Qualitative Spatio-Temporal Representation and Reasoning:

Trends and Future Directions

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Chapter 1
Region-Based Theories of Space:
Mereotopology and Beyond

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ABSTRACT

This chapter focuses on the topological and mereological relations, contact, and parthood, between spatio-temporal regions as axiomatized in so-called mereotopologies. Despite, or because of, their simplicity, a variety of different first-order axiomatizations have been proposed. This chapter discusses their underlying ontological choices and different ways of systematically looking at them. The chapter further gives an overview of the algebraic, topological, and graph-theoretic representations of mereotopological models which help to better understand the model-theoretic consequences of the various ontological choices. While much work on mereotopologies has been primarily theoretical, the focus started shifting towards applications and domain-specific extensions of mereotopology. These aspects will most likely guide the future direction of the field: How can mereotopologies be extended or otherwise adjusted to better suit practical needs? Moreover, the integration of mereotopology into more comprehensive and maybe more pragmatic ontologies of space and time remains another challenge in the field of region-based space.

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1. INTRODUCTION

The very nature of topology and its close relation to how humans perceive space and time make mereotopology an indispensable part of any comprehensive framework for Qualitative Spatial and Temporal Reasoning (QSTR). Within QSTR, it has by far the longest history, dating back to descriptions of phenomenological processes in nature (Husserl, 1913; Whitehead, 1920, 1929)—what we call today 'commonsensical' in Artificial Intelligence. There have been plenty of other motivations to study the topological and mereological relations of space—as an appealing alternative to set theory or point-set topology, or as an region-based alternative to Euclidean geometry. Even beyond QSTR, mereotopology is fairly universal and can be applied to various other fields, where the spatial or temporal character is not its primary purpose.

Mereotopology also often serves for testing and exploring techniques of building qualitative spatial reasoning frameworks. Likewise, central issues of knowledge representation can be tested within it—building reusable, generic ontologies, constructing *upper ontologies*, testing specification and validation of formal semantics for ontologies and not least, coming up with general mathematical frameworks to systematically compare ontologies model-theoretically or axiomatically.

Without doubt, we can say that within QSTR mereotopology encompasses some of the most advanced and best understood spatial theories. This chapter gives a high-level overview of the early and more recent advances in the field, what mathematical tools and techniques are successful, how the theories vary, and what are the challenges remaining within and beyond the field.

The quest for region-based theories of space as alternatives to classical point-based geometry is often driven by the human cognition: how humans perceive their spatial and spatio-temporal environment. An urge for common-sense representations and reasoning systems is given by work on naïve physics (Hayes, 1978; Hayes, 1985b; Smith & Casati, 1994) and naïve geography (Egenhofer & Mark, 1995b). Theories of space and time will be a major component of any common-sense representation of geographic or physical space. For common-sense reasoning, region-based theories of space (and time) are more promising than point-based theories since they are able to draw commonsensical topological or mereological conclusions even in the absence of exact data, or as Egenhofer and Mark (1995b, p. 9) put it: "topology matters, metric refines." Not least, the study of region-based theories contributes towards the understanding of the nature of points—they actually have structure that is not evident in Euclidean geometry (Eschenbach, 1994). Through the inclusion of qualitative models of space, common-sense reasoning but also the next generation of Geographic Information Systems (GIS) and other spatial reasoning software can at least partially bridge the gap between rigid computational models of space and less rigid users that freely navigate between quantitative and qualitative and between low-level and high-level conceptions of space.

1.1. Scope and Structure

Upfront a few words on the scope of this chapter. There have been a few overviews of mereotopology in the context of qualitative spatial reasoning (Bennett, 1997; Casati & Varzi, 1999; Cohn & Hazarika, 2001; Cohn & Renz, 2008; Cohn & Varzi, 2003; Donnelly, 2001; Eschenbach, 2007; Vieu, 1997). We do not simply aim to extend these summaries with more recent work, but we hope to compile a more comprehensive account of mereotopology paying respect to the whole breadth of the field. Overviews covering mereotopology just as one amongst many qualitative spatial reasoning frameworks (Cohn & Hazarika, 2001; Cohn & Renz, 2008) have been unable to allocate sufficient space to cover the many different approaches towards mereotopology. More technical accounts (Casati & Varzi, 1999; Cohn & Varzi, 2003; Eschenbach, 2007) compare axioms and assumptions of different theories, but tend to lose the big picture. Moreover, these accounts focus on a few select theories, leaving out others that do not fit well into the respective frameworks. In particular, work on algebraic theories of region-based space, systematically covered by Vakarelov (2007) as well as topological models and theories are largely left aside. The same defect holds for the summaries given in (Bennett, 1997; Donnelly, 2001; Vieu, 1997). Furthermore, there is often a certain bias towards a particular class of region-based theories: either considering exclusively Whiteheadean space, only RCC-style axiomatizations, only relation calculi (which we admittedly only briefly cover), or only algebraic approaches. We shall attempt here to give a synthesis of the different mathematical formulations used to axiomatize region-based space. We try to guide through the various mereotopologies, their ontological choices, and their mathematical representations. Throughout, parallels between the approaches (despite their differences in the choice of mathematical tools) and their historic development are sketched out. The picture we give relies solely on known results from the literature, however they tend to be spread in various publications and sometimes well hidden by the technical details. We hope the chapter to be accessible to a broad audience, but occasionally we point out technical results of interest to the versed reader in the particular context. However, readers new to the field can safely skip these results without losing the big picture. In general, our presentations are fairly high-level and refrain from using axioms unless absolutely necessary.

After giving a bit of background on topology and mereology, Section 3 constitutes the first of two parts of the chapter. It introduces the wealth of ontological commitments relevant for mereotopology. We follow along the lines of the commitments discussed by Eschenbach (2007) and give examples how the many different mereotopological theories fit into this space of potential mereotopologies. After reviewing some known systematic frameworks of mereotopologies (Section 4), we proceed to the second part of the chapter which presents different mathematical accounts—logical, algebraic, and topological—of mereotopology. In Section 5, three main families of logical theories of mereotopologies are presented: Whiteheadean theories, boundary-tolerant theories, and mereogeometries. Within each family the ontological commitments vary only marginally; they guide the exploration of the space of logical accounts of mereotopologies while commitments manifested in each of the families are pointed out. It turns out that the ontological commitments of Whiteheadean theories have been studied most widely; most other theories lack a comprehensive analysis of ontological commitments. Often it is not trivial to extract the ontological decisions implicit in a particular region-based theory of space. For Whiteheadean space, their algebraic counterparts, contact algebras, treated in Section 6 are well-suited for mapping out the space of possible theories. In general, we believe that ontological commitments of other theories would become more obvious if we study their algebraic representations. Apart from contact algebras, very little work has been done on algebraic representations of logical theories of mereotopologies. The topological models and embeddings in Section 7 once more show the interrelation between the various mereotopologies. As comparison, we review region-based theories directly built from point-set topology. Not surprisingly, it turns out that despite their different mathematical foundations, the resulting region-based theories of space are remarkably close to the axiomatic and algebraic theories. The comparison with the topological models of the latter shows the parallels clearly.

Finally, we point to some work on mereotopology and mereogeometry used in specific application domains. Some specific applications are presented, while for other areas only the necessary expressivity with respect to the ontological commitments is discussed. To complete the chapter, we highlight points that we consider most important for future research in the field. Theoretical and practical challenges within the field as well as challenges in the broad scope of spatial and spatio-temporal qualitative ontologies are briefly discussed.

Of course, we cannot cover the complete literature in the area, but we hope to give the interested reader sufficient starting points where to continue in-depth reading. We see this chapter as a reader's guide to mereotopology accompanied by a comprehensive bibliography useful to start further reading. The chapter shall give a general AI-centered overview of representations of region-based space and its current state and future work. The field of mereotopology and its formal treatise seems to be clustered into separate fields with little awareness and interaction between them. We hope that this chapter helps reconnect the different views on region-based space. Additionally, we hope to point more experienced researchers to some related work, discovering alternative perspectives on methods, tools, and applications of mereotopology and mereogeometry.

2. BACKGROUND

This section prepares the reader for the material in this chapter. Specifically, we briefly cover relevant background on point-set topology and mereology, before giving a short introduction to mereotopology and how it is intrinsically linked to both. This is more of a historical introduction to mereotopology; a more axiomatic overview will be given in Section 5.1. The reader familiar with topology and mereology can easily skip this section, for those who need more background references are given. Although we will make use of lattice theory in Section 6, only little lattice-theoretic background is assumed; it can be found in standard references (Birkhoff, 1967; Grätzer, 1998).

2.1. Point-Set Topology

Point-set topology (from Greek *topos*, "place") is traditionally based on set theory. Open and closed sets of points are distinguished and standard set intersection and union are assumed. A *topological space* (X,τ) can be defined over a set of *open* (or *closed*) sets τ (the topology) where $\varnothing, X \in \tau$ and every set in τ is a subset of X. Moreover, τ is closed under arbitrary unions (finite unions) and finite intersections (arbitrary intersections). If A is an open set, $(X \setminus A)$ is a closed set. If no confusion can arise, the topological space is identified by its base set X. In a topological space the sets $\varnothing, X$ are *clopen* sets, i.e. they are open and closed. In general, there are sets that are neither open nor closed. The interior int(x) of a set x is the union of all open sets contained in it, which is necessarily open (the union and the intersection of open sets is open). Equally, the closure cl(x) of a set x is the intersection of all closed sets that contain the set. In other words, the interior of A is the largest open set contained in A, while the closure of A is the smallest closed set containing A.

The study of topological spaces of regular sets is a distinctive feature of point-free topology, which in traditional topology is not of much relevance. An open set x is called *regular open* iff $x = \mathrm{int}(\mathrm{cl}(x))$ and a closed set x is called regular closed iff $x = \mathrm{cl}(\mathrm{int}(x))$. The set complement of a regular open set is regular closed. It is well known that the regular open sets form a Boolean algebra under the operations $x + y := \mathrm{int}(\mathrm{cl}(x \cup y))$, $x \cdot y := x \cap y$, and $-x := \mathrm{int}(X \setminus x)$, see (Halmos, 1963). Algebras over regular sets have been defined by McKinsey and Tarski (1944). They used the term *closure algebra* (dually *interior algebra*) for a Boolean algebra equipped with a closure operation cl satisfying $\mathrm{cl}(x) \geq x$, $\mathrm{cl}(\mathrm{cl}(x)) = \mathrm{cl}(x)$, $\mathrm{cl}(x) + \mathrm{cl}(y) = \mathrm{cl}(x + y)$, and $\mathrm{cl}(0) = 0$. Such closure algebra can be constructed from a topological space: Let (X,τ) be a topological space, then $(2^{\mathrm{X}}, \mathrm{cl})$ is a closure algebra (McKinsey & Tarski, 1944). For more background on point-set topology, we invite the reader to consult standard references (Engelking, 1977; Munkres, 2000). Specific topological concepts, in particular separation axioms, are introduced in Section 7 as needed.

2.2. Mereology

The origins of mereology (from Greek *méros*, "part") date back to the beginning of the 20th century and the work of Husserl (1901). It received formal treatment by Leśniewski (1931), an English translation appeared in Luschei (1962). Leśniewski is credited with the first development of an extensional part-whole theory, soon Leonard and Goodman (1940) followed with an alternative. We only introduce key concepts that help enable the reader to understand the role mereology plays within mereotopology. For further reading, we refer to Simons (1987) and Varzi (2009). Additionally, Casati and Varzi (1999) give a systematic overview of mereologies deemed relevant for mereotopology.

Common to all mereological theories is a primitive binary relation of *parthood* relating parts to wholes (of which they are part of). Parthood is an anti-symmetric relation that is either reflexive or irreflexive (*proper parthood*). Such basic Mereology (**M**) can be axiomatized as following. We include the definition of proper parthood.

(P1) $\forall x\left[P(x,x)\right]$ (Reflexivity)

(P2) $\forall x,y\left[\left(P(x,y)\wedge P(y,x)\right)\rightarrow x=y\right]$
(Anti-symmetry)

(P3) $\forall x,y\left[\left(P(x,y)\wedge P(y,z)\right)\rightarrow P(x,z)\right]$
(Transitivity)

(P4) $PP(x,y)\equiv_{def} P(x,y)\wedge\neg P(y,x)$
(Proper parthood)

While parthood is usually the only primitive, overlap is the next most important concept. Further relations and operations such as union (or fusion), and intersection are also definable in terms of parthood alone.

(P-O) $O(x,y)\equiv_{def} \exists z\left(P(z,x)\wedge P(z,y)\right)$
(Overlap)

To build extensional mereology (**EM**), strong supplementation is required.

(P5) $\forall x,y\left[\neg P(y,x)\rightarrow \exists z\left(P(z,y)\wedge\neg O(z,x)\right)\right]$
(Strong supplementation)

In the presence of this axiom, the proper parthood relation is rendered extensional, meaning that two distinct entities differ in at least one part.

(P6) $\forall x,y\left[\forall z\left(O(z,x)\leftrightarrow O(z,y)\right)\rightarrow x=y\right]$
(Extensionality of O)

Assuming weak supplementation alone does not renders proper parthood extensional.

(P5')$\forall x,y\left[PP(x,y)\rightarrow \exists z\left(P(z,y)\wedge\neg O(z,x)\right)\right]$
(Weak supplementation)

A further restriction of **EM** to Closure Extensional Mereology (**CEM**) assumes the existence of sums and intersections, the latter conditional on overlap. These closure principles lead in the presence of mereological extensionality to unique sums and intersections.

(P7) $\forall x,y\left[\exists z\forall u\left(O(u,z)\leftrightarrow O(u,x)\vee O(u,y)\right)\right]$
(Sum $z=x+y$)

(P8)
$\forall x, y\big[O(x,y) \rightarrow \exists z \forall u \big(P(u,z) \leftrightarrow P(u,x) \wedge P(u,y)\big)\big]$
(Intersection $z = x \cdot y$)

Opinions differ on whether mereological theories may consist of *atoms*, i.e. individuals without proper parts. This issue is discussed for mereotopologies in Section 3.6.

It is well-known that mereology can be built from algebraic structures. The close relationship between Boolean algebras and mereological structures was first pointed out by Leonard and Goodman (1940), while Grzegorczyk (1955, 1960) coined the term *mereological field*—a complete Boolean algebra with the zero element removed (a *quasi-Boolean algebra*). Grzegorczyk also stressed that the close relationship between mereology and Boolean algebras can be readily exploited for a better understanding of mereology. Extending this argument to mereotopology, we will see in Section 6 how the mereological component of mereotopologies can be regarded as an algebraic structure.

2.3. Origins of Mereotopology

Mereotopology is not only by name intrinsically linked to mereology and topology. From mereology, it inherits the desire to talk about parthood relations amongst entities, while it also aims to capture topological relations between entities. Originally, it has been proposed as a point-free alternative to standard point-set topology which is criticized for countering the human conception of space. Set-theoretic notions are often believed to be at the root of the problem, being an overly complicated abstraction when we deal with commonsense spatial relations. The idea of using regions as primitive entities instead of points was first explored by Whitehead (1920, 1929) and de Laguna (1922). Whitehead proposed *extensive connection* as topological relation between regions of space, though strictly speaking he also formalized mereological relations. However, de Laguna and Whitehead did not eliminate points altogether, they just wanted to replace the unnatural primitive of a point by a more natural one, such as *solid* or *region*. Albeit considering regions as primitive entities, Whitehead suggested the method of *extensive abstraction* to define *abstractive sets*, i.e. infinite sets of regions that are totally ordered with respect to containment. In the limit case, such abstractive sets converge to points. Hence, points were reconstructed as abstract, infinitesimally small regions. We may analogously define other lower-dimensional entities such as lines, surfaces, etc.

Mereotopology is tightly coupled with the idea of region-based space since both mereology and the topological relation of connection rely on the relations between regions. If we accept regions as primitives, mereotopology is a mere combination of mereological concepts of parthood with topological concepts of contact or connection (we use them synonym). Although in principle so simple, it turns out that this combination leaves manifold options to build mereotopological theories. We will study the rich space of these theories in this chapter. First of all, let us summarize what is common to all mereotopologies. This is indeed a rather small core (Varzi, 1998): a reflexive, anti-symmetric, and transitive parthood relation for its mereological component and a reflexive, symmetric connection or contact relation for its topological component. As relation between them, monotonicity shall be obeyed.

(Mon) $\forall x, y\big[P(x,y) \rightarrow \forall z\big(C(x,z) \rightarrow C(y,z)\big)\big]$
(Monotonicity)

Moreover, it is undisputed that in mereotopology, there should be a distinction between (topological) connection and (mereological) overlap. This difference is usually reflected in a relation of external connection, defined as following:

(EC) $EC(x, y) \equiv_{def} C(x, y) \wedge \neg O(x, y)$
(External connection)

Apart from these basic requirements for mereotopological theories, we can choose freely amongst many ontological commitments discussed in the following section.

3. ONTOLOGICAL CHOICES IN MEREOTOPOLOGY

A major motivation for region-based theories of space is the argument that regions are more parsimonious than points in logical formalizations of commonsense spatial knowledge. From the philosophical perspective, this is an important justification. However, from a model-theoretic view where theories with identical models are considered interchangeable, such an argument is only superficial if semantic mappings between a point-based and a point-free theory can be given. Indeed, many so-called mereogeometrical theories equipped with regions as primitive entities have equally expressive point-based counterparts (Borgo & Masolo, 2009; Pratt & Schoop, 1997). Then it becomes a matter of ontological preference which theory to choose.

This section covers some of the core ontological issues that all mereotopologies and mereogeometries have to address—even if only by stating explicitly that certain concepts are simply not definable in a particular mereotopology or class of mereotopologies. Though most mereotopological theories agree on basic terminology and definitions, there is a strong disagreement over these ontological decisions. Eschenbach (2007) systematically studies the axioms, which characterize these ontological decisions for selected theories. To make mereotopology more accessible to readers not familiar with the field, we discuss the controversies and decisions from a more high-level perspective. Hereby no specific order of treatment is intended; instead, we try to follow a natural flow between the issues. Most importantly, none of the issues are independent of one another. When comparing theories of mereotopology, we have to be aware of some unavoidable tradeoffs. If one theory shall encompass all the features discussed (if that is even possible), it would be overly complex for humans and unacceptably inefficient for automated reasoning. There are many other choices we must make when building or choosing a theory of mereotopology. Amongst others, the choice of language or formalism is critical. We leave these questions largely aside, focusing instead on ontological issues that are of concern irrespective of the choice of language. We mostly discuss first-order theories of mereotopology, but this does not mean that the many ontological choices do not apply to other theories as well.

Many ontological decisions e.g. extensionality, identity, and dimensionality are, moreover, influenced by discussions of their commonsensical, cognitive, and philosophical adequacy. Per se, we avoid taking any particular stance on these adequacy issues. Instead, we raise awareness of these ontological commitments to equip readers with sufficient understanding to make acceptable choices in their domain of interest.

3.1. Mereology vs. Topology as Foundation

One of the earliest systematic studies of mereotopologies (Casati & Varzi, 1999; Varzi, 1996a) classified theories by the interaction between mereology and topology within them. Three main ways of building mereotopology from a topological and a mereological component have been identified. Extending mereology by an additional topological primitive is one way, pursued in (Eschenbach, 1999; Pratt & Schoop, 1997; Smith, 1996). Smith employs a reflexive parthood relation for mereology, extended by the mereotopological primitive of interior parthood (comparable to non-tangential parthood in other mereotopologies). Pratt and Schoop (1997) use a Boolean language which implicitly defines parthood, extended by a primitive contact relation. Eschenbach (1997) uses for her 'Closed Region Calculus' (CRC) a standard mereology (based on parthood) equipped with a topological notion of disconnection.

