Model-Theoretic Characterization of Asher and Vieu's Ontology of Mereotopology

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Abstract

We characterize the models of Asher and Vieu's first-order mereotopology RT_0 in terms of mathematical structures with well-defined properties: topological spaces, lattices, and graphs. We give a full representation theorem for the models of the subtheory RT^- (RT_0 without existential axioms) as p-ortholattices (pseudocomplemented, orthocomplemented). We further prove that the finite models of RT^-_{EC}, an extension of RT^-, are isomorphic to a graph representation of p-ortholattices extended by additional edges and we show how to construct finite models of the full mereotopology. The results are compared to representations of Clarke's mereotopology and known models of the *Region Connection Calculus* (*RCC*). Although soundness and completeness of the theory RT_0 has been proved with respect to a topological translation of the axioms, our characterization provides more insight into the structural properties of the mereotopological models.

1 Introduction

Mereotopological systems have long been considered in philosophic and logic communities and recently received attention from a knowledge representation perspective. Mereotopology is composed of topological (from *point-set topology*) notions of connectedness and mereological notions of parthood. Point-set topology (or *General Topology*) relies on the definition of open (and dually closed) sets and extends standard set-theoretic notions of union, intersection, and containment with concepts such as interior, closure, limit points, neighborhoods, and connectedness. Closely related to point-set topology, mereotopology can be considered a generalized, *pointless* version thereof: using regions instead of points as primitive entities.

Uncertainty about differences in mereotopological systems, in particular about their implicit assumptions, seems to be a major source of confusion that hinders forthright application of even well-developed mereotopological theories. This problem arises in the various theories for different reasons: some lack any formal representation, leaving the user unsure about intended interpretations; others are formalized in first-order logic but lack a characterization of the models up to isomorphism. This paper focuses on an instance of the latter problem – we analyze the models of Asher and Vieu's mereotopology RT_0 (Asher & Vieu 1995) in the style of representation theorems using well-understood structures from mathematical disciplines. We want to understand the class of models that the axiomatic system RT_0 describes and what properties these models share. The primary motivation of this work is to give better insight into the axiomatic theory and to uncover problems and assumptions that users of the ontology should be aware of. Although the completeness and soundness of RT_0 has been proved with respect to the intended models defined by $\mathfrak{RT} = RT_T$ over a topology T, this is little more than a mere rephrasing of the axioms. The proofs show that the axiomatic system describes exactly the intended models, but the formulation of the intended models does not reveal structural properties that can be used to learn about practical applicability, implicit restrictions, and hidden assumptions of the theory. For this reason we characterize the models of RT_0 in terms of classes of structures defined in topology, lattice theory, and graph theory and compare the classes to representations of other mereotopological theories. Such representations of the models of the axiomatic theory allow us to directly reuse knowledge about the mathematical structures for the mereotopological theory RT_0. We concentrate on the finite models, since these are dominant in real-world applications. We also compare our findings to the representations of the (always infinite) models of the *Region Connection Calculus* (RCC) conducted in (Stell & Worboys 1997; Stell 2000) and in (Düntsch & Winter 2005; Bennett & Düntsch 2007) which use *Boolean Connection* (or *Contact*) *Algebras* (*BCA*) to describe models of the *RCC*.

Besides the characterization and analysis of RT_0, the main contribution of this work is a comparison of the suitability of different mathematical structures, in particular topological spaces, graph representations, and lattices, for a model-theoretic analysis and comparison of mereotopological frameworks. The long-term objective is an exhaustive comparison of different mereotopological theories within a strictly defined mathematical context. Our results indicate that lattices and related classes of graphs are best suited because they provide an intuitive way to model parthood relations. Notice that we restrict ourselves to a rigid mathematical study that provides the community with a model-theoretic view on mereotopology for the example of RT_0; we do not argue for or against underlying assumptions of different mereotopologies.

Serving the growing interesting in formal ontologies and upper ontologies, this kind of analysis can guide the selection of a generic axiomatization of mereotopology for inclusion in upper ontologies such as SUMO, DOLCE, and BFO.

The remainder of the paper is structured as following: section 2 explains mereotopology and its background and subsections 2.1 and 2.2 briefly introduce the mereotopological system RT_0 of Asher & Vieu together with its intended models. Section 3 presents our different approaches for characterizing the models of subtheories of RT_0. Within each of the subsections 3.1 to 3.4, a representation theorem for some subset of the axioms of RT is proved. Finally, section 4 discusses and compares the representations to its intended models as well as characterizations or known classes of models of other mereotopological ontologies.

