University of Toronto, St. George Campus
Department of Computer Science

CSC 236H
Winter 2007

Assignment #2

Due date: February 14th at 10:00AM

1. One possible recursive definition of a^n is:

$$\begin{aligned} a^1 &= a \\ a^{n+1} &= a^n \cdot a \end{aligned}$$

   Use induction to prove that

   (a) $a^{n+m} = a^n \cdot a^m$

   (b) $(a^n)^m = a^{nm}$

2. (a) Find a closed formula for $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n\cdot(n+1)}$. Please show the work you did in order to determine the formula.

   (b) Using induction, prove that your formula actually works.

3. (a) Prove that for any $n \geq 1$, for any sequence of positive real numbers $a_1, a_2, \ldots, a_n$, and for any sequence $b_1, b_2, \ldots, b_n$ that is a permutation of the a_is, then

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n} \geq n$$

   (b) Prove that $\frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n} = n$ if and only if $b_i = a_i$ for every i.

4. Consider the set of all binary strings S. One possible recursive definition of this set would be

   - The empty string, Λ, is in S
   - For every $s \in S$ and every $x \in \{0, 1\}$, $sx \in S$
   - No other elements are in S

   Two functions used quite often on strings are the length function and the reverse function. A recursive definition of the length function would be

   - $|\Lambda| = 0$
   - For any $s \in S$ and $x \in \{0, 1\}$, $|sx| = |s| + 1$

   A recursive definition of the reverse function would be

   - $rev(\Lambda) = \Lambda$
   - For any $s \in S$ and $x \in \{0, 1\}$, $rev(sx) = xrev(s)$

   (a) Prove that for any strings $x, y \in S$, $|xy| = |x| + |y|$

   (b) Prove that for any string $x \in S$, $|rev(x)| = |x|$

(c) Prove that for any strings $x, y \in S$, $rev(xy) = rev(y)rev(x)$

5. Given a binary tree T, the size of the tree T is the number of nodes in T, and height of T is the length of the longest path from the root of T to a leaf node. (By convention, we will say that the height of the empty tree is -1, the height of a tree that consists of just one node is 0, and so on.) Denote the size of T by $s(T)$, and denote the height of T by $h(T)$.

   Prove that for any binary tree T, $s(T) \leq 2^{h(T)+1} - 1$.