Mereology and topology can also be merged by treating topology as more fundamental and defining mereology in terms of topological primitives. De Laguna (1922) and Whitehead (1929), intrigued by its formal economy, chose this paradigm. It is the most common approach in QSTR. Clarke (1981) chose connection as only primitive for his 'Calculus of Individuals,' while most later work stuck to this choice, e.g. the system RT_0 of Asher and Vieu (1995), the *Region Connection Calculus* (RCC: Cohn, et al., 1997a, 1997b; Randell, et al., 1992; Gotts, 1994). Parthood is expressed in terms of connection alone, i.e. parthood and the topological notion of *enclosure* (Varzi, 1996a) coincide. All sentences in such theories are limited to the expressiveness of contact. The *n-intersection model* (Egenhofer, 1989, 1991) uses likewise only (point-set) topological primitives: interior, boundary, and complement. Mereological relations are then solely defined in terms of these topological concepts.

A third way to combine topology and mereology was employed by Eschenbach and Heydrich (1995). They extend the mereological framework of Leonard and Goodman (1940) by a primitive unary relation 'of being a region.' However, the latter primitive captures a topological idea. Hence, it is questionable whether the theory is purely mereological. Analogously, many mereogeometries (Bennett, 2001; Bennett, et al., 2000; Borgo, et al., 1996; Tarski, 1956a) use a combination of a mereological primitive together with the primitive of 'being a region.' More recently, the algebraic representations of Clarke (1985) and Asher and Vieu (1995) give rise to mereotopologies definable from a single mereological primitive of parthood (compare Section 6.2). The later in particular is either definable from connection or from parthood alone. Hence, it becomes clear that Casati and Varzi's framework is not a real partitioning of mereotopologies.

This classification of mereotopologies distinguishes principles of how to fuse mereology and topology into a common theory. This is a coarse classification; many mereotopologies do not fit well into this framework. For example theories that employ very powerful predicates that are neither strictly of mereological nor topological nature, e.g. (Gotts, 1996b) with a single primitive *INCH(x,y)* meaning '*x includes a chunk of y*,' or some of the mereogeometries (de Laguna, 1922; Donnelly, 2001; Nicod, 1924) cannot be easily classified in this way. In particular, notions of convexity are usually expressive enough to recover the topological structure as demonstrated by Borgo and Masolo (2009).

3.2. Extensionality and Identity

Different strengths of extensionality are a core distinction between mereotopological theories. The following generic axiom captures extensionality of a binary, symmetric predicate *Q*.

(Ext-Q)
$$\forall x, y\Big(\forall z\big[\big(Q(x,z) \leftrightarrow Q(y,z)\big) \rightarrow x = y\big]\Big)$$
(Generic extensionality)

If a theory is extensional with respect to Q, we say that two elements in the theory are indistinguishable with respect to *Q*. If exactly one of the primitives of a theory is extensional, this provides an intuitive notion of identity. Not surprisingly, Whiteheadean theories with a single primitive *C* are extensional with respect to *C* (cf. Section 5.1.1). Other theories require extensionality with respect to the mereological relation of 'overlap' (Eschenbach, 1999; Gotts, 1996b; Randell, et al., 1992; Roeper, 1997). This is equivalent to requiring extensionality with respect to proper parthood

PP, although the axiomatization of extensionality for *PP* would need to accommodate the asymmetry of PP. Any such mereologically extensional theory satisfies the strong supplementation axiom (P5). If *C* and *O* are extensional in a mereotopology, *EC* is also extensional.

Mereotopology allows various weaker assumptions, e.g. weak supplementation (Casati & Varzi, 1999; Varzi, 1996a) where mereological extensionality is not required as in the theories of Asher and Vieu (1995), Clarke (1981), and Roy and Stell (2002). Therein parts of a region *r* do not uniquely identify *r*. There can be multiple, possibly infinitely many, regions r_1, r_2, ... consisting of the same parts, but distinguishable by their connection relation to other regions. This occurs in theories where regions are distinguished from their closure and/or interior with contact depending on this distinction. In contrast, the *Closed Region Calculus* (CRC: Eschenbach, 1999) is mereologically extensional, but not topologically. Instead, Eschenbach (1999) points out that the CRC is extensional with respect to external connection. The RCC is in fact extensional with respect to *C*, *O*, and *EC*. For more discussions on extensionality in different theories see Eschenbach (2007).

3.3. Sums and Fusions

Within mereology, there is a controversy over whether arbitrary (unrestricted) sums of entities should be allowed or even required. From an algebraic perspective, such requirement yields in complete lattices more elegant structures. However, from a philosophical or cognitive perspective, such requirement seems stringent. Abundant examples of irrelevant arbitrary fusions have been given. But most mereotopologies at least assume the existence of binary sums—either mereological sums, denoted here by $\oplus$, or topological sums preserving contact, denoted here by $+$.

(Sum-M)
$$x \oplus y = z \leftrightarrow \forall u \big(O(u,z) \leftrightarrow \big(O(u,x) \vee O(u,y) \big)\big)$$
(Mereological sum)

(Sum-T)
$$x + y = z \leftrightarrow \forall u \big(C(u,z) \leftrightarrow \big(C(u,x) \vee C(u,y) \big)\big)$$
(Topological sum)

Either sum is unique if the relation within the definition (overlap or contact, respectively) is extensional. Mereological sums are found in theories with a mereological primitive while topological sums are standard for theories with a topological primitive, e.g. Whiteheadean theories (Asher & Vieu, 1995; Bennett, 2001; Clarke, 1981; Gotts, 1994; Randell, et al., 1992). Equivalently, binary sums are expressed as upper bounds in lattices, see Section 6 for details. Therefore, we mainly deal with bounded lattices for representing the mereological component of mereotopologies. The existence of *lowest upper bounds* in the lattices indicates the existence of mereological sums. Non-distributive lattices can contain non-unique relative complements and therefore are representative of theories where mereological extensionality is not guaranteed to hold.

Theories (e.g. Clarke, 1981) that go beyond binary, i.e. finite, sums define what Casati and Varzi (1999) call *unrestricted fusion* or what is also known as *infinitary* or *universal fusion*. Notice that unrestricted fusion of arbitrary—possibly infinite sets—of entities is not first-order axiomatizable. Similarly, algebraic or topological theories of region-based space employ either a set-theoretic definition of unrestricted fusion or use complete algebras as basis, e.g. complete Boolean algebras have been considered by Roeper (1997) and used for representing models of Clarke' theory (cf. Biacino & Gerla, 1991).

3.4. Self-Connectedness

Self-connectedness of a region, written as *SelfCon(x)*, is an intuitive property that expresses that a region does not consist of several disconnected, i.e. scattered, parts. *Self-connectedness of the space* (the universe), expressed as *SelfCon(u)* or through the following sentence, means that every region is connected to its complement.

(SC)
$\forall x, y\left[\forall z\left(C(x,z) \vee C(y,z)\right) \to \mathit{SelfCon}(x+y)\right]$
(Self-connectedness)

Depending on the intended (regular) regions we want to capture, two main approaches to enforce self-connectedness are available. If we allow any kind of regular regions in a theory, the topological complement of a regular closed region is regular open. We then obtain a space where every element is not connected to its complement, i.e. $\forall x\left[\neg C(x,-x)\right]$ when using the intuitive definition of Self-Connectedness (SC-S).

(SC-S)
$\mathit{SelfCon}(x) \equiv_{def} \forall y, z\left(y+z=x \to C(y,z)\right)$
(Strong self-connectedness)

The models of such a theory can never be self-connected. To avoid this problem, weaker axioms for self-connectedness have been proposed, in particular the following.

(SC-W)
$\mathit{SelfCon}'(x) \equiv_{def} \forall y, z\left(y+z=x \to C(cl(y), cl(z))\right)$
(Weak self-connectedness)

In the presence of SC-W, a regular closed set and its regular open complement are indeed self-connected. If a theory only allows regular open regions and contact is defined as $C(x,y) \Leftrightarrow x \cap y \neq \varnothing$ (compare C-Weak in Section 5.1.1), then SC-S also results in disconnection between a region and its complement while SC-W would not. For theories that only consider regular closed sets, both definitions of self-connectedness coincide. In such theory, the universe is only not self-connected if there are true disconnected partitions in the universe. Of course, then self-connectedness of the universe can be postulated as an axiom (cf. Egenhofer, 1991; Egenhofer & Franzosa, 1991; Roeper, 1997). There are few theories using an intermediate notion of (self-) connectedness with an open region and its closed complement being connected (Grzegorczyk, 1960; Pratt & Schoop, 1997), compare the discussion of (Cohn & Varzi, 2003) in Section 4.

In atomistic mereotopologies, self-connectedness is trickier to achieve. While it is a widely held view that connectedness between an atom and its complement is desirable to enforce self-connectedness of space, in the context of Whiteheadean space this would lead to the fact that an atom is also part of its complement. Roy and Stell (2002) argue that this is not a defect because discrete space can be seen as approximation of continuous space where extensionality of *P* does not necessarily hold any longer (due to the approximation loss). Hence, we might need to abandon extensionality of *P* in discrete mereotopology. Apart from that, extensionality of *C* conflicts with atomistic mereotopology if connectedness is defined in the weak form, i.e. if atoms are connected to their complement. As Eschenbach (1999, p. 163) notes: “[*C*-extensionality] means to exclude the coexistence of the universe and the complement of an atom because both would be connected to all regions.” This problem does not apply to theories with the open/closed distinction and the stronger form of self-connectedness. However, other options such as a relaxed version of self-connectedness (similar to the second version above, while accommodating atoms, which are usually open) could help overcome the problem.

3.5. Dimensionality

One of the most common simplifications amongst axiomatic theories of mereotopology is the restriction that only entities of equal dimensions can co-exist in a single model. The theories in the Whiteheadean conception of space (Asher & Vieu, 1995; Casati & Varzi, 1994; Clarke, 1981; de Laguna, 1922; Eschenbach, 1999; Nicod, 1924; Randell, et al., 1992; Smith, 1996; Tarski, 1956a; Whitehead, 1920, 1929; Pratt & Schoop, 1997; Galton, 1999) all rely on a single class (sort) of entities. Though all regions must be of equal dimension, this dimension can be chosen freely for each domain except within the theory of Pratt and Schoop (1997). Lower dimensional can only be defined using higher-order constructs, e.g. in a three-dimensional domain, lower dimensional entities such as points, lines, and surfaces can be reconstructed through *extensive abstraction* (de Laguna, 1922; Whitehead, 1929), an idea dating back to Lobačevskij (1834). Indeed, most region-based theories define so-called *abstract points* as limits of infinitely many nested regions or sets of regions (cf. Clarke, 1985; Eschenbach, 1994; Menger, 1940; Tarski, 1956b). Equally, points can be recovered as prime ideals, ultrafilters (maximal filters), or generalizations thereof from many classes of lattices in a second-order way (cf. Asher & Vieu, 1995; Roeper, 1997).

Galton (1996) argues that we should neither assume regions nor points, nor any other kind of spatial entities as more fundamental than the other. In this spirit and irrespective of the philosophical or cognitive adequacy of regions or points, Galton (1996) and Gotts et al. (1996) have proposed frameworks that accommodate entities of any kind of dimension (in particular points, lines, surfaces) through a binary predicate of *equi-dimensionality* and separate parthood relations between equi-dimensional and non-equi-dimensional entities. This comes close to what Hayes (1985a) envisioned for commonsense reasoning in physics. Points and indivisible atomic regions can then theoretically co-exist (Galton, 1996). This also lets us define boundaries elegantly: boundaries are defined as entities of a dimension one lower than the entities they bound. Key to the axiomatization of Galton (1996) is the insight that entities of a lower dimension cannot be part of a higher-dimensional entity, but can only lie within such. In the result, two separate parthood relations are distinguished: one exclusively between equi-dimensional regions, whereas a separate relation *IN(x,y)* relates a lower-dimensional entity x to a higher-dimensional y. We would say 'a point lies in a region' instead of 'a point is part of a region.' In a similar framework, Gotts (1996b) uses the *INCH(x,y)* primitive to state that '*x includes a chunk of y*' whereas the overlap relation *OV(x,y)* only applies to objects of equal dimensions. The use of cell complexes, i.e. collections of discrete objects of different dimensions, is another solution, which can accommodate objects of different dimensions (cf. Winter & Frank, 2000; Roy & Stell, 2002). Cell complexes are frequently used in GIS (cf. Burrough & Frank, 1995; Frank, 2005).

In strictly topological theories of region-based space (Egenhofer, 1989, 1991; Egenhofer & Herring, 1991; Egenhofer & Sharma, 1993b; McKenney, et al., 2005), a natural topological distinction between points, lines (and hence boundaries), and regions exists. However, these frameworks employ full point-set topology avoided by the previously mentioned theories. This trade-off is characteristic for region-based space: we can either resort to classical point-set-theoretic axiomatizations allowing entities of any dimension and relations between them, or chose a cognitively or ontologically more appropriate approach without points as primitives, but then need to overcome inherent difficulties when defining entities of different dimensions and their relation to one another.

3.6. Atoms and Continuous vs. Discrete Mereotopologies

While points can be defined as abstract entities, explicit definitions of so-called *concrete points* (Eschenbach, 1994) are also common. Concrete points—to prevent confusion usually referred to as *atoms*—are the smallest, indivisible regions without proper parts. They are found in the mereotopologies of Nicod (1924) and Smith (1996) and in the point-free geometry (mereogeometry) proposed by Huntington (1913). Atoms are extended—contrary to the definition of points as limits, which have no extension.

(PT) $Pt(x) \equiv_{def} \forall y \big(P(y,x) \rightarrow y = x\big)$
(Concrete points)

Historically, de Laguna and Whitehead implicitly required hat every region has a proper part, defining extensionless abstract points through higher-order constructs. Their understanding of *atomless*, i.e. *continuous*, mereotopology was adopted by Tarski (1956a), Menger (1940), Clarke (1981, 1985), and the RCC. E.g. in the RCC space is required to be Atomless (AL) because AL is a theorem of the remaining axioms. In AL, the relation *NTPP* (Non-Tangential Proper Part) is a specialization of proper parthood in which boundaries are not shared.

(AL) $\forall x \exists y \big[NTPP(y,x)\big]$ (Atomless)

Hence, we immediately conclude $\forall x \exists y \big[PP(x,y)\big]$ and with *O* being extensional, we can derive the principle of infinite divisibility (see also Masolo & Vieu, 1999).

(Div) $\forall x \exists y,z \big[PP(y,x) \wedge PP(z,x) \wedge \neg O(y,z)\big]$
(Infinite divisibility)

Though the original RCC theory is atomless, changes to the RCC axioms to allow models with atoms (*atom-tolerant*) have been proposed (Roy & Stell, 2002; Dong, 2008). The other extreme—requiring the existence of atomic parts for each region—results in *atomistic* (or *discrete*) mereotopologies (Galton, 1999; Masolo & Vieu, 1999; Nicod, 1924; Smith, 1996). Notice that atomicity does not imply that all models are finite since such restriction cannot be expressed in first-order logic.

(AT) $\forall x \exists y \big[P(y,x) \wedge \neg \exists z PP(z,y)\big]$
(Atomicity)

There are a number of arguments in favor of non-continuous theories of region-based space (and time). For example, the data recorded in geospatial applications is always of limited granularity so that we have some smallest set of entities that we can treat as atoms. These might be parcels of land when modeling land use or counties or municipalities within GIS, or molecules, atoms, or smaller elements in physics—whatever is appropriate for the domain. Here the choice of atoms is always dependant on the particular domain, however for most domains we can come up with some set of atoms. This view matches early ideas of atomism in space, e.g. as Masolo and Vieu (1999) remark: “Aristotle held that one can always divide a magnitude any finite number of times but that infinite divisibility is only potential” (p. 236). For any particular domain, the potential is barely relevant; however, for a general qualitative theory of space, it must be taken into account. We can also build discrete mereotopologies using graph-theoretic concepts and their relationships to (discrete) tessellations of space (cf. Section 8).

Other theories (Asher & Vieu, 1995; Li & Ying, 2004) allow atomistic and atomless models. These *atom-tolerant* theories generalize both atomistic and atom-less theories. For an in-depth discussion of the relationship between atomicity, divisibility, and density in mereologies and mereotopologies, we invite the reader to consult Masolo and Vieu (1999). Moreover, Masolo and Vieu (1999) have surveyed several mereotopologies from Casati and Varzi (1999) and Varzi (1996a) with respect to their consistency with axioms of atomicity and axioms of divisibility and atomicity.

3.7. Boundaries

Associated to the issue of dimensionality and the reconstruction of lower-dimensional entities is the treatment of boundaries. One of the criticisms of mereotopology raised by Breysse and De Glas (2007) and within the field itself is that point-set topological interpretations of regions leaves us with three unsatisfying options of how to treat boundaries. See also Fleck (1996) for a discussion of the problematic topological nature of boundaries. Restricting ourselves to closed regions, we have to accept that there are points of a region and its complement that overlap which seems discomforting. Retreating to an option where the complement of a closed region is open and vice versa (cf. Asher & Vieu, 1995) seems arbitrary. E.g. cutting a region then leaves one part of it with a boundary at the cut, while the other part does not have that boundary. Many more examples have been given to demonstrate the problematic nature of boundaries, e.g. for a black area on a sheet of white paper, is the boundary between the black area and the remainder of the paper black or white? When restricting ourselves to open regions, there remain points that belong to neither region. Fleck (1996) suggested deleting these points to obtain a more intuitive topological structure and to make a proper distinction between real contact and touching (so-called *weak contact*).

An axiomatic theory incorporating boundaries as special kind of extended but dependent (depending on the region they bound) entities has been proposed by Smith (1996). With an appropriate definition of boundaries and by allowing both open and closed regions, a boundary of a region is also a boundary of its complement. Most importantly, boundaries are not part of the bounded region itself. In this way, Smith (1996) provides for one of the rare accounts of boundaries within mereotopology while abstaining from the arbitrary choice of to which region the boundary actually belongs. Contact can then be defined as two regions sharing a boundary. The discrete version of the RCC of Roy and Stell (2002) resembles Smith's theory in that boundaries are defined as regions without interiors. Other theories (Galton, 1996; Gotts, 1996b) treat boundaries as separate sort of entities. From an abstract view, this makes perfect sense, although arguments treating boundaries as thin layers of space instead have their justifications especially for modeling physical objects. Similarly, Eschenbach (1994) handles topological entities such as boundaries and points separately from the mereological regions.

In general, the nature of boundaries is philosophically disputed. It is questionable what kind of entity a boundary is—either spatial object or abstract entity (cf. Varzi, 2008). Another philosophical distinction has been made between natural (*bona fide*) and artificial (*fiat*) boundaries (Smith, 1996; Smith & Varzi, 1997). Fiat boundaries seem to always be perceived as lower-dimensional artifacts, while bona fide boundaries are more appropriately modeled as 'thin layers.' Following the serious doubts of the existence of bona fide boundaries, we could argue that the abstraction of all boundaries as lower-dimensional entities seems indeed the most viable option for commonsense space.

