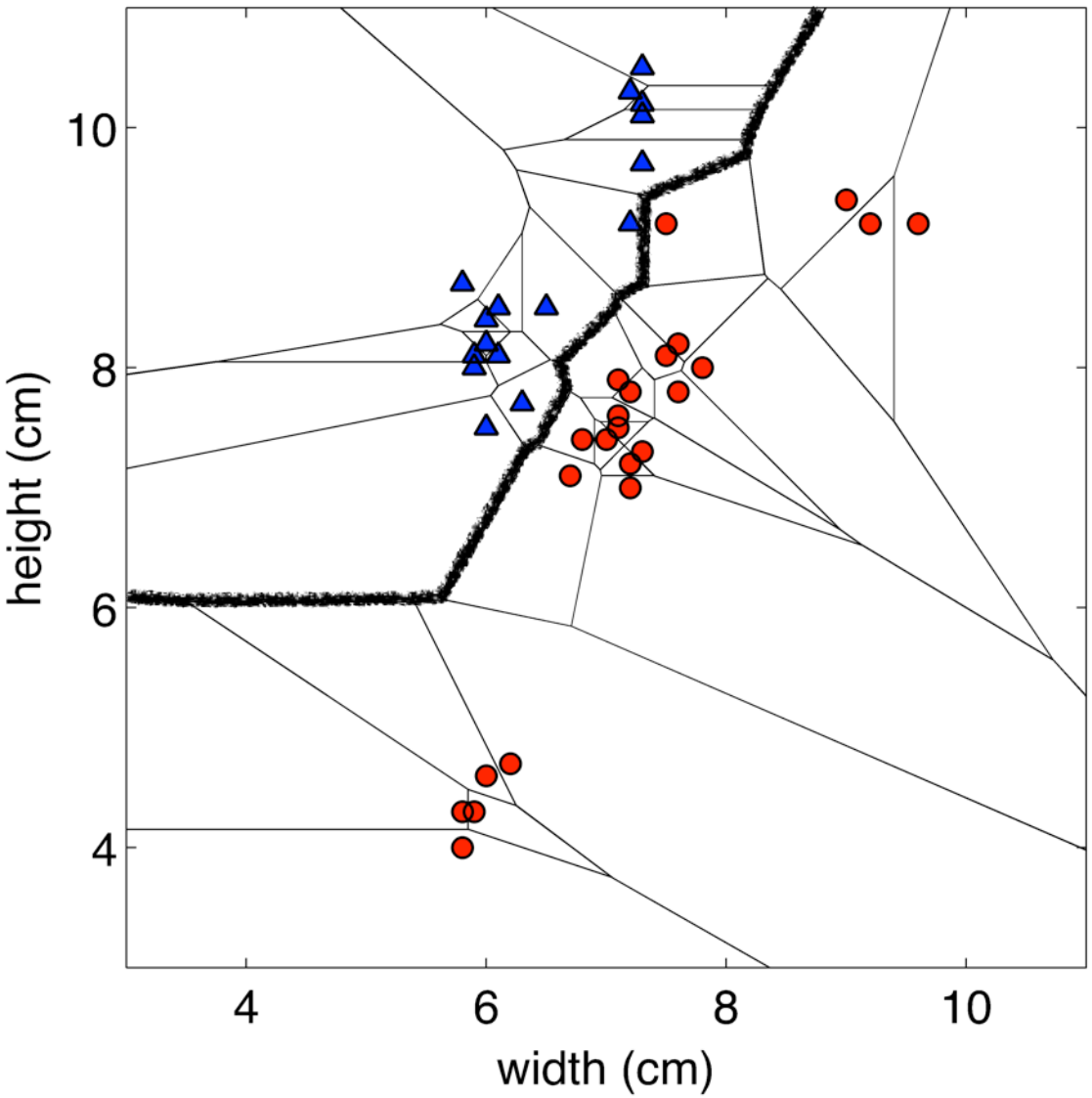
     - Form efficient search tree (kd-tree), use Hashing (LSH), etc

Decision Boundary K-NN



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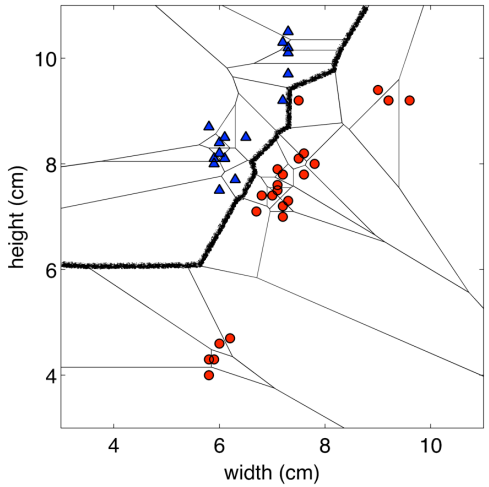

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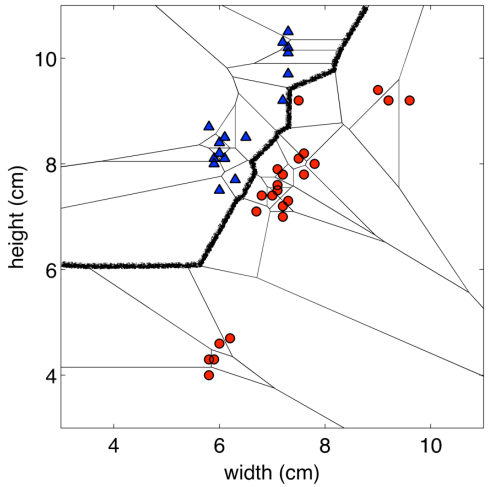

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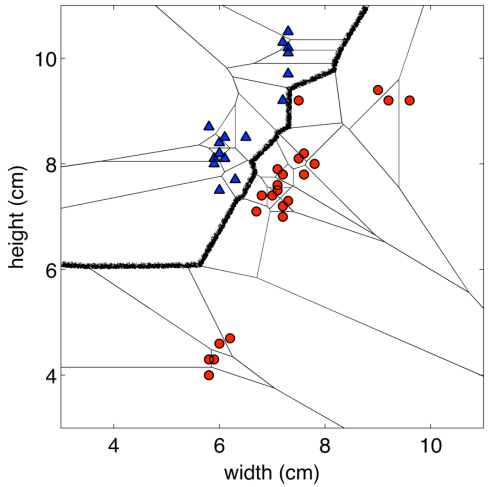

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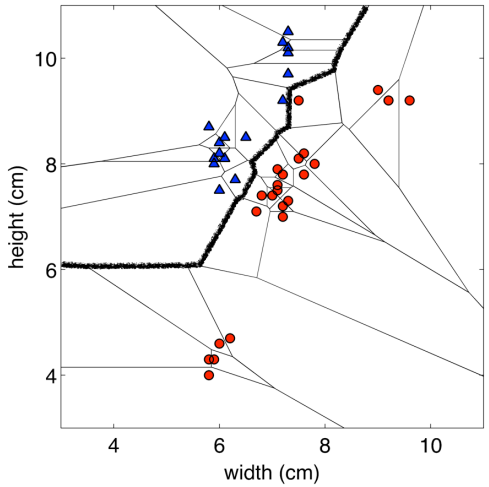

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- Inductive Bias: What kind of decision boundaries do we expect to find?