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A DEFICIENCY OF NATURAL DEDUCTION

J.G. WILTINK

Department of Mathematics and Computing Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

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Natural languages are ill-suited to express mathematical reasoning. The purpose of an artificial language—predicate logic—with a deduction system is to enable us to express reasoning clearly and concisely. In this article we give two examples illustrating how Gentzen's system of 'natural deduction' [2] fails to meet the case.

In [1], the following problem is stated and solved:

> Given two caskets, gold and silver, one of which contains a portrait of a lady. Both caskets bear an inscription: on the gold casket is written
>
> "The portrait is not in here",
>
> and on the silver one
>
> "Exactly one of these inscriptions is true".
>
> Which of the two caskets contains the portrait?

The problem is solved by proving that the portrait is in the gold casket. The proof is conducted in the formal system of natural deduction and comprises 29 steps.

Here is another formal proof: let G and S be the inscriptions on the gold and silver caskets, respectively. Then we write S formally as

$S \equiv \neg G$

and derive

$= \text{true}$

$= \{\text{definition of S}\}$

$S \equiv (S \equiv \neg G)$

$\{\text{associativity of } \equiv\}$

$(S \equiv S) \equiv \neg G$

$= \{\text{reflexivity of } \equiv\}$

$\text{true} \equiv \neg G$

$= \{\text{identity element of } \equiv\}$

$\neg G$

$= \{\text{definition of G}\}$

"The portrait is in the gold casket".

This proof is 5 steps long.

Although the proof in [1] is a bit longer than necessary—the present author knows of a proof of 21 steps—, the difference in length between the two proofs illustrates a defect of Gentzen's system: it does not handle equivalence efficiently. Equivalence of A and B can only be expressed by something like

$(A \Rightarrow B) \wedge (B \Rightarrow A)$

or

$(A \wedge B) \vee (\neg A \wedge \neg B);$

since A and B occur twice, these formulae are twice as long as desirable. In case of more than one equivalence, this leads to exponential growth of the formulae: for example, the approximately shortest way to describe the predicate S is to postulate

$$(S \Rightarrow (S \wedge \neg G) \vee (\neg S \wedge G)) \\ \wedge ((S \wedge \neg G) \vee (\neg S \wedge G) \Rightarrow S).$$

Furthermore, different formulations of S give rise to different proofs; which formulation should we choose?

One might argue that predicates involving more than one equivalence—like $S \equiv (S \equiv \neg G)$ in our proofs—are rather uncommon, and that hence the objection against Gentzen's system is a bit far-fetched. Let us therefore give another example: how do we formulate and prove in Gentzen's style that $A \equiv B$ implies $A \vee C \equiv B \vee C$? The shortest formulation the present author can think of is

$$(A \Rightarrow B) \wedge (B \Rightarrow A) \\ \Rightarrow (A \vee C \Rightarrow B \vee C) \wedge (B \vee C \Rightarrow A \vee C);$$

its proof comprises at least 16 steps, whereas a simple appeal to the rule of Leibniz—for function f, $(x \equiv y) \Rightarrow (f.x \equiv f.y)$—gives

$$(A \equiv B) \Rightarrow (A \vee C \equiv B \vee C)$$

in one step. A shrill contrast.

Gentzen states in [2] that one of the advantages of his system is "The close affinity to the actual reasoning (...). Hence, the calculus lends itself in particular to the formalisation of mathematical proofs."

Since natural languages are ill-suited to express mathematical reasoning, such an affinity to 'actual reasoning—in a natural language' is rather a disadvantage of a deduction system: our two examples at least show that the use of the system of natural deduction may lead to rather inefficient proofs.

The problem in [1] is stated under the title "Natural Deduction?", which could be read as "Is Natural Deduction suited to express mathematical reasoning?". If efficiency of proofs is of any interest, the answer to that question is clearly "Not in its present form".

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