3.8. Holes, Discontinuities, and Superficialities

Mereotopology is able to express self-connectedness of solids through the connectedness of its parts in a first-order way (Section 3.4) while topology can express self-connectedness only by quantifying over all subsets of a set (a region): set A is self-connected if for all $B, C \subset A$ $A = B \cup C \Rightarrow cl(B) \cap C \neq \varnothing \vee B \cap cl(C) \neq \varnothing$ (Masolo & Vieu, 1999). The question arises to what extent mereotopology can define concepts such as holes and other discontinuities. Casati and Varzi (1994) give a remarkably fine-grained, though informal, classification of different kinds of discontinuities: cavities, tunnels, hollows, ridges, cracks, fissures, and every imaginable combination thereof, but only the appendix of Casati and Varzi (1994) gives an axiomatic treatment of these discontinuities, also found in Varzi (1996b). Unfortunately, a primitive notion *H(x,y)* meaning '*x is a hole in (or through) y*' is assumed. No attempt to define holes in terms of connection and parthood is made. (Gotts, 1994) tested the definability of holes using the RCC notions (basically connection) with the example of a 'doughnut' (a torus) and derivations thereof. Similarly, the definability of holes in the n-intersection model was researched in Egenhofer et al. (1994). Critical to Gott's investigations is the definition of *finger-connectivity* and *separation numbers*, which depends on the actual dimensions and is not dimension independent. In particular, a theory of the natural numbers is required while the separation number can only be defined recursively. More limiting is that the RCC and other Whiteheadean axiomatization of space can capture only holes of the same dimension as the regions, inhibiting the definition of lower-dimensional superficialities (Hahmann & Gruninger, 2009). Again, mereotopologies allowing entities of different dimensions are necessary in such cases, emphasizing the argument in Section 3.5. A more general definition of holes within RCC can be found in Mormann (2001), where *ECN(x,y)* is the external connection relation of *x* without the complement of *y*, i.e.

$$ECN = \left\{(x,y) \mid EC(x,y), x \neq y'\right\}.$$

(ECN)
$$ECN(x,y) \equiv_{def} EC(x,y) \wedge \exists z\left[\neg C(x+y,z)\right]$$

(H) $$H(x,y) \equiv_{def} EC(x,y) \wedge \exists z\Big[PP(x+y,z)\Big] \wedge \forall v\Big[ECN(v,x) \to O(v,y)\Big]$$
(Hole)

This definition of a hole in RCC has been useful in constructing non-standard (Düntsch & Winter, 2004a) and maybe even undesirable (cf. Li & Ying, 2003) models of the RCC.

Occasionally, proposals to restrict the intended semantics of mereotopology so that all regions are hole-less have been made (Egenhofer, 1991; Egenhofer & Franzosa, 1991).

3.9. Convexity and Mereogeometries

So far, we have been chiefly concerned with purely mereotopological theories and their ontological commitments. However, analogous ideas also apply to theories that define geometrical notions in addition to mereological and topological ones, but what exactly distinguishes mereotopology from mereogeometry? Both are classes of region-based theories of space. The name mereogeometry suggests the inclusion some geometrical relation. Indeed, Gerla (1995) proposes the ability to reconstruct points as criteria for a region-based theory to be geometrical. Such theories are usually called *pointless geometries,* or nowadays, *point-free geometries* (Gerla, 1995); similar approaches can be found in the early works by Lobačevsky (1934) and Huntington (1913) trying to build Euclidean geometry from regions or solids. In contrast, Borgo and Masolo (2009) take the sheer inclusion of a geometrical primitive such as convexity, e.g. *Conv(x)*, *Congr(x,y)*, *SPH(x)*, or relative size, e.g. *CanConnect(x,y,z)*, as distinctive feature lifting a mereotopology to the geometrical level (though not necessarily to Euclidean geometry). Here we follow the later interpretation. While (mereo-)topological relations are required to be invariant under all continuous transformation, (mereo-)geometrical relations are required to be invariant to the strength of the desired geometry. Usually this is elementary geometry, but can also be weaker forms thereof such as affine geometry (invariant under affine transformations) or projective geometry (invariant under transformations of the projective group). From Borgo and Masolo (2009) we learn that the mereogeometrical theories of Bennett et al. (2000), Borgo et al. (1996), de Laguna (1922), Donnelly (2001), Nicod (1924), and Tarski (1956a) all have equivalent standard topological model in $\mathbb{R}^n$. Borgo and Masolo (2009) introduce the term *full mereogeometries* for these theories. However, the non-standard models have not been systematically studied; it remains open whether the models without topological interpretation are also equivalent for the full mereogeometries.

With Tarski's two primitives *sphere* and *parthood* (Tarski, 1956a; see Gruszczyński & Pietruszczak, 2008 for a full development of his theory), we can define a ternary relation of co-linear points with equal distances between them. Thus, defining the predicate of *betweenness* between three points, it is easy to reconstruct elementary geometry as axiomatized in Tarski (1959). Indeed, Tarski (1956a) states the equivalence between his geometry of solids and three-dimensional Euclidean geometry as a theorem. Hence, all full mereogeometries are region-based equivalents of Euclidean geometry.

The only theory including a notion of convexity but not constructing a full mereogeometry that we know of is the RCC extended by a convex hull primitive (cf. Cohn, 1995; Cohn, et al., 1994, 1997b; Randell, et al., 1992). Borgo and Masolo (2009) conjectured and Cohn and Renz (2008) confirmed that this theory is strictly weaker than full mereogeometries. It is suspected that the RCC together with a convex hull (or convexity) primitive is a point-free equivalent of affine geometry. For possible future work in this direction, see Section 10.3.

Within topological theories of mereotopology, it is even possible to define convexity in Boolean terms, compare the discussion of Roeper (1997) in Section 7.3. This however relies on points as smallest entities. It remains an open question whether such a convexity notion can be defined without using points. Separately, it remains a challenge to build theories in between mereotopology and mereogeometry that can express basic morphological distinctions without constructing a full mereogeometry—if any such theories exist.

3.10. Vagueness, Location, and Granularity

Beyond the analysis of ontological commitments within mereotopologies, several other aspects are of practical relevance when constructing such theories of space. In particular, vagueness has to be dealt with in reality, since information is often incomplete. Input data is usually not perfect (e.g. from satellite pictures, maps, descriptions, etc.) or the exact outlines of regions and boundaries are unclear for other reasons. Furthermore, our world is very dynamic in the large and small scale, e.g. gravitational effects and winds let shorelines vary, while also atoms and molecules are constantly in movement and, physically speaking, constantly alter the surface of any physical object. Finally, available information about spatial configurations usually comes in a wide range of granularity—especially in the realm of geographic information. We have high-level maps such as political maps but also very detailed maps such as topographical maps of small areas. Integrating mereotopological information of diverse granularity is far from obvious. These three aspects, namely vagueness, the distinction between objects and their location, and granularity, are crucial in real-world applications. However, there are not specific to mereotopology. Nevertheless, we cover them briefly in this subsection. For the sake of brevity, we will not address these issues in later sections.

Within QSTR, there is an abundance of work on notions of vagueness using *vague* or *rough sets*, i.e. approximations of crisp sets (cf. Pawlak, 1991), in so-called *rough relation algebras* (cf. Düntsch, 1994; Düntsch, et al., 2001b; Düntsch & Winter, 2006), algebras of approximate regions (Düntsch, et al., 2001*a*), approximate topological relations (Clementini & Di Felice, 1997b), fuzzy relations/set (Burrough, 1996), or more recently used for the definition of the fuzzy region connection calculus (Schockaert, et al., 2009). One specific way to look at vagueness in regions is the use of indeterminate boundaries. Regions with indeterminate boundaries commonly occur when dealing with geographic features (lakes, mountains, mountain ranges, deserts, etc.) as well as rapidly changing objects such as clouds, crowds, swarms, etc. The groundwork for objects with indeterminate boundaries was laid by Burrough and Frank (1995, 1996) distinguishing the region an object certainly occupies from its maximal possible extent, the vague region. A classification of boundaries and their properties has been attempted by Galton (2003), while more philosophical discussions are found in Varzi (2008). The *egg-yolk calculus* (Cohn & Gotts, 1996b) is the most prominent logical theory of vague regions. The yolk represents the certain region ('*definitely in*'), while the whole egg represents the maximal possible extent ('*possibly in*'). There is a direct relation to spatio-temporal reasoning, where the yolk represents the area an object is always in, while the remainder of the egg represents the spatial region where the objects is sometimes located (Stell, 2003). Guesgen (2002) extended the egg-yolk theory to spatio-temporal regions by using fuzzy sets, resulting in the *scrambled-egg calculus*. Ibrahim and Tawfik (2004) apply a similar fuzzy-logical approach to Muller's qualitative spatio-temporal framework (Muller, 1998a, 1998b, 2002). In general, vagueness is orthogonal to other ontological commitments within mereotopological theories.

In reality, humans distinguish physical objects and the location, i.e. the space they occupy. Many mereotopological theories assume that either only regions of space or only physical objects are in the domain of discourse. To incorporate both within one theory, we need to define a theory of space (ground) and one of physical objects (figure) while additionally axiomatizing the relationships between them (*figure-ground relations*). Such theory of localization is discussed in Casati and Varzi (1996), presenting the idea of a topological theory of space combined with a mereological theory of physical objects—therefore redefining the interaction between topology and mereology in a rather unorthodox way. *Rough locations* have been studied extensively, e.g. by Bittner (1999, 2004). Later, a series of papers extended the RCC with notions of vagueness and rough locations (Bittner & Stell, 1998, 2000a, 2000b, 2002). For space occupied by physical objects, Schmolze (1996) considers a topological account of space extended by axioms governing the interrelation between space and objects.

Dealing with spatial information of different granularity is closely related to vagueness in the following way: less fine-grained information leads to a higher degree of vagueness. For that reason, granularity, granular partitions, and hierarchical theories of mereotopology have been usually studied in the context of vagueness (Bittner, 2002; Bittner & Smith, 2001, 2003; Bittner & Stell, 2003). Major aspects of granularity and location influenced the design of the *Basic Formal Ontology* (BFO: Grenon, 2003). Another hierarchical approach integrating discrete and continuous mereotopology using Generalized Boolean Contact algebras (see Section 6.1) has been presented by Li and Nebel (2007). Cohn and Gotts (1996a) use 'crisping' as a way to capture granularity for dealing with spatial vagueness while Cohn (1995) presents another hierarchical theory based on the RCC and convexity. Finally, a congruence primitive has been combined with the RCC by Cristani et al. (2000) which in turn allows defining spheres and four JEPD relations of congruence defining the MC-4 calculus (cf. Cristani, 1999). Location is expressed in terms of parthood, but the essential new idea is the definition of *mobile parts*, which

Table 1. Set of ontological commitments and their choices for mereotopology

Ontological aspect	Set of possible choices
Foundation	· Mereology · Topology · Mereology and Topology
Extensionality	· Choice of one or many extensional primitives, e.g. *C*, *O*, *P*, and/or *EC*
Sums	· Mereological sum · Topological sum · Mereological and topological sum
Fusion (restriction of sums)	· Restricted to finite fusions · Allows infinite (unrestricted) fusions
Connectedness of space	· Regions and their complements connected · Regions and their complements disconnected
Dimensionality and Boundaries	· Only equi-dimensional regions (as in Whiteheadean space) · Inclusion of one sort of entities, but allowing regions without interiors as boundaries (boundary-tolerant theories) · Multiple sorts of entities (Multi-dimensional mereotopology)
Atomicity	· Atomless (continuous) · Atomic (discrete) · Atom-tolerant theories
Dimensionality of holes	· No holes · Only equi-dimensional holes (same dimension as the host) · Equi-dimensional holes and holes of one dimension lower than the host ('cracks') · Any kind of holes (missing points, etc.)
Types of holes	· No holes · Only interior holes (cavities) · Only superficialities · Both cavities and superficialities acceptable
Geometric expres-sivity	· No geometric notion such as convexity (pure mereotopology) · Full mereogeometry (restoration of Euclidean geometry possible)

can move freely within their host body. In this light, the proposed theory is an egg-yolk theory where the maximal extent is the location extent of a mobile part. However, with the ability to define spheres in the tradition of Tarski, the resulting theory is just another full mereogeometry (see Sections 3.9 and 5.2).

3.11. Summary of Ontological Choices

The ontological choices we presented in this section are summarized in Table 1. Each aspect is presented as a partitioning of the available choices with the exception of the aspect of geometric expressivity and the aspect of the foundation as explained in Section 3.1. We use the aspects to discuss ontological choices within families of mereotopology, especially of Whiteheadean space and their algebraic counterparts in later sections of this chapter. Notice although dimensionality and boundaries were discussed separately, they are both related choices of a single ontological decision. The aspect of geometric expressivity is quite different from the remaining commitments in that we only name two extremes of a scale of possible choices. At the lower end of the scale of geometric expressivity, which have pure mereotopologies that cannot express any convexity or other geometric notions, while on the other extreme full mereogeometries are able to express anything that can be expressed in Euclidean geometry without reference to points. Filling the other potential choices in between pure mereotopology and full mereogeometry remains one of the tasks for future research (compare Section 10.3).

Not surprisingly, the ontological aspects summarized in the table resemble those discussed by Eschenbach (2007) for a set of five mereotopological theories. One aspect we have not treated separately is the possible distinction of open and closed regions. Though it is important especially for understanding the difference between some of the Whiteheadean theories of space, it is more a technical than an ontological distinction. Practically what matters is whether we can explicitly model boundaries and the connectedness of elements to their complements. Both distinctions are covered by the set of ontological commitments extracted here. For the remaining chapter, the aspect of holes will be discussed only marginally. Whiteheadean mereotopologies allow all types of equi-dimensional holes, with some theories requiring the intended regions to be hole-free. However, many theories have not been explicitly analyzed in that respect.

4. SYSTEMATIC TREATMENTS OF MEREOTOPOLOGY

Beyond the more informal analysis of ontological commitments of mereotopologies, categorizing the various theories in more formal ways helps us understand the theories, their limitations, and their relations to one another. For mereotopology, simple classification methods as well as more complex, partially-ordered hierarchies of theories with respect to certain ontological aspects (e.g. extensionality, connectedness of complements, existence of atoms, etc.) have been used to organize the wealth of theories. This section gives an overview of four different ways of systematically comparing mereotopological theories. These differ on what they compare—logical statements such as axioms and theorems, the models of theories, the semantics of key predicates, or the set of primitives. Apart from these frameworks, algebraic representations have been prolific in systematically analyzing mereotopologies. These representations are covered in-depth in Section 6. Equally, comparisons of topological models of Whiteheadean theories of space are given in Section 7, relating them also to purely topological theories of region-based space.

4.1. Logical Statements

A first systematic study of axiomatic systems of mereology and topology has been conducted in Casati and Varzi (1999) and Varzi (1996a). The space of theories is divided into mereology-based and topology-based theories of space as outlined in Section 3.1. The theories are further analyzed with respect to their strength of extensionality. Following up on these results, a purely axiomatic study of the relationship between Whiteheadean theories, in particular the theories of Asher and Vieu (1995), Clarke (1981), Eschenbach (1999), Randell et al. (1992), and Roeper (1997), has been conducted by Eschenbach (2007). In her work, the space of (Whiteheadean) mereotopological theories is mapped out according to several ontological aspects which significantly influenced our discussions in Section 3 and the comparison of axiomatic theories of Whiteheadean space in Section 5.1.1. Amongst the ontological commitments considered by Eschenbach (2007) varying degrees of extensionality, kinds of universal regions, self-connectedness of space, and open/closed distinctions are explored. Moreover, different kinds of complement and sum/fusion operators are used to chart the investigated mereotopologies.

4.2. Models

The comparative framework of Borgo and Masolo (2009) studies axiomatic theories with respect to their standard topological models. Though their work is limited to mereogeometry, the approach has a much wider applicability as we will see in Section 7.1. Borgo and Masolo (2009) utilize the interdefinability of primitives of several so-called *full mereogeometries* (Bennett, et al., 2000; Borgo, et al., 1996; de Laguna, 1922; Donnelly, 2001; Nicod, 1924; Tarski, 1956a) to prove the equivalence of their standard topological models in $\mathbb{R}^n$ when limited to only regular closed subsets or to only regular open subsets. The theories presented in Borgo et al. (1996), de Laguna (1922), and Donnelly (2001) are further restricted to those subsets with finite diameter. As a consequence of Borgo and Masolo's (2009) work, we can freely pick amongst these theories depending on our preferred primitives. For example the sphere primitive *SPH(x)* from Tarski (1956a) and congruence *CG(x,y)* from Borgo et al. (1996) are equally express in the presence of the parthood relation, while the primitives *CCon(x,y,z)*—'*x can connect y and z*'—from de Laguna (1922) and Donnelly (2001) or Conjugate *Conj(x,y,z,w)* from Nicod (1924) are alone sufficient to construct full mereogeometry. That allows us in Section 5.2 to discuss a single axiomatic theory representative for all full mereogeometries. However, the possible non-standard models, i.e. the models without interpretation in traditional point-set topology, of these mereogeometries might still differ. Further follow-up work is required to investigate this possibility.

4.3. Semantics of Connection and Parthood

A very different framework by Cohn and Varzi (1998, 2003) explores mereotopologies with respect to a three-dimensional space of the type of their connection relation, parthood relations, and fusion operation. This focuses on a select set of ontological commitments from the larger set covered in Section 3, but treats them within a formal framework. It allows us to better understand what combinations of distinct connection and parthood relations are feasible in mereotopological theories. However, some ontological choices such as, for instance multi-dimensionality, are incompatible with this framework, while choices such as atomism, holes, and self-connectedness of space are compatible, but independent from the dimensions of Cohn and Varzi's framework. Cohn and Varzi (2003) shows the difference between boundary-tolerant (they refer to them as boundary-based) and boundary-free theories. Boundary-free ones are all uniformly typed, e.g. their connection and parthood relations are based on the same notion of 'contact.' Of course, Whiteheadean theories (see Section 5.1.1) must be uniformly typed since they are only equipped with a single connection primitive and parthood is defined in terms of that. On the contrast, boundary-based theories are not uniformly typed. In other words, the boundary-based theories use a different notion of connection for their parthood relation than what is assumed for their own connection relation. In addition, Cohn and Varzi (1999, 2003) investigate the strength of the connection as an orthogonal dimension in their framework. It refines the coarse separation of weak and strong self-connectedness found in Section 3.4 and in Asher and Vieu (1995) to allow more fine-grained notions of connectedness. A mereotopology incorporating several different notions of connection could resolve the implausibility of several boundary-based theories that Cohn and Varzi (2003) and Breysse and De Glas (2007) mention. For instance, an interior can be in weak contact (adjacent) to its exterior in the presence of boundaries, compare also Fleck (1996) if a definition of weak contact independent of the standard contact relation (implying the sharing of a point) exists.

4.4. Sets of Primitives

Finally, we devise a categorization based upon the set of primitives of mereotopologies. Although being highly informal, it contributes to a deeper understanding of different kinds of mereotopological theories. As already observed by Robinson (1959), a single binary relation is insufficient to construct a theory of elementary geometry, such as Tarski's theory (Tarski, 1959). With the establishment of equivalences of a large set of mereogeometrical theories (Borgo & Masolo, 2009), it becomes clear why these theories either use some ternary relation (de Laguna, 1922; Donnelly, 2001; Nicod, 1924) or a combination of two relations of which at least one is binary (Bennett, et al., 2000; Borgo, et al., 1996; Nicod, 1924; Tarski, 1956a). As demonstrated by Clarke (1981) and others, if we are willing to stay within certain restrictions, a single binary primitive is sufficient to construct a mereotopology: Whiteheadean mereotopology (compare Section 5.1.1).