2 The Mereotopology *RT* of Asher and Vieu

Mereology, dating back to Whitehead (Whitehead 1929) and Leśniewski (Luschei 1962), investigates parthood structures and relative complementation. A first specification of extensional mereology was presented in (Leonard & Goodman 1940). For an overview of extensional mereology, we invite the reader to consult (Simons 1987). The primitive relation in mereology is *parthood* (an entity being part of another) expressed as irreflexive *proper parthood*, $<$ or *PP*, or as reflexive *parthood*, $\leq$ or *P*. The latter is usually a standard partial order that is reflexive, anti-symmetric, and transitive, coined *Ground Mereology* **M** in (Varzi 2007). Moreover, most mereologies define concepts of *overlap*, *union*, and *intersection* of entities. General sums (fusion, i.e. the union of arbitrarily many individuals) are also widespread. In all mereological theories a *whole* (universe) can be defined as the entity that everything else is part of. If *differences* are defined, a *complement* exists for every individual relative to the mereological whole. More controversial is whether mereology should allow *atoms*, i.e. individuals without proper parts that are the smallest entities of interest. Some theories are atomless while others explicitly force the existence of atoms (Simons 1987); mereotopology inherits this controversy: it can be defined atomless, atomic, or make no assumption about atomism at all.

Neither classic mereology nor classic topology (see point-set topology in section 1) are by themselves powerful enough to express part-whole relations without defining supplemental concepts of connection or parthood, respectively. Connection does not imply parthood between two individuals and, similarly, mereological wholes do not imply topological (self-connected) wholes. To reason about integral, self-connected individuals, a combination of mereology with topology is necessary to bridge the gap between them. The different options to merge the two independent theories are presented in (Casati & Varzi 1999) to classify mereotopologies. First, mereology can be supplemented with a topological (Smith 1996) or geometrical primitive (Tarski 1956; Bennett 2001). More widespread is the reverse: assuming topology to be more fundamental and defining mereology using only topological primitives ("Topology as Basis for Mereology"). The majority of mereotopological approaches such as (Whitehead 1929; Clarke 1981; Asher & Vieu 1995; Pratt & Schoop 1997) and the *RCC* (Randell, Cui, & Cohn 1992; Cohn *et al.* 1997a) use this method with a connection (or contact) relation as the only primitive – expressing parthood in terms of connection. A third, less common way to merge topology and mereology, applied in (Eschenbach & Heydrich 1995), extends the mereological framework of (Leonard & Goodman 1940) by quasi-mereological notions to define topological wholeness.

As mentioned earlier, our focus lies on first-order mereotopologies. Unfortunately, most of these theories either entirely lack soundness and completeness proofs, e.g. (Clarke 1985; Smith 1996; Borgo, Guarino, & Masolo 1996), or the proof is based on a rephrased model definition as in (Asher & Vieu 1995). Only the theory of (Pratt & Schoop 1997), which is limited to planar polygonal mereotopology, provides formal proofs that exhibit the possible models. For the *RCC* (Cohn *et al.* 1997a) the intended models are thoroughly characterized but no full representation theorem exists yet[1]. But to compare mereotopologies solely by their models, we first need to characterize the models only from the axioms (or a definition for which equivalence to the axioms is proved). Clarke's theory has received significant attention, but some problems have been identified with it. We focus on Asher & Vieu's revised version of Clarke's theory; their completeness and soundness proofs with respect to a class of intended structures ease the model-theoretic analysis. Notice that Clarke's and Asher & Vieu's theories explicitly allow to distinguish between different regions with identical closures. The *RCC* considers entities as equivalence classes of regions with identical closures (Cohn *et al.* 1997b), claiming that Clarke's distinction is too rich for spatial reasoning. On the other hand, tangential and non-tangential parts as well as regular overlap and external connection – which all rely on open and closed properties – are distinguished in *RCC*.

2.1 Axiomatization RT_0

The first-order theory RT_0 of (Asher & Vieu 1995) uses the *connection* relation C as only primitive. The theory is based on Clarke's *Calculus of Individuals* (Clarke 1981; 1985), with modifications that make the theory first-order definable: the explicit fusion operator is eliminated, and the concept of *weak contact, WCont*, is added. To eliminate trivial models, RT_0 requires at least one *external connection* and one *weak contact* (A11, A12). Some previous ontological and cognitive issues are also addressed, see (Asher & Vieu 1995) for details. RT_0 follows the strategy "Topology as Basis for Mereology" for defining mereotopology and hence does not contain an explicit mereology. Consequently, the parthood relation P is sufficiently defined by the extension of the primitive relation C, which limits the expressiveness of the whole theory to that of C. For consequences of such kind of axiomatization, see (Casati & Varzi 1999; Varzi 2007).