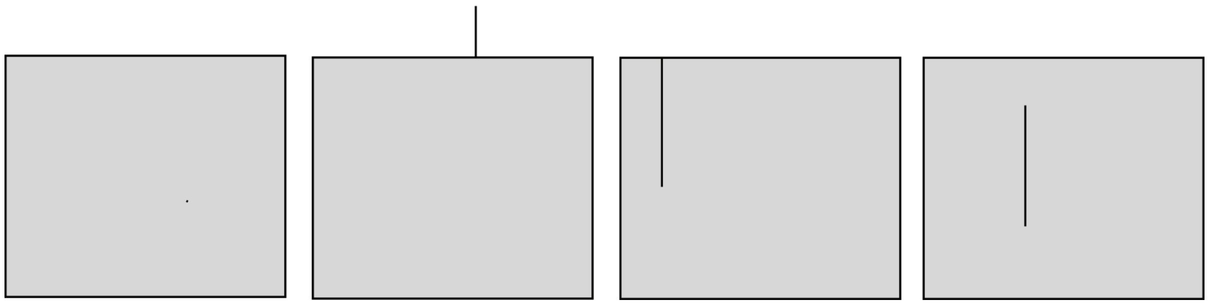
Mereotopology can be extended with additional geometric primitives without necessarily obtaining a full mereogeometry. For example, the RCC has been extended in with a binary predicate *convex hull, conv(x)* (cf. Cohn, 1995; Cohn, et al., 1997b) which gives a theory strictly weaker than full mereogeometry, conjectured by Borgo and Masolo (2009) and verified by Cohn and Renz (2008) using earlier results from Davis et al. (1999). It remains open whether mereotopological theories more expressive then the ones currently known, but strictly weaker than full mereogeometry, exist. This is especially important, since unary functions or predicates such as 'convex hull of x' or 'x is convex' seem to be difficult to extract from real-world applications (e.g. a computer vision system) and should be replaced by notions that are more primitive. Furthermore, it is unclear whether we can define such a theory using a single binary predicate or whether some additional predicate is necessary.

As Smith (1996) remarks, formal ontology is no longer primarily concerned with devising spatial ontologies with minimal sets of non-logical primitives of low arity (ternary relations proposed to build mereogeometry seem in general less intuitive in comparison with a combination of unary/binary predicates). Nevertheless, thinking about minimal sets of primitives is still useful when studying and comparing different axiomatizations of common real-world information, e.g. of qualitative space. For practical applications, it might well be the case that the number of primitives will increase.

5. LOGICAL AXIOMATIZATIONS OF MEREOTOPOLOGY

After identifying the set of choices for different ontological commitments in mereotopology and after reviewing earlier comparative studies of mereotopology, we are now in a position to take a closer look at some specific axiom sets for mereotopology and mereogeometry. Based on earlier studies about mereotopological theories and their assumptions, we present three main families of logical axiomatizations of mereotopology and mereogeometry. Many theories from the literature fall within one of these families. We present axiomatizations in first-order logic, outlining alternative sets of primitives, definitions, and axioms where appropriate. Most importantly, for each family we identify the common ontological commitments on the one hand and explore the differing ontological choices on the other hand. In the subsequent sections, we study alternative ways (alternative to logical axiomatizations) of treating mereotopology: Section 6 the algebraic counterparts of some mereotopology are interrelated, before turning in Section 7 to topological interpretations and topological specifications of mereotopologies. It is followed by a short section on graph theoretic ways of building mereotopology—especially discrete mereotopology. Altogether, we hope to give the reader an overview of the different styles of axiomatizations and what kinds of theories exist in the space of possible theories (along the ontological commitments). At the same time, we hope to convince the reader that studying the different mathematical frameworks (logic, algebra, topology, and graph theory) for

Figure 1. Examples of non-regular regions: in the first a point is missing while the others have so-called 'cracks'—positive or negative



specifying mereotopologies allows us to fully understanding the rich space of mereotopologies.

In this section, we will first look at two families of what we call *classical mereotopology*—Whiteheadean and boundary-based accounts of region-based space. Both families have been studied quite exhaustively (Casati & Varzi, 1999; Cohn & Varzi, 1998, 1999, 2003; Eschenbach, 2007; Varzi, 1996a). Though the immediate results are related directly only to continuous mereotopology, most of the results readily extend to discrete theories as Section 6.1 will demonstrate. In the subsequent Section 5.2, we give an overview of full mereogeometry using the axiomatization of Borgo et al. (1996) in combination with Tarski's 'Geometry of Solids' (Tarski, 1956a). Due to the results of Borgo and Masolo (2009) other mereogeometries can be reformulated using his primitives and axioms. The first subsections are formulated in first-order logic, while the last subsection discusses alternative, logical approaches with their differences, advantages, and disadvantages.

5.1. Classical Mereotopology

What we call classical mereotopology comprises pure mereotopological theories restricted to regular, equi-dimensional regions (see Figure 1 for examples of non-regular regions). This means three of our ontological aspects are fixed: the dimensionality of entities, indirectly also the dimensionality of holes, and the geometric expressivity. Notice that there is no restriction to any particular dimension, but instead each model is restricted to regions of equal dimensions. For instance, if a model contains three-dimensional entities like spatial regions, it cannot contain entities of any other dimension, e.g. two-dimensional surfaces, one-dimensional lines, or zero-dimensional points. Consequently, all entities must be regular. That means the domain of discourse is either a set of regular closed regions, i.e. regions that satisfy $x = \mathrm{cl}(x) = \mathrm{cl}(\mathrm{int}(x))$, a set of regular open regions, i.e. regions that satisfy $x = \mathrm{int}(x) = \mathrm{int}(\mathrm{cl}(x))$), or a set of regular regions (not necessarily open or closed) that satisfy both $\mathrm{int}(\mathrm{cl}(x)) = \mathrm{int}(x)$ and $\mathrm{cl}(\mathrm{int}(x)) = \mathrm{cl}(x)$.

5.1.1. The Whiteheadean Approach

Whitehead (1920, 1929) and de Laguna (1922) pioneered mereotopology by proposing the relation *extensive connection* to qualitatively describe the topological relations between regions of space. Extensive connection is what we call today *connection* or *contact*, or more generally *proximity*. Such an economical framework built around a single topological primitive distinguishes their work from the mereological approach of their contemporaries (Husserl, 1913; Leonard & Goodman, 1940; Leśniewski, 1927, 1931). Apart from regions being primitives instead of points, assumptions of Whiteheadean theories include the following (Mormann, 1998). The first two assumptions apply to all equi-dimensional theories, i.e. to all theories of classical mereotopology.

a. The dimension of all regions coincides with the dimension of space
b. Regions can be only part of regions and regions have only regions as parts
c. Regions can be interpreted as point sets (topological representability)
d. The theory is based on a single connection primitive

We now concentrate on assumption d) while the interpretation of regions as point sets is discussed in more detail in Section 7.3. Notice that often another assumption is added restricting the topological representability to representability by regular regions, which follows from a) and b). Because of assumption d), Whiteheadean theories are extensional with respect to the contact relation *C*. In fact, Whiteheadean theories are extensions of Strong Mereotopology (**SMT**) (Casati & Varzi, 1999). Other ontological commitments vary across the different Whiteheadean theories; we will discuss them as appropriate. Notice that some theories appearing as Whiteheadean do not satisfy d), for example, the theory of Roeper (1997) uses an additional mereological primitive hidden in the Boolean structure. However, it can be redefined as shown in (W-P') below.

Interest in Whiteheadean space was sparked by the axiomatic treatment presented by Clarke (1981, 1985). Now, perhaps most prominently the Region Connection Calculus (RCC: Cohn, et al., 1997a, 1997b; Gotts, 1994; Gotts, et al., 1996; Randell, et al., 1992), but also Asher and Vieu (1995) and Roeper (1997) give axiomatizations of region-based space based on the Whiteheadean assumption of extensionality of *C* as required by W-Ext. Notice that W1 and W2 are common to all mereotopologies characterizing contact as reflexive and symmetric relation.

(W1) $\forall x\big[C(x,x)\big]$ (Reflexivity of *C*)

(W2) $\forall x,y\big[C(x,y)\rightarrow C(y,x)\big]$
(Symmetry of *C*)

(W-Ext) $\forall x,y\Big[\forall z\big(C(z,x)\leftrightarrow C(z,y)\big)\rightarrow x=y\Big]$
(Extensionality of *C*)

Alternative to W3, the axiom W-P can be posited (Masolo & Vieu, 1999). This is indeed the common definition of parthood for Whiteheadean space.

(W-P) $\forall x,y\Big[\forall z\big(C(z,x)\rightarrow C(z,y)\big)\leftrightarrow P(x,y)\Big]$
(Definition of *P*)

Notice that despite the similarities between the theory of Roeper (1997) and other Whiteheadean theories, the former is a purely topological account using a topologically inspired complement $-x := \bigcup_y \neg C(x,y)$ inspired by Clarke's definition. This complement is not necessarily a mereological complement as in the RCC. The complement itself serves to define the relation of proper parthood.

(W-P') $PP(x,y)\equiv_{def}\neg C(x,-y)$
(Roeper's definition of *PP*)

In algebraic terms, this is equivalent to $\neg C(x,-y)\Leftrightarrow x<y$ which holds in the RCC. Similarly, $\neg C(x,-y)\Leftrightarrow x\leq y$ induces the contact relation from parthood for Clarke (1981) and Asher and Vieu (1995) (cf. Biacino & Gerla, 1991; Hahmann, et al., 2009). Notice the $\leq$ in the second equivalence; this is caused by a definition of complements (interpretable as true set complements) in the theories of Clarke and Asher and Vieu that differs from complements in RCC.

The work of Roeper (1997) will be discussed in more detail in Section 7.3. For now, we concentrate on the purely logical theories of Whiteheadean space. For the other theories, overlap *O* and external connection *EC* are defined as following.

(W-O) $O(x,y) \equiv_{def} \exists z\big(P(z,x) \wedge P(z,y)\big)$
(Definition of *O*)

(W-EC) $EC(x,y) \equiv_{def} C(x,y) \wedge \neg O(x,y)$
(Definition of *EC*)

In addition, Whiteheadean theories define binary topological sums and intersection of regions, as well as complements. Additionally, concepts such as tangential (proper) part and non-tangential (proper) part can be defined. Most importantly, the notion of self-connectedness is definable through axiom SC-S, compare Section 3.4 and Randell et al. (1992). For Asher and Vieu (1995), self-connectedness is defined using the weaker variant SC-W—accommodating the fact that the space is not self-connected.

The theories differ in their axiomatization. Clarke (1981) utilizes second-order notions (set-theory or definite descriptions) to describe infinitary fusions, while Asher and Vieu (1995) and the RCC are first-order theories limited to finite sums. Thus, the latter two are first-order axiomatizable. Both the RCC and Clarke (1981, 1985) are atomless, while the account of Asher and Vieu (1995) is atom-tolerant. The Generalized Region Connection Calculus (GRCC: Li & Ying, 2004) has been proposed as atom-tolerant generalization of the RCC theory.

The main difference between these theories lies in the domain of discourse. Clarke (1981) and Asher and Vieu (1995) allow any kind of regular regions, while the RCC and (Roeper, 1997) only deal with regular closed regions. This in turn is reflected in the definition of the contact relation and thus in the self-connectedness of space. Either kind of theory defines the Whiteheadean contact relation as following.

(C-Weak) $C(x,y) \Leftrightarrow x \cap y \neq \varnothing$
(Weak contact)

Notice that even for theories over regular closed regions the standard topological interpretation comes closer to that of (C-Strong) if we consider regular closed regions as equivalence classes of all elements that have the same closure. For instance in the system of Roeper (1997), C-Strong is a theorem. In theories over regular open regions, contact is also characterized by C-Strong.

(C-Strong) $C(x,y) \Leftrightarrow cl(x) \cap cl(y) \neq \varnothing$
(Strong contact)

In theories allowing all kinds of regular regions, C-Strong could be used to ensure that a region and its complement are connected and thus self-connected spaces are possible.

The algebraic structure and extensionality also depend on the accepted type of regular regions. If only regular closed regions are considered, the models are quasi-Boolean algebras. The algebraic structure is more general if all kinds of regular regions are acceptable (see Section 6). With regard to extensionality, the algebraic representations show that theories concerned only with regular closed regions are *O*-extensional, while others are not necessarily, e.g. Asher and Vieu (1995) is not *O*-extensional.

Theories related to Whiteheadean space. The RCC, the most studied Whiteheadean theory, has been modified with regard to several ontological commitments resulting in new theories. Many of these modifications are studied in the algebraic counterparts of the RCC, see Section 6.1. These include mainly weakening of the atomless and extensional nature of RCC. Another modification of interest pertains to the dimensionality restriction. The INCH calculus (Gotts, 1996b) absorbs most of RCC's ontological commitments, but relaxes the limitation to a single primitive to accommodate entities of diverse dimensions in single model. Thus, the INCH calculus violates the assumptions a) and b); hence, we do not call the resulting theory Whiteheadean. In fact, the algebraic representation most likely consists of a set of contact algebras with each of the contact algebras captures the entities of a single dimension and their topological contact.

To a limited extend, Pratt and Schoop (1997) also uses Whitehead's style but is restricted to polygonal regions. The theory employs a single, unary primitive of self-connectedness, but their first-order logic comes equipped with the Boolean operations meet $\cdot$ and join $+$ which are equivalent to a partial order $\leq$ defined in the following way: $x \leq y \Leftrightarrow x \cdot y = x$. However, this partial order can be directly used to define a mereological primitive. Therefore, implicitly a mereological and a topological primitive are used.

In many respects similar to Whiteheadean spatial theories, the CRC (Eschenbach, 1999) differentiates itself in one important ontological choice. Though based on a mereological primitive of parthood and a topological primitive of disconnection, the CRC could be axiomatized using parthood and connection. However, Eschenbach (2007) showed that in the CRC the mereological relation *CoveredBy* can have a different extension than the topological relation of enclosure, Encl. In contrast, in Whiteheadean theories enclosure and parthood coincide, compare axiom W-P.

(Encl) $Encl(x,y) \equiv_{def} \forall z \left[C(z,x) \rightarrow C(z,y) \right]$
(Enclosure)

5.1.2. Boundary-Tolerant Approaches

Within Whiteheadean and any other classical, i.e. equi-dimensional axiomatizations of region-based space, regions are the only entities considered in the domain of interest. Moreover, all regions are of the same dimension; hence, boundary elements cannot be in the domain of discourse. For instance, Clarke (1981) and Asher and Vieu (1995) require that all regions have non-empty interiors which must be regions themselves. Hence, boundaries are excluded. In theories restricted to regular closed regions, there is no difference between a regions' interior and closure, i.e. boundary elements cannot be modeled either. This is a fundamental ontological commitment. Since boundaries often play an important role, other authors (Casati & Varzi, 1999; Galton, 1996, 2004; Gotts, 1996b; Smith, 1996; Smith & Varzi, 1997) incorporate them into their theories. Two different approaches have been pursued. Either boundaries are considered as being of the same dimension as regions (Smith, 1996). Other authors go a step further and treat boundaries as entities of a lower dimension (Galton, 1996, 2004; Gotts, 1996b), hence dismissing the Whiteheadean assumptions a) and b) altogether. In particular, these fall outside the scope of classical mereotopology and are not further treated in this section. We refer to Section 3.5 for a brief discussion of these theories.

Smith's (1996) theory combines a mereological primitive of parthood *P* and a topological primitive of interior parthood *IP* which together define the contact relation *C*. Within Smith's theory, points are defined as regions without proper parts using axiom (PT) from Section 3.6. Unrestricted fusions and the topological operations sum, intersection, complement, and difference are defined using definite descriptions; thereby avoiding second-order notions but not really giving a first-order axiomatization either. What makes it unique amongst the logical region-based theories of space is that in addition to regions, Smith allows boundaries as special regions without interior. Every boundary region is part of the region they bound. Moreover, boundaries are self-bounding. Using fusion, a maximal boundary *bdy(x)* can be defined.

Although Smith (1996) sticks to regions as only entities in the domain, it is clear that boundaries have special properties that distinguish themselves from other regions. However, no topological interpretation or algebraic representation of the models has been given. Without formal semantics it is, however, far from obvious what the whole set of ontological commitments are.

In an attempt to capture the notion of boundaries using only standard topological notions, Fleck (1996) comes to an understanding similar to that of Smith. However, no axiomatization is proposed by Fleck (1996) that allows an explicit comparison.

5.2. First-Order Full Mereogeometry

Classical mereotopology does not deal with any geometric notions at all. On the other end of the scale of the ontological aspect of geometric expressivity, we have several theories classified as full mereogeometries. As noted before, these theories share a common topological interpretation, which allows the reconstruction of points. Hence, we present only one set of axioms for full mereogeometry. All of the full mereogeometry consider only regions of a single dimension.

Tarski's categorical Geometry of Solids (Tarski, 1956a) is probably the best known mereogeometry, later incorporated into the *Region-based Geometry* (RBG: Bennett, 2001; Bennett, et al., 2000), a categorical first-order theory. Borgo et al. (1996) have proposed an alternative first-order theory (we refer to the theory as *BGM*) using three primitives: parthood *P* for the mereological part, the unary, quasi-topological predicate 'simple region' *SR*, and the morphological primitive of congruence *CG*. In style, the axiomatization of BGM is closer to the axiomatizations of Whiteheadean space we saw before, so we use this theory to exemplify constructing a mereogeometry from mereology, topology, and morphology. Nevertheless, we will be able to outline Tarski's theory within the scope of these axioms. The primitives of other full mereogeometries can be expressed as definitions reusing the axiomatization we present here.

The mereological part of BGM consists of a standard **CEM** (compare Section 2.2) without infinitary fusion operation (cf. Section 3.3). The topological definition of connection relies on the primitive notion of *simple regions*, *SR*.

(G-C)

$$C(x,y) \equiv_{def} \exists z[SR(z) \wedge O(z,x) \wedge O(z,y) \wedge \forall u\Big(P(u,z) \rightarrow O(u,x) \vee O(u,y)\Big)]$$

In addition, the concepts of interior part *IP* (compares to *NTPP*), maximally connected part *MCP*, and strong connection *SC* are defined in BGM. Strong connection only holds between two regions if a simple region exists that consists of parts of each region.

(G-IP)

$$IP(x,y) \equiv_{def} PP(x,y) \wedge \forall z\Big(SR(z) \wedge PO(z,x) \rightarrow O(z,y-x)\Big)$$

(Interior part)

(G-MCP)

$$MCP(x,y) \equiv_{def} P(x,y) \wedge SR(x) \wedge \neg\exists z\Big[P(z,y) \wedge SR(z) \wedge PP(x,z)\Big]$$

(Maximally connected part)

(G-SC)

$$SC(x,y) \equiv_{def} \exists uv\big[P(u,x) \wedge P(v,y) \wedge SR(u+v)\big]$$

(Strong connection)

The following axioms complete the topological structure of the full mereogeometry. They force the existence of simple regions with interiors that are simple regions as well and the existence of a maximal connected part. (G3) requires all regions to be an interior part of some simple region. Thus no universal region can exist, hence excluding atomistic models.

(G1)
$$\forall x, y, z[SR(z) \wedge z = x + y \rightarrow \exists u\big(SR(u) \wedge O(u,x) \wedge O(u,y) \wedge IP(u,z)\big)]$$

(G2) $\forall x \exists y\big[MCP(y,x)\big]$

(G3) $\forall x \exists y\big[SR(y) \wedge IP(x,y)\big]$

In the presence of a definition of complementation the topological primitive *SR* can be defined using connection. *C*, in turn, can be defined in terms of overlap and congruence *CG* (Borgo & Masolo, 2009).

(G-Comp)
$$Compl(x,y) \equiv_{def} \forall z\big[C(z,y) \leftrightarrow \neg IP(z,x)\big]$$
(Complement)

(G-SR)
$$SR(x) \equiv_{def} \forall y,z,w\big[\big(y+z = x \wedge Compl(w,x)\big) \rightarrow \exists v\big(SC(v) \wedge O(v,y) \wedge O(v,z) \wedge \neg C(v,w)\big)\big]$$
(Simple region)

(G-C')
$$C(x,y) \equiv_{def} \forall z \exists z'\big[CG(z',z) \wedge O(z',x) \wedge O(z',y)\big]$$

On the geometrical (or morphological) part, BGM uses the primitive relation *congruence, CG,* to define spheres as special kinds of simple regions. Notice that without a notion of congruence, BGM is closely related to Whiteheadean theories of space, although two primitives are necessary, e.g. parthood and connection or parthood and simple region. For its place relative to other Whiteheadean theories, see Eschenbach (2007).

(G-SPH)
$$SPH(x) \equiv_{def} SR(x) \wedge \forall y\big[CG(x,y) \wedge PO(x,y) \rightarrow SR(x-y)\big]$$
(Sphere)

This enables the theory to reuse Tarski's definition of spheres (cf. Tarski, 1956a), such as *externally tangent*, *internally tangent*, *externally diametrical*, and *internally diametrical*. That leads to a definition of two spheres being *concentric*, which in turn allows defining the ternary relation of '*sphere x is in between the spheres y and z*,' *BTW(x,y,z)*. The core notion of equidistance of two points to a third can then be defined by two pair of congruent spheres having equidistance centers (Tarski, 1956a). Assume *x* and *x'* congruent and *y* and *y'* congruent. The spheres *x* and *y* have the same center, and so do *x'* and *y,'* i.e. the centers of *x*, *y* and *x,' y'* are equidistant as follows (Borgo & Masolo, 2009). Since the first-order BGM does not reconstruct points, Tarski's notion of equidistance (cf. also Bennett, 2001; Bennett, et al., 2000) of two points from a third is not definable.

(G-SCG)
$$SCG(x,y) \equiv SPH(x) \wedge SPH(y) \wedge CG(x,y)$$
(Congruent spheres)

(G-EqD)
$$EqD(x,y,x',y') \equiv_{def} SCG(x,x') \wedge SCG(y,y') \wedge \neg P(x,y) \wedge \neg P(y,x) \wedge \neg P(x',y') \wedge \neg P(y',x') \wedge \exists z,z'\big[ID(z,x,y) \wedge ID(z',x',y') \wedge SCG(z,z')\big]$$
(Equidistance)

Now it is easy to imagine how the *between* or the *equidistance* relation can be used to define a metric system, therefore reconstructing elementary geometry (cf. Tarski, 1959). For more details on the full mereogeometries we refer to Bennett (2001), Bennett et al. (2000), Borgo et al. (1996), and Tarski (1956a). A comprehensive algebraic/topological analysis of Tarski's geometry of solid can be found in Gruszczyński and Pietruszczak (2008).

For the other mereogeometries, e.g. the primitive *CCon(x,y,z)* meaning '*x can connect y and z*' of de Laguna (1922) and Donnelly (2001) can be defined in terms of *CG* and *P*. Vice versa, *CCon* is sufficient to define *C*, *P*, and *CG*. For details see Borgo and Masolo (2009).

(G-C”) $C(x,y) \equiv_{def} \forall z \big[CCon(z,x,y)\big]$

5.3. Beyond First-Order Theories

So far, we have focused our discussion on ontological commitments of mereotopologies axiomatized in first-order logics and compared them amongst each other. For actual reasoning applications, concerns about the computational complexity of first-order mereotopologies have been raised. In general, we know that the language of first-order logic is undecidable, while reasoning with specific first-order theories might be decidable but is in most cases still highly intractable. A core issue here is the tradeoff between expressivity and tractability: a theory might be able to express a large variety of concepts, but reasoning with it is intractable, while another theory with a more limited expressiveness might be tractable with respect to the fewer sentences it can express. Hence, the development of less expressive, but more tractable theories is of great interest. This section showcases constraint calculi and modal logics as widespread approaches to build computationally more efficient mereotopological systems. First of all, we summarize the complexity results obtained for some of the first-order theories previously mentioned.

5.3.1. Computational Complexity and Decidability

Amongst the classical mereotopologies, the RCC has perhaps received the widest attention. Computational properties have been explored in much detail, especially when using the RCC as a relation calculus and reasoning with composition tables (Düntsch, et al., 2001*b*; Li & Ying, 2003; Li, et al., 2005; Renz, 1999, 2002, 2007; Renz & Li, 2008; Renz & Nebel, 1998, 1999; Xia & Li, 2006). Similarly, Smith and Park (1992) investigated the complexity of reasoning with the topological relations from Egenhofer’s n-intersection. Although the full first-order theory of RCC is undecidable (cf. Dornheim, 1998; Gotts, 1996c; Grzegorczyk, 1951), tractable segments of the compositional calculi of the RCC have been identified (Renz, 1999; Renz & Li, 2008; Renz & Nebel, 1999). Approaches based on the RCC but using languages such as constraint calculi or propositional/modal logics (cf. Section 5.3.3) also yield more efficient reasoning frameworks.

If we go beyond pure mereotopology, Davis (2006) showed that a region-based theory consisting of a connection primitive extended by a primitive of convexity is already able to express any analytical relation (which contains a very broad class of relations) invariant under affine transformations. Loosely speaking, a Whiteheadean theory of space extended by a notion of convexity is equally expressive as affine geometry. Then it is only small step to elementary/Euclidean geometry (Borgo & Masolo, 2009). However, if our only concern is the definability of concepts within an ontology (cf. Hahmann & Gruninger, 2009), we can resort to Euclidean geometry which is quite expressive and decidable (cf. Tarski, 1959), though intractable. However, elementary geometry is more a geometric than a qualitative spatial framework. On the other side if we feel mereotopology to be too restrictive for some applications, theories in between mereotopology and full mereogeometry might be a solution (cf. Section 10.3). Besides the RCC with a convexity primitive *convex hull* (Cohn, 1995; Cohn, et al., 1994, 1997b; Randell, et al, 1992), we do not know of any theory filling this gap on the expressiveness scale.

5.3.2. Composition Tables and Constraint Calculi

A common alternative to axiomatizations of mereotopology in first-order logic are qualitative spatial frameworks using *composition tables* or *Binary Constraint Networks* (BCN), short *constraint calculi*, as representation. Composition tables might be most familiar from work on the temporal interval calculus (Allen, 1983), but have also been used for spatial reasoning, for instance with the RCC (cf. Renz & Ligozat, 2004; Renz & Nebel, 2007) and the n-intersection model of Egenhofer and colleagues (see Section 7.3.2). Any set of binary relations complete with respect to composition can be compactly represented by composition tables (cf. Düntsch, 1999; Düntsch, et al., 1999; Ligozat, 2001). Composition tables and their representations as BCNs are especially well-suited for reasoning with constraint propagation mechanisms. As prerequisite, the logical theory must consist of a set of Jointly Exhaustive and Pairwise Disjoint (JEPD) binary relations. For example, the relations of RCC-8 form a JEPD lattice (Li & Ying, 2003, Randell, et al., 1992) from which we can construct a composition table. We have to be careful here to distinguish between weak and proper composition. The latter is required to obtain a relation algebra (cf. Düntsch, 1999; Renz & Ligozat, 2004, 2005) while the former is commonly used to define a composition table. Composition-based reasoning methods, e.g. determining path consistency for constraint networks or algebraic closure for relation algebras, are usually more tractable than reasoning with a full first-order language.

For representational purposes, it is easy to translate a composition table into an axiomatic theory (Eschenbach, 2001). Conversely, for many first-order mereotopology the composition table can be easily constructed from a first-order theory if we can identify a set of binary base relations. In that respect, composition tables are just compact representations of axiomatic theories. Hence, it is usually sufficient to consider (first-order) axiom systems of mereotopologies and mereogeometries for a study of ontological commitments of related constraint calculi. We do not cover relation algebraic and composition-based representations, since they can be transformed into axiomatic theories with identical ontological commitments if we follow the straightforward methods of Düntsch (1999) and Eschenbach (2001). As Schlieder (1996, p. 124) argues:

"Even though it is not always practicable to give a spatial representation formalism in this strict logical form, the idea of axiomatization is generally thought of as the ideal to achieve because it is a prerequisite for any further analysis of the formalism's properties."

Notice that the known tractable (or at least decidable) relational calculi for region-based reasoning are incapable of expressing self-connectedness (Bennett & Düntsch, 2007), a core concept when bringing mereology and topology together, but theories unable to express self-connectedness can be barely recognized as full-fledged mereotopologies.

5.3.3. Modal Logics

Undecidability of first-order logic has sparked interest in other logics for spatio-temporal reasoning. Amongst them encodings of spatial theories in terms of propositional modal logics (Bennett, 1997; Wolter & Zakharyaschev, 2000, 2002; Balbiani, et al., 2008) seem to be most popular. Balbiani et al. (2008) summarize the modal logic approaches towards mereotopology, giving topological and relational semantics for several decidable propositional theories, for example for the modal theory BRCC-8 (Wolter & Zakharyaschev, 2000, 2002), a propositional version of RCC and GRCC. This work implicitly establishes the relationship between the propositional versions of continuous and discrete RCC theories. Computational aspects of these and various other propositional spatial and temporal logics have also been analyzed by Gabelaia et al. (2005).

The accessibility relations of modal operators have also been directly employed as spatial relations in modal logics (Cohn, 1993; Lutz & Wolter, 2004) and multi-modal logics (Bennett, et al., 2002). More generally, every spatial logic is just a logic interpreted over spatial, i.e. geometrical, metric, or topological, structures. However, spatial logics—even those employing regions as primitive entities—are beyond the scope of this chapter. The interested reader may consult (Aiello, et al., 2007). In this chapter, we focus on first-order theories of space, whereas some of these ontological decisions are representative for a much larger set of region-based theories of space.

Algebraic logic (cf. Andréka, et al., 2001) has been prolific in researching the interpretation of algebraic structures within logical systems. Best known for establishing algebraic representations of sentential logic (as two-element Boolean algebra) or first-order logic (as cyclindric algebras or as relation algebras with quasi-projections), similar relationships can be exploited to construct spatio-temporal logics from algebraic structures, i.e. by giving a modal logic a topological interpretation. We, however, continue by focusing on the algebraic representation of logical theories of region-based space.

6. ALGEBRAIC REPRESENTATIONS OF MEREOTOPOLOGY

While lattice theory was still in its early development Menger (1940) realized that lattice concepts as developed by Birkhoff can be readily used for constructing pointless point-free topology. Algebraic theories share an important aspect with the logical theories discussed so far: they directly yield axiomatic theories—sometimes even equational theories (so-called lattice varieties). It has been long known that certain mereological theories have a quasi-Boolean algebraic structure (cf. Leonard & Goodman, 1940; Tarski, 1935). Tarski (1935, 1956b) showed that General Extensional Mereology (**GEM**)—a CEM with an unrestricted fusion operation added—is isomorphic to a mereological field, another name for a quasi-Boolean algebra. In that sense, mereological fields are probably the earliest algebraic structures associated with mereology and mereotopology.

In AI, many mereotopological theories are known to be representable by so-called (Boolean) contact algebras—algebraic structures consisting of a lattice and a contact relation. As we will see shortly, these are closely related to the logical system of RCC and to proximity spaces known from topology. We present different classes of contact algebras, their axiomatizations, and their relations to logical theories of mereotopology. Topological representations of contact algebras and the relationship between contact algebras and proximity spaces are discussed subsequently in Section 7.2. In both sections, we follow the terminology and axiomatizations of Vakarelov (2007).

A great benefit of algebraic theories is that they overcome a criticism by Smith (1996) that in Whiteheadean space "the mereological and topological components […] are difficult or impossible to separate formally" (p. 288). In contact algebras the lattice structure defines the mereological component while the topological component is captured by the contact relation. Apart from this achievement, algebraic representations of logical theories of mereotopologies help to relate mereotopological structures to more established mathematical structures. Eventually, algebraic representations can help us to rewrite axioms for mereotopological systems and create more computationally efficient theories, for instance equational theories of region-based space.

One comment in order concerns relation algebras (cf. Düntsch, 1999, 2005). Though applicable to compositional spatio-temporal reasoning (Egenhofer, 1994; Egenhofer & Sharma, 1993), relation algebras are distinct from the algebraic representations used in this section. The former are concerned with the algebra of relations of a theory, while we just treat the models of a theory algebraically. In this chapter, relation algebras are not in the focus; we solely concentrate on algebraic representations of models of logically specified mereotopologies.

6.1. (Boolean) Contact Algebras

Contact algebras of the form $\langle A, C\rangle$ are algebraic structures consisting of a lattice $\langle A; 0; 1; +; \cdot\rangle$ defining a partial order ≤ and a contact relation *C*. They are the algebraic counterparts of logical theories of Whiteheadean space and are also closely related to proximity spaces (cf. Section 7.2.3) known from topology. Here, we present different strengths of contact algebras and show their relationship to logical theories of mereotopology. Boolean connection algebras first appeared in Stell (2000) as counterpart of (strict) RCC models. Later, the term *Boolean Contact Algebra* (BCA) has been established instead. In a series of papers (Düntsch, 2005; Düntsch & Winter, 2004a, 2004b, 2005b), Düntsch and Winter examined contact algebras with axiomatic extensions guaranteeing extensionality, interpolation/normality/density, and connectivity of space. They also studied the relationships between contact algebras and RCC theories which led to indirect topological representations of RCC models through the topological representtation of their corresponding algebraic structures (Düntsch, et al., 2006). BCAs have been generalized to classes of contact algebras that correspond to other mereotopologies. Amongst others, *weak contact structures* (cf. Düntsch & Winter, 2005b), *contact algebras* (CA: Roy & Stell, 2002), *Generalized Boolean Contact Algebras* (GBCA: Li & Ying, 2004), *Distributive Contact Algebras* (DCA: Düntsch, et al., 2008), and *Precontact Algebras* (PCA: Dimov & Vakarelov, 2005; Düntsch & Vakarelov, 2007) have been defined and their properties studied.

The lattice within a contact algebra can be seen as algebraic description of its mereological component. The overlap relation between two regions can be defined as non-empty meet, thus expressing $O(x,y) \Leftrightarrow x \cdot y \neq \varnothing$ in algebraic terms:

$$O(x,y) \equiv_{def} \exists z\left[z \neq 0 \wedge z \leq x \wedge z \leq y\right]$$

Different strengths of mereologies correspond to various classes of lattices, e.g. pseudocomplemented lattices, distributive lattices, or Boolean lattices. Unless otherwise stated, we restrict ourselves to non-trivial (non-degenerate) lattices, i.e. lattices that contain at least another element besides 0 and 1. Moreover, we consider only bounded lattices. A contact algebra is called *complete* if the underlying lattice is complete, i.e. for arbitrary, possibly infinite, sets of lattice elements there exists a *supremum* (lowest upper bound) and an *infimum* (greatest lower bound). Completeness of lattices directly corresponds to the existence of unrestricted fusions in the logical axiomatizations, (cf. Section 3.3, Biacino & Gerla, 1991; Mormann, 1998) but is not first-order definable. Finite bounded lattices are always complete. We use general lattice theory as found in standard literature. We mainly follow Blyth (2005); more background can be found in Birkhoff (1967) and Grätzer (1998). The purely axiomatic treatment of lattices and Boolean algebras in Padmanabhan and Rudeanu (2008) is also a helpful guide.

For the remainder of this subsection we consider the following axioms. Many alternative notations and alternative sets of axioms occur throughout the literature.

(C0) $\forall x\left[\neg C(0,x)\right]$ (Null disconnectedness)

(C1) $\forall x\left[x \neq 0 \rightarrow C(x,x)\right]$ (Reflexivity)

(C2) $\forall x,y\left[C(x,y) \rightarrow C(y,x)\right]$ (Symmetry)

(C3) $\forall x,y,z\left[C(x,y) \wedge y \leq z \rightarrow C(x,z)\right]$
(Closure/Monotonicity)

(C3’) $\forall y,z\left[y \leq z \rightarrow \forall x\left(C(x,y) \rightarrow C(x,z)\right)\right]$

(C4) $\forall x,y,z\left[C(x,y+z)\to C(x,y)\vee C(x,z)\right]$
(Topological sum)

(C4') $\forall x,y,z\left[C(x+z,y)\to C(x,y)\vee C(z,y)\right]$
(Symmetric topological sum)

(Ext) $\forall x\left[x\neq 1\to \exists y\left(y\neq 0\wedge \neg C(x,y)\right)\right]$
(Disconnection)

(Ext') $\forall x\,,y\left[\forall z\left(C(x,z)\leftrightarrow C(y,z)\right)\leftrightarrow x=y\right]$
(Extensionality)

(Ext") $\forall x,y\left[\forall z\left(C(x,z)\to C(y,z)\right)\to x\leq y\right]$
(Compatibility)

Notice that C0–C3 are satisfied by all mereotopologies according to our definition in Section 2.3. Moreover, axioms C1, C2, and Ext' correspond to the logical axioms W-1, W-2, and W-Ext of Whiteheadean mereotopology introduced in Section 5.1.1.

Obviously, C3 and C3' are equivalent. In the presence of C0–C4, all of Ext, Ext', and Ext" are equivalent (Vakarelov, 2007). In general, Ext" implies Ext' which implies Ext but none of the reverse directions hold in general; more such implications are studied in Düntsch and Winter (2005b). Alternatively, Ext can be stated as '*C* is anti-symmetric.' Then, because C3 establishes the reverse direction of the implication in C4, i.e. $\forall x,y,z\left[C(x,y)\vee C(x,z)\to C(x,y+z)\right]$, a partial order can be defined on *C* (Düntsch & Winter, 2004b), which is usually the parthood relation *P* used in many Whiteheadean mereotopologies.

Now we define different strengths of contact structures that arise from mereotopological theories. Where appropriate, we reference the corresponding logical theories of mereotopology from Section 5.1.1. Irrespective whether such correspondences exist, all the classes of lattices used here are first-order definable—indeed they are definable as equational theories. Thus, the algebraic definitions directly lead to axiomatic theories.

Definition 1. (Düntsch & Winter, 2005b) A weak contact algebra $\langle A,C\rangle$ is a bounded distributive lattice $\langle A;0;1;+;\cdot\rangle$ equipped with a binary relation *C* satisfying C0–C3.

Definition 2. (Düntsch, et al., 2008; Düntsch & Vakarelov, 2007) A distributive contact algebra $\langle A,C\rangle$ satisfying C4.

Definition 3. (Roy & Stell, 2002) A contact algebra $\langle A,C\rangle$ is a distributive contact algebra where the lattice is equipped with a dual pseudocomplementation * operator, i.e. the lattice is a structure $\langle A;0;1;^{*};+;\cdot\rangle$.