[1] Only for Boolean Contact Algebras (*BCA*) there exists a full representation theorem

To construct models of the theory RT_0, the following definitions are necessary. Except for $WCont$, all these were already defined in (Clarke 1985) and are similar to those of other mereotopological systems.

(D1) $P(x,y) \equiv \forall z\,[C(z,x) \rightarrow C(z,y)]$ (Parthood as reflexive partial order satisfying the axioms of **M**)

(D3) $O(x,y) \equiv \exists z\,[P(z,x) \wedge P(z,y)]$ (Two individuals overlap iff they have a common part)

(D4) $EC(x,y) \equiv C(x,y) \wedge \neg O(x,y)$ (Two individuals are externally connected iff they are connected but share no common part)

(D6) $NTP(x,y) \equiv P(x,y) \wedge \neg\exists z\,[EC(z,x) \wedge EC(z,y)]$ (Non-tangential parts do not touch the border of the larger individuals)

(D7) $cx \equiv -i(x)$ (Closure defined through complements $-x$ and unique interiors $i(x)$, both guaranteed for all x by A7 and A8 below)

(D8) $OP(x) \equiv x = i(x)$ (Open individuals)

(D9) $CL(x) \equiv x = c(x)$ (Closed individuals)

(D11) $WCont(x,y) \equiv \neg C(c(x), c(y)) \wedge \forall z\,[(OP(z) \wedge P(x,z)) \rightarrow C(c(z),y)]$ (Weak contact requires the closures of x and y to be disconnected, but any neighborhood containing $c(x)$ to be connected to y)

The concepts proper part PP (the irreflexive subset of the extension of parthood, i.e. $PP(x,y) \equiv P(x,y) \wedge x \neq y$), tangential part TP ($TP(x,y) \equiv P(x,y) \wedge \neg NTP(x,y)$), and self-connectedness CON (see (Asher & Vieu 1995)) are defined in RT_0, but are irrelevant for the model construction, since they are not used in the axioms. RT_0 is then defined by:

(A1) $\forall x\,[C(x,x)]$ (C reflexive)

(A2) $\forall x,y\,[C(x,y) \rightarrow C(y,x)]$ (C symmetric)

(A3) $\forall x,y\,[\forall z\,(C(z,x) \equiv C(z,y)) \rightarrow x = y]$ (C extensional)

(A4) $\exists x \forall u\,[C(u,x)]$ (Existence of a universally connected element $a^* = x$)

(A5) $\forall x,y \exists z \forall u\,[C(u,z) \equiv (C(u,x) \vee C(u,y))]$ (Sum for pairs of elements)

(A6) $\forall x,y\,[O(x,y) \rightarrow \exists z \forall u\,[C(u,z) \equiv \exists v\,(P(v,x) \wedge P(v,y) \wedge C(v,u))]]$ (Intersection for pairs of overlapping elements)

(A7) $\forall x\,[\exists y\,(\neg C(y,x)) \rightarrow \exists z \forall u\,[C(u,z) \equiv \exists v\,(\neg C(v,x) \wedge C(v,u))]]$ (Complement for elements $\neq a^*$)

(A8) $\forall x \exists y \forall u\,[C(u,y) \equiv \exists v\,(NTP(v,x) \wedge C(v,u))]$ (Interior for all elements; the interior $y = i(x)$ is the greatest non-tangential (not necessarily proper) part y of x)

(A9) $c(a^*) = a^*$ (Closure c defined as complete function)

(A10) $\forall x,y\,[(OP(x) \wedge OP(y) \wedge O(x,y)) \rightarrow OP(x \cap y)]$ (The intersection of open individuals is also open)

(A11) $\exists x,y\,[EC(x,y)]$ (Existence of two externally connected elements)

(A12) $\exists x,y\,[WCont(x,y)]$ (Existence of two elements in weak contact)

(A13) $\forall x \exists y\,[P(x,y) \wedge OP(y) \wedge \forall z\,[(P(x,z) \wedge OP(z)) \rightarrow P(y,z)]]$ (Unique smallest open neighborhood for all elements)

For the representation theorems, we will consider subtheories of the axioms of RT_0, which we refer to as RT_C, RT^-, and RT^-_{EC}. The subtheory RT_C is the topological core of the theory consisting of axioms A1 to A3; it corresponds to extensional ground topology (**T**) or Strong Mereotopology (**SMT**) (Casati & Varzi 1999) and to extensional weak contact algebras (which satisfy axioms C0 - C3, C5e of (Düntsch & Winter 2006)). Hence, C is a contact relation in the sense of (Düntsch, Wang, & McCloskey 1999). The subtheory $RT^- \equiv RT_0 \setminus \{A11, A12\}$ excludes the existential axioms that eliminate trivial models, but its models have the same structural properties as those of RT_0. Hence, a representation theorem for the models of RT^- elegantly captures important properties of RT_0 as well. Finally, we consider models of $RT^-_{EC} \equiv RT^- \cup \{A11\}$ and show how external connections prevent certain lattices.