Notice that because of the duality of lattices, a contact algebra can also be obtained from a pseudocomplemented distributive lattice. If $\langle A,C\rangle$ is a contact algebra where the dual pseudocomplementation * is indeed a unique complementation operation, denoted by ', the lattice $\langle A;0;1;^{*};+;\cdot\rangle=\langle A;0;1;';+;\cdot\rangle$ must be Boolean. If the lattice is Boolean and satisfies C0–C4, it is a GBCA. If it additionally satisfies Ext, we obtain a BCA.

Definition 4. (Li & Ying, 2004) A generalized Boolean contact algebra is a Boolean algebra $\langle A;0;1;';+;\cdot\rangle$ equipped with a binary relation *C* satisfying C0–C4.

Definition 5. (Vakarelov, 2007) A Boolean contact algebra is a generalized Boolean contact algebra satisfying Ext.

Notice that we deviate from the common definition of a Boolean Connection Algebra which needs to satisfy Con, e.g. in Stell (2000). We stick to the naming of Düntsch and Winter (2004b) and Vakarelov (2007) by referring to the BCA that satisfy Con as RBCAs, compare Definition 7 below. Notice further that the class of BCAs as we define them here allows overlap as only contact relation if the underlying Boolean algebra is finite-cofinite (Düntsch & Winter, 2005a). Therefore, BCAs are unsuited to construct discrete mereotopology because BCAs cannot adequately capture external connection. The more general classes of contact algebras yield mereotopologies that are not necessarily atomless (cf. Li & Ying, 2004, Stell & Roy, 2002). In general, since lattices describe only the mereological component of a contact algebra, more than a single contact relation can be defined—in particular for Boolean lattices (Düntsch & Winter, 2008).

Definition 5 generalizes to arbitrary distributive (dually) pseudocomplemented lattices: In the presence of Ext, Ext', or Ext" the lattice must be Boolean (cf. Düntsch, et al., 2006; Düntsch & Winter, 2005b).

Finally, the precontact algebras were introduced to further generalize BCAs to adjacency spaces useful for discrete mereotopology (Galton, 1999). PCAs need both axioms C4 and C4' since symmetry of *C* as required by C2 is not assumed.

Definition 6. (Dimov & Vakarelov, 2005) A precontact algebra $\langle A, C \rangle$ is a Boolean algebra $\langle A; 0; 1; '; +; \cdot \rangle$ equipped with a binary relation *C* satisfying C0, C4, and C4'.

For bounded lattices $\langle A; 0; 1; +; \cdot \rangle$ with some unary operation $'$, such as (dual) pseudocomplementation, orthocomplementation, or unique complementation (as in Boolean lattices), the following additional axioms are of importance:

(Con) $\forall x \left[\left(x \neq 0 \wedge x \neq 1 \right) \rightarrow C(x, x') \right]$
(Connection)

(Nor)
$\forall x, y \left[\neg C(x, y) \rightarrow \exists u, v \left(\neg C(x, u) \wedge \neg C(y, v) \wedge u + v = 1 \right) \right]$
(Normality)

(Nor')
$\forall x, y \left[PP(x, y) \rightarrow \exists z \left(PP(x, z) \wedge PP(z, y) \right) \right]$
(Density)

(Int)
$\forall x, y \left[\neg C(x, y) \rightarrow \exists z \left(\neg C(x, z) \wedge \neg C(y, z') \right) \right]$
(Interpolation)

(Int') $\forall x, y \left[\forall z \left(C(x, z) \vee C(y, z') \right) \rightarrow C(x, y) \right]$
(Complement. interpolation)

Assuming C0–C4, it can be easily verified that Int and Int' are equivalent, while Nor and Int are only equivalent if Con is also present; it suffices to choose $z = u$ and $z' = v$ so that $z + z' = 1$ follows. Notice, however, that in the presence of Con, $C \neq O$ must hold.

Indeed, strict RCC models correspond to the BCAs satisfying Con (Düntsch & Winter, 2004b; Stell, 2000). Algebraic variants of RCC can then be defined as *RCC BCAs* (RBCA) and *Proximity BCAs* (PBCA), compare (Düntsch & Winter, 2004b; Vakarelov, et al., 2002). The notion of a *proximity BCA* establishes the close relationship to the proximity spaces which relax the notion of contact to that of *proximity*. Notice that PBCAs are always compact, or dense, since they satisfy Nor. Contact algebras involving Heyting lattices were used by to represent the non-strict models of RCC by the complete, regular, connected Heyting algebras (Stell & Worboys, 1997). Heyting algebras are more general than Boolean algebras but more restricted than distributive pseudocomplemented lattices.

Definition 7. (Düntsch & Winter, 2004b) An RCC algebra is a BCA satisfying Con.
Definition 8. (Düntsch & Winter, 2004b) A proximity BCA is a BCA satisfying Int.

The algebraic representations in this section are often favored for their elegance over the corresponding logical theories—in particular as a way to separate mereology (the lattice structure) from topology (the contact relation) clearly, thereby addressing the criticism of Smith (1996). Moreover, studying these algebraic structures seems more manageable than studying large sets of entangled axioms.

6.2. Other Algebraic Representations

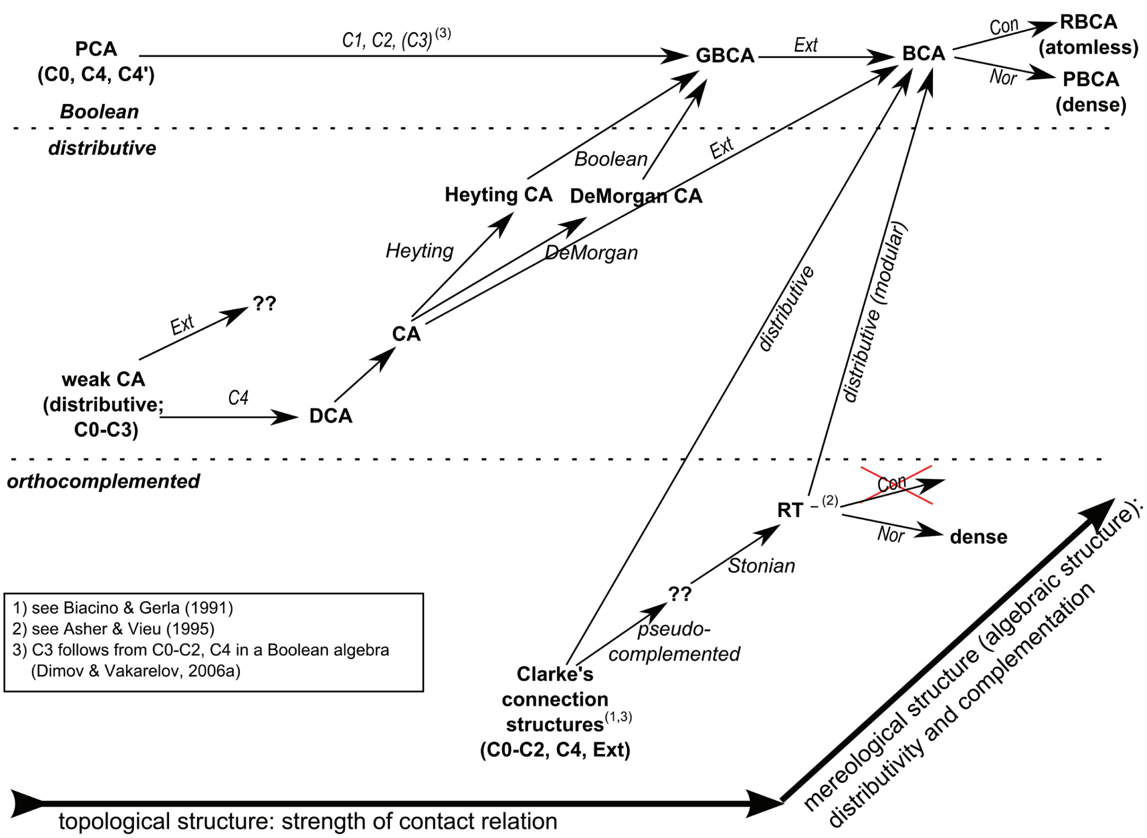
While in contact algebras, a lattice represents the mereological component and some contact relations represents the topological component of a mereotopology, in some special cases the contact relation is completely defined by the algebraic structure. The theories of Clarke (1981) and Asher and Vieu (1995) have representations where contact *C* can be solely defined in terms of the partial order of the lattices as $\neg C(x, y) \Leftrightarrow x \leq y^{\perp}$. Hence, either theory can be completely reconstructed using mereology alone. In contrast, the choice of a contact relation in contact algebras is only limited by the underlying algebra but is not completely eliminated (Düntsch & Winter, 2008).

The results from Biacino and Gerla (1991) and Hahmann et al. (2009) construct the relevant representations and prove the definition of the contact relation. The first one represents the *connection structures* of Clarke (1981)—satisfying C1, C2, Ext, and an unrestricted fusion variant of C4—as complete orthocomplemented lattices (cf. Kalmbach, 1983). The contact relation defined as mentioned together with orthocomplementation allow deriving C3 as theorem. The connection structure are complete because Clarke's use of unrestricted fusion. No assumption about atomicity is made; the structures are atom-tolerant. For Clarke's complete theory with points (Clarke, 1981, 1985) the orthocomplemented lattice is in fact a complete atomless Boolean algebra. Then the connection relation collapses to overlap, so no external connection can exist—something that Clarke was most likely unaware of. Similarly, Hahmann et al. (2009) represents a generalized theory RT^{-} of Asher and Vieu (1995)—itself a continuation of Clarke's work, but fully axiomatized in first-order logics—by Stonean portholattices, i.e. lattices that are orthocomplemented and (dual) pseudocomplemented while satisfying the Stone identity $(x \cdot y)^* = x^* \cdot y^*$. While the latter identity holds in Boolean algebras, Stonean p-ortholattices are modestly weaker than Boolean algebras; in fact, any of modularity, distributivity, unique complementation, or non-existence of external connection (as in Clarke's theory) immediately requires the lattice to be Boolean (Hahmann, et al., 2009). Therefore, RT^{-} exhibits all desired algebraic properties of the similar RCC, except for distributivity. The failure of distributivity is owed to the inclusion of regular open and regular closed sets in the theory, which prevents mereological extensionality. With RT^{-} making no assumption about continuity, it can be seen as open-closed variant of GBCAs. The precise relationship between RCC and RT^{-} via the skeleton has been established in Winter et al. (2009). A theory such as RT^{-} directly gives an equational theory which might lend itself to answer certain queries more efficiently. It needs to be further investigated whether even some standard first-order theorem proves provide an advantage when reasoning with such an equational theory compared that an equivalent non-equational theory such as the original axioms of RT^{-}. In addition, the question whether certain other contact algebras can be expressed in terms of equational theories remains open.

6.3. Map of Algebraic Theories of Whiteheadean Space

Figure 2 maps the algebraic theories into a two-dimensional space. The first dimension measures the strength of the contact relation (the topological component), while the second dimension displays the strength of the underlying lattice (the mereological component). The latter is further divided into distributivity, a property often assumed but not further discussed, and the existence of different strengths of unary complementation operations on the lattices (orthocomplemented, pseudocomplemented, uniquely complemented). Interestingly enough not so much distributivity seems essential for the large set of contact algebras,

Figure 2. Map of the algebraic theories of Whiteheadean space within several dimensions



but instead pseudocomplementation. For finite lattices, pseudocomplementation follows directly from distributivity. More generally, it forces the existence of topologically closed complements. Notice that as soon as the connection relation satisfies either of Con or Int, all the models are infinite. All other theories allow atomistic variants by adding axiom AT from Section 3.6.

The map exhibits the relationship between the few non-distributive theories (at the bottom) and the large set of distributive theories. It makes explicit that lattices with a unique contact relation are no more than a special case of other contact algebras; the contact relation is just directly dependent upon the lattice structure. As another benefit, this map highlights few 'missing' theories such as p-ortholattices (a weaker variant of RT^{-}), extensional weak contact algebras, and Heyting, DeMorgan, or Ockham contact algebras (all weaker variants of GBCAs where the complementation is not unique). These theories have to our knowledge not yet been fully analyzed. Further analysis might reveal that these either collapse to one of the stronger theories or that they are inadequate for construction mereotopological theories for other reasons. Devising similar maps of other region-based theories will also be of interest for future work (compare the discussion in Section 10.2). Notice further the close relationship between the extensionality of the contact relation and the mereological structure, e.g. extensionality forces a CA and anything stronger to be Boolean. Notice further the inconsistency of RT^{-} with Con.

All the contact algebras could be also interpreted as proximity algebras by changing the interpretation of *C*. In that way, proximity algebras are not special compared to contact algebras and there is no reason to assume that all proximity algebras must be Boolean, as often found in the literature.

7. TOPOLOGY FOR MEREOTOPOLOGY

Recall one of the original motivations of mereotopology: finding a suitable, cognitive adequate formalization of space using regions instead of points as primitives. In that light, point-set topological models of space seem inadequate—although they might use sets of points as primitives, standard operations such as interior, boundary, complement, neighborhood, etc. are still defined using points (cf. Roeper, 1997). Additionally, the use of any set theory raises suspicion in parts of the ontology community given its second-order nature. Though we might restrict ourselves to the first-order definable fragment of set theory, a great benefit of mereotopology would be an adequate theory of space that can replace point-wise and set-theoretic axiomatizations. In that regard, topological theories of mereotopology are only of limited use for QSTR. Moreover, unlike algebraic formalisms, topological theories do not directly yield efficient axiomatic theories.

Nevertheless, there is a need for (point-set) topological models within the study of mereotopologies. Topological representations help to understand the models of mereotopology. Often we think about mereotopological models in the traditional topological sense and judge a theory based on their topological interpretation. Oddly enough, such an argument implies that we actually understand point-based topological models better than mereotopological models. Though possibly true for mathematicians, it does not apply to a broader audience, or do humans tend to conceive space in a point-based way since most of us are taught classical Euclidean geometry? However, it is naïve to assume that the point-based models can give us a complete understanding of a mereotopology. We have no proof that only the topological models are relevant. Non-standard models are often forgotten. Only through a careful investigation of these models, can we understand whether there are intuitive mereotopological models without corresponding point-set interpretations. As far as we know, only few studies of the non-standard models of mereotopologies have been conducted, identifying some counter-intuitive models such as space-filling curves or 'completely holed regions' (Düntsch & Winter, 2004a), but obtaining topological representation does not guarantee the existence of corresponding spatial interpretations. Indeed, we can construct 'abstract' topological spaces from algebraic structures without a meaningful spatial interpretation (Johnstone, 1983). Since the topological representation of Boolean algebras by Stone (1936), the duality between topological spaces and algebraic structures has been known—leading to the fact that topological representations are just another way of looking at algebraic structures. On the other side, such representations emphasize that topology does not depend on points (Mormann, 1998). Nevertheless, topological representations can assist us in finding meaningful spatial interpretation of all models of a mereotopology. However, it is only one of the tools available.

This section discusses some of the key ideas in using topology for mereotopology. First, work on topological models of mereotopologies is presented in Section 7.1, while so-called embeddings are discussed in Section 7.2. Both sections focus on the role of topological models in comparing mereotopologies. In particular, the relationship between axiomatic theories of Whiteheadean space (cf. Section 5.1.1), contact algebras (cf. Section 6.1), and point-set topological models is outlined. Subsequently, Section 7.3 briefly introduces work on purely topological accounts of mereotopology.

7.1. Topological Models

When thinking about topological interpretations of region-based theories of space, there are two main directions pursued. The easier way constructs standard point set models, i.e. showing that a certain logical or algebraic system admits a classical interpretation of regions as point sets (*satisfiability*). Most commonly, models in Euclidean (vector) spaces $\mathbb{R}^n$, in spaces of rational numbers $\mathbb{Q}^n$, or in $\mathbb{Z}^n$ (for discrete mereotopologies) are given. We discuss these so-called *standard models* of region-based theories in this subsection, while the next subsection concentrates on the generally more challenging task of giving a full representation for all the models of a region-based theory.

Constructing models, i.e. showing satisfiability, for a certain logical or algebraic theory helps us verify that some intuitive class of (intended) models are covered by the theory. For mereotopological theories, these intended models usually rely on point sets where contact means either $\mathrm{cl}(x) \cap \mathrm{cl}(y) \neq \varnothing$ for theories limited to regular open sets or $x \cap y \neq \varnothing$ for theories allowing arbitrary regular sets or theories limited to regular closed sets (where $x \cap y = \mathrm{cl}(x) \cap \mathrm{cl}(y) \neq \varnothing$). For logical axiomatizations such topological models in $\mathbb{R}^2$ have been considered for Whitehead's theory (Gerla & Miranda, 2009), for polygonal mereotopology (Pratt & Schoop, 1997), and for the RCC (Gotts, 1996a). For algebraic theories it is usually shown that the set of all regular closed (or open) sets of some (restricted) topological space is in fact a model of a mereotopology of interest (cf. Stell, 2000; Vakarelov, 2007). This argument dates back to the famous construction of a Boolean algebra over the regular open sets of a topological space (cf. Halmos, 1963). For discrete mereotopologies and their algebraic counterparts, discrete (raster) models of GBCAs and the theory of Galton (1999) have been constructed in the digital plane $\mathbb{Z}^2$ (Li & Ying, 2004). Likewise, the full mereogeometries have been compared with respect to their models in $\mathbb{R}^n$ (Borgo & Masolo, 2009), see Section 5.2 for details. However, Fleck (1996) investigated boundaries of topological models of mereotopology and concludes that $\mathbb{R}^n$ interpretations fail to provide intuitive models.

Unintended models are more challenging to find, we often have to think outside the box of standard spatial models. For example completely holed regions (all regions have infinitely many holes) as RCC algebras (Düntsch & Winter, 2004a) seems counter-intuitive, prompting (Li & Ying, 2003) to recommend disallowing holed regions in RCC altogether. An interpretation of the RCC contact relation as 'distance-less-than-or-zero-meter' is given in Dong (2008). This interpretation reminds us of proximity relations as defined in Section 7.2.3. However, it is unclear whether such proximity interpretation can be avoided without referring to points or set-theoretic notions explicitly.

7.2. Topological Representations

It is well known that the open subsets τ of a topological space form a complete lattice. If the lattice is distributive it is called a *frame* or *local* (cf. Gerla, 1995; Johnstone, 1983). In distributive lattices, points may be defined in terms of maximal ideals (or ultrafilters), as shown in Stone's representation theorem for Boolean algebras (Stone, 1936), extended to distributive lattices (Priestley, 1970; Stone, 1937) and complete lattices (Urquhart, 1978). Already, earlier, it was earlier known that algebraic structures give rise to topological spaces (McKinsey & Tarski, 1944; Wallman, 1938). Studying these topological representations gives insight into the corresponding algebraic structures. Equally, we can study the topological spaces arising from algebraic counterparts of mereotopologies to understand the mereotopological theories better, e.g. Grzegorczyk (1960) was interested in the topological spaces of his theory of a mereological field (quasi-Boolean algebra) with a contact relation—a predecessor of today's BCAs. Here we give a taste of more recent results that focus on the different classes of contact algebras and their equivalent logical theories. First, we introduce some terminology for topological spaces necessary for the remainder of the section.

7.2.1. Properties of Topological Spaces

Classical point-set topology often characterizes topological spaces by their adherence to separation axioms. We restate the ones necessary for our discussions, for more background see Munkres (2000).

(Axiom T_0) Given two points of a topological space, at least one of them is contained in an open set not containing the other.
(Axiom T_1) Of any two points one lies in an open sets not containing the other.
(Axiom T_2) For any two points, there are disjoint open sets, each containing just one of the two points.
(Axiom T_3) For a closed set S and a point p not in S, there are disjoint open sets, one containing S and the other containing p.
(Axiom T_4) For any two disjoint closed sets, there are two disjoint open sets each of which contains one of the closed sets.

A topological space satisfies T_1 if all finite point sets are closed. In point-free topology, this axiom is tricky considering that arbitrarily many points could be in a region. A space satisfying T_2 (T_1 and T_3; T_4) is called *Hausdorff* (*regular*; *normal*). A regular and normal space is called a *T_4-space*. Assuming T_1, a normal space is always regular. If a space has a basis of regular open sets, it is called *semi-regular*.

Topological representations of region-based theories of space often use weaker properties unfamiliar to traditional topologists. This is owed to the fact that separation is point-based, while point-free topological representations need to be more general. A topological space is *weakly regular*, a point-free version of regularity, if it is semi-regular and for each non-empty set *S*, there is a non-empty set S' with $cl(S') \subseteq S$. Restricting T_4 to the regular closed sets yields a special kind of normal space, called *κ-normal*. From Düntsch and Winter (2004b), we know the following implications:

X is normal → X is κ-normal → X is regular
→ X is weakly regular → X is semi-regular

Moreover, a space *X* is *connected* if it is not representable as the sum of two non-open disjoint empty sets. *X* is *compact* if for every non-empty family of closed sets $\{A_i \mid i \in I\}$ with every finite subset $J \subseteq I$ having a non-empty intersection $\bigcap\{A_i \mid i \in J\}$, the intersection $\bigcap\{A_i \mid i \in I\}$ is also non-empty (*finite intersection property*, cf. Düntsch, et al., 2008). *X* is *locally compact at a point P* if there exists a compact subspace $S \subseteq X$ that contains an open neighborhood of *P*. If *X* is locally compact at every point, it is simply called *locally compact*. Compact spaces are always locally compact (Munkres, 2000).

7.2.2. Topological Embeddings of Contact Algebras

Several variants of contact algebras have been embedded into topological spaces with the standard topological contact relation defined on the set of regular closed regions. We give some of these results, but refer for details to Dimov and Vakarelov (2006a, 2006b) and Vakarelov (2007). We only want to demonstrate how logical and algebraic theories can be grounded in topological interpretations in principle.

Theorem 1. (Düntsch & Winter, 2004a) Let $\langle X, \tau \rangle$ be a topological space with the contact relation *C* defined on $B = \mathrm{RegCl}(X)$. Then,

a. If X is semi-regular, X is weakly regular if and only of B satisfies Ext.
b. X is κ-normal if and only if B satisfies Nor.
c. X is connected if and only if B satisfies Con.
d. If X is compact and Hausdorff T_2, then B satisfies Ext and Nor.
e. If X is Hausdorff T_2 and normal, then $\bar{B} = \mathrm{RegOp}(X)$ satisfies Nor.

These results let us deduce topological representations of classes of contact algebras as in f) to i), while j) and k) give embeddings into more restricted spaces.

Theorem 2. (Düntsch & Winter, 2004a, 2004b) Let $\langle X, \tau \rangle$ be a topological space with the contact relation *C* defined on $B = \mathrm{RegCl}(X)$. Then,

f. B is a GBCA if and only if X is any such topological space.
g. B is a BCA if and only if X is weakly regular.
h. B is an RBCA if and only if X is weakly regular and connected.
i. B is a PBCA if and only if X is weakly regular and κ-normal.
j. Every BCA is isomorphic to a dense substructure of some $\langle \mathrm{RegCl}(X), C_\tau \rangle$ for a weakly regular T_1-space.
k. Each RCC model (or alternatively RBCA model) is isomorphic to a substructure of some $\langle \mathrm{RegCl}(X), C_\tau \rangle$ for a connected weakly regular T_1-space.

However, not every model of RBCA is directly embeddable into a regular T_1-space (Düntsch & Winter, 2004b). Instead, weak regularity is necessary and sufficient, compare Theorem 2k). In general, the separation axioms are not applicable to topological representations of BCAs. Though every BCA might be embeddable in a certain T_1-space, there are embeddings into spaces that are not T_1 or even T_0 (cf. Eschenbach, 1994). Results for weaker, but still distributive CAs show that U-extensional (underlap $U(x,y) \Leftrightarrow x + y \neq 1$, a dual to overlap, is extensional) distributive contact algebras are embeddable in BCAs over a semi-regular T_0-space (Düntsch, et al., 2006, 2008). Notice that not all distributive CAs are topologically representable. This hints that distributivity alone is too weak to build classical mereotopology. Only the combination with pseudocomplementation ensures representability. For all finite models, pseudocomplementation automatically follows from distributivity. Topological embeddings of precontact algebras have been studied by Düntsch and Vakarelov (2007).

7.2.3. Proximity Spaces

Proximity relations appeared independently from point-free topology in work by Efremovič (1952); see also Naimpally and Warrack (1970). Proximity spaces are constructed from a set and a binary relation expressing that two entities are 'close' to each other. Proximity spaces are intermediates to topological spaces and contact algebras as we outline here. We follow the account of Vakarelov (2007) on the use of proximity spaces for mereotopologies and contact algebras.

A *proximity* (or *nearness*) relation δ as defined by Efremovič (sometimes called an *Efremovič proximity*) must satisfy the condition Prox'. The relation δ is only used in the context of proximity spaces; we rewrite the axioms in terms of the familiar relation *C*, e.g. Prox' can be easily rewritten as Prox using the lattice terms and *C*.

(Prox') $A \cap B \neq \varnothing \Rightarrow A \delta B$

(Prox) $x \cdot y \neq 0 \Rightarrow C(x,y)$

The contact relations we considered as part of contact algebras trivially satisfy Prox. However, the reverse implication as common for the standard topological contact relation is not required here. All proximity spaces must further satisfy the axioms C0–C4 defining contact relations in the previous section.

Such a system $\langle X, C\rangle$ with X being a non-empty set and a binary relation C satisfying C0–C4 and Prox is a *Čech proximity space* (cf. Čech, 1966). The relation between CAs and Čech proximity spaces has been pointed out by Vakarelov (2007). If a Čech space additionally satisfies E', we obtain an *Efremovič proximity space*. We can rewrite it as the first-order sentence E. This is equivalent to the interpolation axiom Int considered in the context of CAs. Int, Int', or Nor are equivalently sufficient to turn a Čech proximity space into an Efremovič proximity space.

(E') $A\overline{\delta}B \Rightarrow \exists C\left[A\overline{\delta}C \wedge (X-C)\overline{\delta}B\right]$

(E)
$\forall x, y\left[\neg C(x,y) \rightarrow \exists z\left(\neg C(x,z) \wedge \neg C(y,z')\right)\right]$

We can give semantics to contact algebras by constructing proximity spaces that do not define a traditional contact relation, i.e. not both directions of $A \cap B \neq \varnothing \Leftrightarrow C(A,B)$ are satisfied, as shown in Vakarelov (2007). A proximity spaces induces a topology when using the topological closure $Cl(A) = \left\{x \in X \mid \{x\}\delta A\right\}$ of the proximity relation. If we consider the subset of regular closed sets thereof, the regular closed sets not only satisfy the axioms C0–C4 of contact algebras, but also Nor and Ext (cf. Vakarelov, 2007). Thus, we obtain a model of a Proximity BCA (PBCA) defined earlier. A proximity space is called separated if and only if $A\delta B$ implies $A = B$. In a separated proximity space, the topology induced by the closure $Cl(A) = \left\{x \in X \mid \{x\}\delta A\right\}$ is in fact Tychonoff (completely regular Hausdorff), and if the space is not separated, the space induced by the closure is still completely regular—a stronger notion than regularity.

The relationship between (separated) proximity spaces and contact algebras has been investigated in great detail in Vakarelov et al. (2001). In principle, proximity spaces behave like contact algebras, only their intended topological interpretation differ. Although typically Boolean lattices are considered in combination with proximity relations, proximity algebras can be built from bounded lattices in combination with a proximity relation that satisfies a subset of all the axioms considered for contact algebras.

7.3. Purely Topological Theories

After our treatise of topological models and representations of mereotopology, we now shift the attention even further to consider frameworks that define mereotopological relations using topology alone. Stone (1936) fostered an interest in point-free topology based on the insight that topology does not depend on points (Mormann, 1998). Equally, Menger (1940) anticipated the development of point-free topology early on. Johnstone (1983) interest in point-free (he calls it—pun intended—pointless) topology was motivated by an attempt to obtain a generalization of point-set topology to a point-free theory of *locales* (see Gerla, 1995 for a detailed introduction). Within locales, points may be defined as prime ideals (or dually as filters) very similar to the construction used by Stone (1936) and to the topological representations of contact algebras in Section 7.2. However, these ideas have not resonated much in the mereotopological community. Instead, besides logical and algebraic theories of mereotopology, the main purely topological theories in the field are based on traditional point-set topology. We will review two of them here. First, we go into more depth of Roeper's theory (Roeper, 1997) as topological alternative of building Whiteheadean mereotopology. Not only are its ontological assumptions very similar to those of the RCC, but it can also be considered as a proximity space. We follow this idea (due to Vakarelov, 2007) in the next subsection. Subsequently, we present the n-intersection model, a topological theory created by Egenhofer and colleagues. Some variations thereof as well as an extension to multi-dimensional mereotopology by the Dimension Extended Method (DEM) are outlined. Again, this is a topological formulation adhering to the ontological commitments of Whiteheadean space. Another point-set topological account of mereotopological relation (Schmolze, 1996) is based on regular open sets, similar in the idea to Tarski's work (Tarski showed that regular open sets are models of his theory).

7.3.1. Roeper's Region-Based Topology

A purely topological, but first-order axiomatizable (without the infinite join and meet operations) theory of region-based space was presented by Roeper (1997). Despite being built from topology, there is a close resemblance to the algebraic and logical theories of Whiteheadean space, cf. Section 5.1.1. It is based on a primitive contact relation, denoted by ∞, and a primitive unary predicate of *limitedness*. The underlying mereological structure is a Boolean algebra of regions—indeed the first five axioms, i.e. A1–A5 of Roeper (1997) correspond to C0–C4 from Section 6.1. For that reason, we omit them here and use C0–C4 instead. Given the intended interpretation of $x\infty y$ as '*x and y are at least infinitesimally close*,' the similarity to proximity spaces which satisfy exactly the same axioms comes at no surprise. Indeed, postulate Prox should always hold in the intended interpretation of ∞. In addition to the contact relation, all regions are *limited*, i.e. bounded. We use the following axiomatization, using *C* instead of ∞.

(B1) $Lim(0)$

(B2) $\forall x, y\left[Lim(x) \wedge y \leq x \rightarrow Lim(y)\right]$

(B3) $\forall x, y\left[Lim(x) \wedge Lim(y) \rightarrow Lim(x+y)\right]$

Contact and limitedness are closely related in Roeper (1997) by the following two axioms. B4 states that contact between regions requires contact between limited regions (interpolation), while B5' is some sort of compactness condition, as we will see shortly.

(B4)
$\forall x, y\left[C(x, y) \rightarrow \exists z\left(Lim(z) \wedge z \leq y \wedge C(x, z)\right)\right]$

(B5') $\forall x, y[\left(Lim(x) \wedge y \neq 0 \wedge x < y\right)$
$\rightarrow \exists z\left(Lim(z) \wedge z \neq 0 \wedge x < z < y\right)]$

So-called *local proximity spaces* (Leader, 1967) satisfy besides B1-B4 also B5 and E (see Vakarelov, 2007 for details).

(B5) $\forall x\left[Lim(x) \rightarrow \exists y\left(Lim(y) \wedge \neg C(x, y')\right)\right]$

In algebraic terms, Vakarelov (2007) defined a *Local Contact Algebra* (LCA) as a CA with a predicate of boundedness, *Lim(x)*, where the extension of *Lim* contains a subset of regions satisfying B1–B5. If a LCA additionally satisfies Ext and Nor, we obtain Roeper's axiom B5' as theorem. Hence the axioms B1–B5 together with Ext and Nor give and equivalent definition of Roeper's region-based topology.

The notion of limitedness allows defining points as limit of sets; in particular, points are *collocated ultrafilters* (Roeper, 1997). Besides this marginal difference to the usual use of arbitrary ultrafilters, his construction of points resembles the previously discussed ones for logical and algebraic theories. An important part of Roeper's work is establishing a one-to-one correspondence between the *limited regions* in his region-based topology and *compact regular closed sets* in traditional point-based topology. That leads to a grounding of Roeper's theory of region-based space in the locally compact T_2-spaces. The theory can be extended by requiring infinite divisibility, simply postulated by:

(B6) $\forall x\left[x \neq 0 \rightarrow \exists y\left(y \neq 0 \wedge y < x\right)\right]$

In addition, Roeper (1997) defines what it means for a region to be *coherent* (self-connected) and *convex* in terms of the primitives. We restate them in algebraic terms. These definitions go beyond what is usually found in Whiteheadean mereotopology.

(B-SC)
$$Coherent(x) \equiv_{def} \forall y, z[(y \neq 0 \wedge z \neq 0 \wedge y + z = x) \rightarrow C(y, z)]$$

(B-Conv)
$$Convex(x) \equiv_{def} \forall y, z[y + z = x \rightarrow \exists v\left(v < x \wedge C(v \cdot y, v \cdot z)\right)]$$

Obviously, a convex region must be coherent (self-connected). Roeper's theory is accompanied by two simple existential axioms. B7 requires the mereology being a non-degenerate Boolean algebra as in most of the region-based theories of space, while B8 demands the universal region to be self-connected.

(B7) $\exists x\left[x \neq 0 \wedge x \neq 1\right]$

(B8) $Coherent(1)$

In the presence of these two axioms, B6 is in fact a theorem. Roeper (1997) further explores continuity and the restriction of region-based topologies to continua. Unfortunately, continuity, standard contact, and boundaries seem do not seem to fit well together in a single theory, compare the discussion of Breysse and De Glas (2007) and Fleck (1996).

As a side note, the primitive notion of limitedness can be substituted by the topological notion *compactness*, reducing Roeper's theory to a mereology where the contact relation can be expressed in terms of the Boolean algebra as $\neg C(x, -y) \Leftrightarrow x < y$ (Mormann, 1998). This relates the topological theories to the algebras in Section 6.

For LCAs embeddings in locally compact semi-regular T_0-spaces exist (for details see Vakarelov, 2007). Topological properties for the additional axioms Con, Ext, and Nor are analogue to Theorem 1 from Section 7.2.2. Hence, Roeper's region-based topology is accounted for by the *locally compact Hausdorff* spaces (Roeper, 1997).

7.3.2. Models of n-Intersection

GIS has been traditionally a major force in the advancement of region-based theories of space. Long before the majority of work in QSR, Egenhofer (1989) and Egenhofer and Franzosa (1991) proposed the *4-intersection* model based on the set intersections of regions interior, A°, and region boundaries, δA, resulting in four feasible topological relations between two regions. However, this turned out to be insufficient to capture all possible topological relations of $\mathbb{R}^n$ with complements being especially difficult to define. However, Egenhofer (1991) extended this model to *9-intersections* taking the set intersection of region interiors, boundaries, and complements, A^{-1}, into account. Then, it is easily verified that indeed all topological relations are expressible in these terms—for each point set A the sets A°, δA, and A^{-1} partition the underlying space completely. Of the theoretically possible 81 combinations, with some additional restrictions these can be reduced to eight topological relations that are directly mappable to the JEPD relations *EQ*, ¬*C*, *O*, *EC*, *TP*, TP^{-1}, *NTP*, NTP^{-1} of RCC-8 and RT^{-}. Notice that indeed the 4-intersection model is already sufficient to describe the five RCC-5 relations (Cohn & Renz, 2008). Three of the restrictions imposed on the 9-intersection model are similar to those in Whiteheadean space: equi-dimensionality of all regions (and space), regions are always non-empty, and only regular closed regions are considered (Egenhofer, 1991). The fourth condition requires each of the interiors, boundaries, and complements to be self-connected, going beyond what Whiteheadean space demands. This last assumption excludes holed objects; holes have only been considered within the scope of the 4-intersection model in Egenhofer et al. (1994). Contrary to Whiteheadean space, regions assumed primitive, instead standard point-set topology is used as underlying formalism, with an assumed interpretation in $\mathbb{R}^n$. Therefore, we can see the n-intersection model as a mereological framework build from topology. The necessary changes to obtain a discrete version of the 9-intersection model have been discussed in Egenhofer and Sharma (1993b). In the discrete version, adjacency needs to be accounted for separately (compare the use of adjacency by Galton, 1999), whereas boundaries need to have some 'thickness.' This results in a total of 16 relations, while a discrete 4-intersection model contains only five JEPD relations.

The set intersection approach has been transferred to include points, lines, and regions in Egenhofer and Herring (1991). It results in groups of possible topological relations between elements of different dimensionality. However, this set of possible relations is prohibitively large. In the *Dimension Extended Method* (DEM: Clementini, et al., 1993), instead of only distinguishing between empty and non-empty intersection, the dimension of the intersection is also taken into account, resulting in a more fine grained classification of possible relations while also reducing the number of feasible relations to 52 consisting of 9 area/area, 17 line/area, 3 point/area, 18 line/line, 3 point/line, and 2 point/point relations. A three-dimensional framework disregarding points would therefore be limited to 44 relations—still quite complex and most likely an important factor why many theories restrict themselves to equi-dimensional entities. Line/line (Clementini & Di Felice, 1998) and region/line relations (Egenhofer & Mark, 1991, 1994, 1995a) received special attention. This again relates to the idea of a boundary-toleration theory (cf. Section 5.1.2) as compromise between the simplicity of classical mereotopology and complexity of multi-dimensional mereotopology.

Clementini and Di Felice (1995) compared the point-based n-intersection model to an alternative calculus based model, proposed in Clementini et al. (1993). The calculus-based model is indeed very similar to the RCC-5. It builds on the five relations 'touch,' 'in,' 'cross,' 'overlap,' and 'disjoint,' but allows lines and/or points for the relations where they make sense, e.g. overlap is applicable to area/area and line/line relations only. Clementini et al. (1993) show that their calculus is expressive enough to represent all relations of the 9-intersection model combined with the DEM.

8. GRAPHS FOR MEREOTOPOLOGY

For certain applications such as route finding, route optimization, etc. graph-based representations of space have been proven to be of great help. Tessellations of space such as raster, triangulations, and Voronoi diagrams are of widespread use within GIS (see Chen, et al., 2001; Li, et al., 2002). In this context, the basic entities are *cells* (usually thought of as smallest elements in the sense of indivisible regions) and *cell complexes* (cf. Frank & Kuhn, 1986). The concepts are borrowed from algebraic topology (cf. Faltings, 1995), which defines *simplices* (the n-dimensional equivalent of triangles) and *(simplicial) complexes* (triangulations of space as collections of simplices (Frank, 2005). Modeling discrete space based on such tessellations of space is common in the GIS community. We refrain from covering this large area of research; instead, we just outline the relationship to axiomatic theories of continuous and discrete mereotopology. In particular, the graph representations of such tessellations shall be of interest here, since they exhibit interesting properties that might be helpful for building intuitive theories of discrete mereotopology that avoid the paradoxical nature of atoms experienced in discrete axiomatic theories of Whiteheadean space.

Tessellations of space are directly representable as graphs where every cell is a vertex in the corresponding graph while adjacency of cells is modeled by edges. Such graph representations are independent of the particular kind of adjacency or contact relation between cells. To keep the graph simple, adjacency shall only be modeled on the cell level. Arbitrary (non-atomic) regions made up of cells can then be thought of as subgraphs. For a grid/raster interpretation, the use of either 4-adjacency, 6-adjacency, or 8-adjacency is most common and natural (cf. Roy & Stell, 2002).

A mereotopological theory along these lines has been proposed by Galton (1999) using two sorts: one for cells (vertices) and one for regions (subgraphs) including a null region (the empty set), atomic regions (singletons), the universal region (the full graph), and arbitrary subgraphs. Not considering cells, the algebraic structures of regions are specialized GBCAs (cf. Section 6.1; Galton, 1999; Lin & Ying, 2004). Galton (1999) also constructs standard topological concepts for his theory of cell adjacency. However, the properties of the interior and closure operations differ significantly from their topological counterparts, i.e. $cl(x) \subseteq y \Leftrightarrow x \subseteq \mathrm{int}(y)$ holds—an impossible theorem for point-set topology (consider the example $\mathrm{int}(y) < y = x = cl(x)$ in a topological space). Problems arise because Galton (1999) wants to ensure atoms are connected to their complements. Then the Whiteheadean definition of parthood in terms of connection fails unless we allow atoms to be part of their complements. It would be interesting to see whether we can alter the theory to differentiate two notions of self-connectedness: a weak (using adjacency, indeed an atom and its complement should be adjacent) and a strong one (using connection in the sense of sharing a point; an atom would then not be strongly self-connected). Galton's so-called *adjacency spaces* resemble the distinction between connection and weak contact in the theory of Asher and Vieu (1995). Moreover, they are captured by the precontact algebras mentioned briefly at the end of Section 7.2.2.

9. APPLICATION DOMAINS OF MEREOTOPOLOGY

Though theoretical work on mereotopology is often motivated by practical applications, these applications remain sparse. Only recently, more work on specific applications of mereotopology and mereogeometry emerged. Most of the work known chooses a set of ontological commitments reasonable in the domain and then customizes the ontologies to fit the envisaged applications. The benefits of studying practical applications do not only lie in proofs of the usefulness of the theoretic work, but more importantly, we can gain more insight into the ontological choices relevant in real world. In practice, it has turned out that mereotopology by itself is rarely useful; instead integration into larger ontologies or reasoning frameworks is necessary.

Amongst the main areas for applications of mereotopologies GIS, navigation, computer vision applications, biological and medical ontologies, and applications in (computational) linguistics, e.g. for language understanding, are widely recognized. Less known is the work using mereotopologies for product engineering and product modeling.

Apart from these specific areas of applications, it is undisputed that upper ontologies need to incorporate spatial and spatio-temporal concepts. Most proposals use some mereotopological component, e.g. the BFO (Grenon, 2003), DOLCE, SUMO, and openCyc. However, it is often not clear what kind of ontological assumptions are made in the respective mereotopologies—the relation to earlier theoretical work is usually vague while the actual axiomatizations are rarely scrutinized with respect to their models. More recently, the OntoSpace project surveyed the spatial ontologies of the previously mentioned upper ontologies in Bateman and Farrar (2005).

9.1. GIS and CAD

Traditionally, the GIS community has been a driving force in the advancement of qualitative theories of space with the objective of expressing topological relations that humans use and of applying high-level reasoning to it. For a discussion of the role of ontologies—including spatial ontologies—in GIS we refer to Fonseca et al. (2000, 2002). In the context of built structures/environments, Bittner (2000) used a mereotopological theory with rough location relations to model parking lots. He explored the necessity of boundaries in general, but also the necessary distinction of different kinds (bona fide and fiat) of boundaries to capture the space naturally.

One particularly promising field for applications of mereotopologies are ontologies for CAD (Computer-Aided Design) software. Not only are many representations of space in such software systems region-based, but there is further a need to exchange models between different CAD systems. For such a model exchange, meaning of terms must be preserved and translations between different ontological commitments must be bridged while avoided much loss of information, e.g. one software system treating all regions as open but having lines as separate entities and another system supporting lines only as boundaries of regions must overcome their ontological differences to enable data exchange. On a higher level, we want to know when a lossless translation is possible or what specific data will be lost.

9.2. Bio-Ontologies

Biological, biomedical, and medical research has shown considerable interest in ontologies to represent various relations, e.g. anatomical, genetical, or simple spatial and spatio-temporal relations for describing medical images (X-rays, tomographic images, etc.). Not surprisingly, many relations occurring in these fields are of mereological and mereotopological nature. The ontologies in the Open Biomedical Repository (OBO) use basic spatial and spatio-temporal relationships defined in the BFO and the OBO *relation ontology*. The mereotopological and mereogeometrical concepts of the OBO relation ontology have been explored in Smith et al. (2005) and Bittner (2009). The relation ontology framework for general biological and medical ontologies also contains location relations, while an explicit distinction between contact and adjacency (external connection) is made. Moreover, all mereotopological and mereogeometrical relations are temporal, thus allowing for change over time. Bittner (2009) gives an example how anatomical relations can be expressed in this framework. Body parts are also considered by Donnelly (2004) from a mereotopological perspective where holes play an important role. A comprehensive ontology of anatomy based on mereological, topological, and orientation relations also exists (Rosse & Mejino, 2003). Related works in Description Logics includes the clinical GALEN ontology (Rector, et al., 1996; Rogers & Rector, 2000; Schulz & Hahn, 2001a, 2001b; Schulz, et al., 2005). However, most of the literature dedicates most of their research energy on taxonomical relations. First-order logical axiomatizations are rare with BFO being a notable exception.

9.3. Robot Navigation and Linguistics

Robot navigation through unknown or partially known territory can profit from (mereo-)topological representations of space. Examples are exploring the connectivity of rooms in an unknown building to learn which rooms/hallways/staircases are connected (Kuipers & Byun, 1991; Kuipers & Levitt, 1988; Levitt & Lawton, 1990; Remolina & Kuipers, 2004) or which rooms belong to certain floors, etc. This provides a high-level spatial model for a robot to search for things in a building (e.g. search-and-rescue robots), find their way out again, or backtrack once trapped in a dead end. Learning topological maps directly from the environment can be achieved used mereotopological representations where the maps usually consist of entities of multiple dimensions including regions, lines, and points, supplemented by orientation information about the robot. After learning topological maps, these can be refined by geometrical information. More recent work in that direction additionally uses landmarks to define fiat boundaries demarcating regions. However, the amount of mereotopology used is often marginal; complex qualitative reasoning on the topological maps is rarely done. Instead, most of the work uses graph-based approaches such as Voronoi diagrams or connectivity graphs, sometimes in connection with region partitioning (Thrun, 1998), while reasoning is usually algorithmic and not symbolic. It is interesting to see that work on robot navigation using topological maps predates the growth of interest in qualitative representations of space in AI. There still seems little work that actually uses mereotopological or mereogeometrical theories discussed in this chapter in practical navigation applications.

Interesting problems in a similar direction include qualitative route finding where traditional graph-based route finding is combined with region properties, e.g. instead of finding the shortest or fastest route between some points, we might be interested in the most scenic route (going through forests, along a like, outside a city) where the different properties are represented as regions (from geographic maps) instead of assigning each link in the network an individual value for such properties. Other navigation problems such as translating a route description into a map are directly linked to natural language processing of spatio-temporal relations. Because of the variability and ambiguity of language in expressing mereological and topological relations, understanding of mereotopological relations or spatio-temporal relations in general is more of an extraction challenge. We need to identify the proper interpretation of terms such as 'is a part of,' 'adjacent,' etc. to build mereotopological models. This has been done for temporal relations (Verhagen, et al., 2005), but the methodology is applicable to spatial relations as well. For instance the language presented in Chaudet (2004)—an extension of event calculus with mereotopological relations—can help track epidemics by capturing and understanding the language of epidemic outbreak reports.

9.4. Product Modeling, Design, and Engineering

Mereotopology has been customized to a so-called *Design Mereotopology* (DMT) by Salustri (2002). He believes that regions are perfect because engineers tend to take them as primitive anyways. The DMT is an atomistic mereotopology based on CRC (Eschenbach, 1999) and Smith's mereotopology (Smith, 1996). DMT uses a mereological and a topological primitive, P and C, and postulates mereological extensionality. Though not directly an applications, Salustri mentions many possible applications of DMT: CAD, semantic knowledge bases of materials to distinguish between matter (steel, lubricants, etc.) and objects and provide properties of different materials, and configuration, function, and system modeling (cf. Salustri, 2002; Yang & Salustri, 1999).

In engineering, mereotopological and mereogeometrical relations can also be used for representing the assembly of parts. Kim et al. (2006, 2008, 2009) have developed an ontology using the Semantic Web Rule Language (SWRL) to distinguish different kinds of assembly joints obtained by welding, gluing, brazing, fastening, soldering, stitching/stapling, etc. common product design and manufacturing. As basis, they use the boundary-tolerant mereotopology of Smith (1996), but introduce additional geometrical predicates such as angles and offsets of the joined objects. The basic mereotopological definitions are translated into SWRL rules. The contact is further refined to distinguish the morphology of the contact. This documents that for practical applications, mereotopology is usually only a basis and needs to be extended by domain-specific terminology.

10. REMAINING CHALLENGES

As we previously emphasized, testing region-based theories of space in real-world domains is an essential task that needs to be undertaken. Only feedback from practice will spur the development of new theories that can cope with the peculiarities of individual domains and applications. We just outlined some research addressing this ‘practice challenge.’ Apart from that ongoing work, there is theoretical work left in the field. We briefly explain some directions of future research and discuss how these could help to advance the field of mereotopology. However, we do not discuss the many computational issues in need for future research. Amongst them, the question of decidability or the nature of tractable fragments is still of unanswered for many theories. Equational theories (Bennett, 1997) could be of great help in that pursuit. To our knowledge, there has been little work on equational theories for region-based spatial reasoning despite the fact that contact algebras are naturally equational theories. It needs to be explored whether specific equational theories have more efficient reasoning behaviors than comparable first-order theories. Even if complexity is not reduced, efficient equational solvers and theorem provers likely fare better in practice compared to first-order theorem provers.

Now, we will look at representational challenges, first within the scope of mereotopology before proceeding to the integration challenge for spatial ontologies at large.

10.1. Dimensionality and Boundaries

The discussion of the ontological commitments made it quite clear that still only little is known of how to integrate boundaries and entities of varying dimensions in a region-based mereotopology/-geometry. Though we pointed out the existing work in this direction, these can be only seen as starting points. Their models and formal semantics are often not yet completely captured, thus preventing the adaptation of these ontologies for applications or larger, more generic ontologies. The complexity of these mereotopological systems goes way beyond those of Whiteheadean space, but because of this complexity, there is an even greater urgency to capture the models thereof in a more familiar mathematical framework (cf. Section 10.2). This could also spur the development of new mereotopologies that integrate multi-dimensional mereotopologies with other aspects, e.g. holes, other discontinuities, or geometric properties such as convexity.

10.2. Formal Semantics

As outlined in Section 7, there is a need for identifying all the models of mereotopological and mereogeometrical theories. Even though the standard models are equivalent for large set of theories, the non-standard models are decisive to select an appropriate theory. Out of two theories with equivalent standard models, one could still be the preferable one because it is more restricted and allows fewer unintended models. A model-theoretic study of all the region-based theories along the lines of the algebraic/topological representations of Whiteheadean theories as contact algebras is necessary. A starting point could be the study of other mereotopologies as algebraic systems.

10.3. Region-Based Equivalents of Geometry

It is suspected that the RCC together with a convex hull (or convexity) primitive is a point-free equivalent of affine geometry, thus being strictly weaker than the full mereogeometries. This analogue to point-based geometry provokes questions whether region-based equivalents of other point-based geometries, e.g. projective or ordered geometry, can be constructed. The idea of such research in the style of Klein's *Erlanger programm*' is discussed by Clementini and Di Felice (1997a) who propose an approach combining topological, projective, and metric properties to describe shapes. A survey on algebraic representations of projective geometries building on the work of Menger, Birkhoff, and Faltings can be found in Greferath and Schmidt (1998) while Wehrung (1998) considers algebraic representations of von-Neumann's continuous geometries. Both representations rely on orthocomplementation and modularity properties of the underlying algebras.

Of particular interest is the question whether other theories exist in between mereotopology and mereogeometries. The RCC with convexity is so far the weakest theory extending pure mereotopology with some morphological predicate, but perhaps, other predicates such as relative size can be used to extend mereotopology without reaching full mereogeometry or even the equivalent of affine geometry. This question remains unexplored as far as we know and needs to be looked at more carefully in the future.

10.4. Integrated Ontologies

10.4.1. Integrated Ontologies of Qualitative Space

Recently, combinations of qualitative properties have received increased attention. Amongst them, one of the most challenging qualities of spatial objects and regions is their shape. In this chapter, we limited ourselves to convexity as only shape attribute. It extends purely mereotopological theories rather naturally. The equally important concept of relative size turns out to be only superficially different from convexity. Relative size has been integrated into mereogeometries, in particular the RBG, in Bittner and Donnelly (2007). Relative size is linked to parthood: proper parts must always be smaller (cf. Gerevini & Renz, 2002). Relative distances (cf. Bittner, 2009; Bittner & Donnelly, 2007; Gahegan, 1995) are another aspect of qualitative space that seems to seamlessly integrate with region-based space, but as in the case of the mereogeometry of Donnelly (2001) with a ternary '*can connect*' relation, it might result in a full mereogeometry.

Qualitative theories about relative positions, directions and orientation have been the focus of the work on cardinal directions (Frank, 1996; Hernández, et al., 1995; Renz & Mitra, 2004), the single and double cross calculus (SCC, DCS: Freksa, 1992; Schockaert, et al., 2008). Their combination with regions would present a more powerful and practical framework. Two examples thereof are a combination of cardinal directions with RCC-5 (Chen, et al., 2007b) and a rectangular cardinal direction calculus (Navarrete, 2006; Skiadopoulos & Koubarakis, 2004, 2005). Relative positions of extended objects are tricky to apply since they either rely on some center of regions (centre of mass, geometrical centre, etc.), or are expressed in terms of minimal bounding objects, such as rectangles, blocks, cubes (Balbiani, et al., 1998; Chen, et al., 2007a), or spheres.

The main challenge here is to use only primitives of very limited expressiveness to avoid the definition of a full mereogeometry. In that way, integrating region-based theories of space with other qualitative properties is a challenge that goes hand in hand with the exploration of theories between mereotopology and mereogeometry, compare Section 10.3.

10.4.2. Integration of Qualitative and Quantitative Space

The misfit of continuous mereotopologies for representing discrete data from the real-world has been singled out as a major shortcoming limiting the applicability of continuous mereotopologies in practice. However, discrete mereotopologies have been proposed to overcome this problem in principle. However, there is a broad scale of possible discreteness, i.e. data from different sources has different granularity. Hierarchical approaches of qualitative spatial reasoning with mereotopological relations might be of great help (cf. Li & Nebel, 2007). Nevertheless, there is still some distance to go to fully integrate qualitative approaches towards space with quantitative models, i.e. integrating region-based theories with point-based theories of space. In one direction, mereotopological relations have been extracted from metric representations of space or directly from the environment (cf. Galata, et al., 2002; they learn Markov models from the environment; it should in principle be possible to obtain logical predicates of qualitative spatial relations instead), while reconstructing metric space from qualitative space is still open. Of course, since full mereogeometries are able to reconstruct Euclidean space, their models are implicitly translated into metric models. However, for general mereotopology, this is not as straightforward. The framework outlined by Clementini and Di Felice (1997a) proposes to incorporate varying levels of granularity for such integration. However, extensions thereof need to overcome its agnosticism towards connection relations. Alternatively, we can construct minimal point-based embeddings of mereotopological models. For this task, the experience with topological embeddings comes handy, since these are usually embeddings into intuitive point-based topological models.

10.4.3. Integration into Upper Ontologies

From a broader perspective, upper ontologies able to express common-sense knowledge need to incorporate qualitative and quantitative spatial and spatio-temporal predicates. However, many of today's upper ontologies (compare the introduction in Section 9) have on informal formulations of their spatial relations. A study of their ontological commitments and considering replacement by stronger or weaker theories is necessary to foster reuse of these ontologies. Otherwise, their reliability is doubted, resulting in limited reuse.

11. SUMMARY

This chapter gives an overview of the main ontological commitments in mereotopological and mereogeometrical theories. We gathered and organized the knowledge about ontological commitments for the wealth of region-based theories that have been proposed for representing space qualitatively. Further, we reviewed the methods and formalisms that have proven useful for analyzing and comparing mereotopologies. Altogether, we hope that this gives a broader picture of mereotopology and how the different theories fit into this bigger picture regardless of their concrete axiomatization. In particular, we mapped out the space of algebraic theories corresponding to the logical theories of Whiteheadean space. Despite the overwhelmingly disperse set of theories, we hope to have made it clear that the actual differences are usually only minor and that there are only a handful of substantially different approaches. We hope this chapter inspires more work on mathematical representations of mereotopological theories not yet fully understood. Moreover, we hope the chapter provides sufficient information for choosing and integrating a mereotopological theory into applications or more general ontologies.

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KEY TERMS AND DEFINITIONS

Atom: An indivisible region, i.e. a region that has no proper part.

Algebraic Representation: Representation of the models of a mereotopology by a class of algebraic structures, usually by some class of contact algebras or class of lattices.

Boolean Contact Algebra: A contact algebra where the underlying lattice is Boolean, i.e. is a complemented distributive lattice and the unique sums with respect to the extension of contact exist.

Contact Algebra: An algebraic structure consisting of a bounded lattice defining a partial order P and a binary reflexive and symmetric contact relation C that is monotone with respect to P. This is an algebraic representation of a Whiteheadean mereotopology.

Extensionality: A (binary) predicate is extensional in a logical theory if any two objects in the domain of the theory are distinct if and only if their extension of that particular predicate is different.

Full Mereogeometry: a region-based theory of space that is expressive enough to define every concept definable in Euclidean geometry.

Regular Set: A set x in a topological space is regular if and only if cl(x) = cl(int(x)) and int(x) = int(cl(x)). We distinguish regular closed sets which satisfy x = cl(x) = cl(int(x)) and regular open sets which satisfy x = int(x) = int(cl(x)).

Topological Embedding: The embedding of the models of an algebraic structure into the sets of some topological space. These sets must be closed at least under union and set intersection. An embedding theorem ensures that every model of a certain algebraic structure can be embedded into a topological space in such a way. A special case is that of a topological representation in which case the reverse also holds: from an arbitrary topological space from a certain class of topological spaces it is always possible to construct a model of the algebraic structure.

Whiteheadean Mereotopology: Region-based theories of space assuming regions as only elements in the domain of discourse, thus requiring that all regions are of the same dimension. Whiteheadean mereotopologies must be topologically representable, i.e. there must exist an embedding into some topological space. Usually this embedding must be into the regular sets of a topological space. The term 'Whiteheadean mereotopology' is used in a loose sense in this chapter; the more strict sense requires that such theory can be based on a single primitive (extensional) relation of contact, C.