2.2 Intended Models $\mathfrak{RT}$

Asher & Vieu provide soundness and completeness proofs for RT_0 with respect to the class of structures RT_T defining the intended models of the mereotopology. Each intended model is built from a non-empty topological space (X,T) with T denoting the set of open sets of the space. Standard topological definitions of interior int and closure cl operators, open and closed properties, and $\sim$ as relative complement w.r.t. X are assumed. The intended models are then defined as structures $RT_T = \langle Y, f, [\![]\!] \rangle$[2] where the set Y must meet the conditions (i) to (viii). To avoid confusion with the axiomatic theories, we use the notation $\mathfrak{RT}$ to denote the class of all structures RT_T. The models of the axiomatic theory RT_0 are thus by (Asher & Vieu 1995) exactly isomorphic to the models in $\mathfrak{RT}$. However, the conditions constraining the intended models in $\mathfrak{RT}$ are a mere rephrasing of the axioms A4 to A13 of RT_0 that – although motivated by common-sense – give no useful alternative representation of the models of RT_0 in terms of known classes of mathematical structures. Only the connection structures defined by RT_C are not directly linked to the conditions (i) to (viii).

(i) $Y \subseteq \mathscr{P}(X)$ and $X \in Y$; X is the universally connected individual a^* required by A4 and all other elements in a model of RT_0 are subsets thereof;

full interiors (ii) and *smooth boundaries* (iii):

(ii) $\forall x \in Y\,(int(x) \in Y \,\&\, int(x) \neq \emptyset \,\&\, int(x) = int(cl(x)))$; requires non-empty interiors for all elements equivalent to A8;

(iii) $\forall x \in Y\,(cl(x) \in Y \,\&\, cl(x) = cl(int(x)))$; requires closures for all elements which is implicitly given by D7 as closure of the uniquely identified interiors and complements (by A7 and A8); A9 handles a^* separately;

(iv) $\forall x \in Y\,(int(\sim x) \neq \emptyset \rightarrow \sim x \in Y)$; requires unique complements equivalent to A7;

(v) $\forall x,y \in Y\,(int(x \cap y) \neq \emptyset \rightarrow (x \cap^* y) \in Y)$; for pairs of elements with non-empty mereological intersection an intersecting element is guaranteed equivalent to A6;

[2] For definitions of f and $[\![]\!]$, see (Asher & Vieu 1995)

(vi) $\forall x,y \in Y\,((x \cup^* y) \in Y)$; guarantees the existence of sums of pairs equivalent to A5;

(vii) $\exists x,y \in Y\,((x \cap y) \neq \emptyset \,\&\, int(x \cap y) = \emptyset)$; requires a pair of externally connected elements equivalent to A11 together with def. D4;

(viii) $\exists x,y \in Y\,((cl(x) \cap cl(y)) = \emptyset \;\&\; \forall z \in Y\,[(open(z) \;\&\; x \subseteq z) \rightarrow y \cap cl(z) \neq \emptyset])$; requires a pair of weakly connected elements equivalent to A12 with def. D11;

where $x \cup^* y = x \cup y \cup int(cl(x \cup y))$ and
$x \cap^* y = x \cap y \cap cl(int(x \cap y))$.

Since the interplay of the conditions and resulting implicit constraints are not clear, this description of the intended models of RT_0 is insufficient for understanding the properties and structure of the mereotopological models of RT_0. Hence, in the next section our goal is to better understand the models by characterizing them as classes of well-understood mathematical structures.

3 Characterization

This section presents a new characterization of the models of RT_0 and its subtheories in terms of topological spaces, lattices, graphs, and a combination of lattices and graphs. We are the first to characterize the models of a mereotopological or any spatial reasoning framework using all these different structures. Previously, (Biacino & Gerla 1991) characterized the models of Clarke's system from (Clarke 1981) in terms of lattices, showing that the *connection structures* defined by the axioms A0.1, A0.2, and A1.1 of (Clarke 1981) are isomorphic to the complete orthocomplemented lattices. Furthermore (Biacino & Gerla 1991) proved that the models of Clarke's system from (Clarke 1985), which includes axiom A3.1 requiring the existence of a common point of two connected individuals, are equivalent to the complete Boolean algebras. (Düntsch & Winter 2006) also represent the *contact relation* of Clarke as atomless Boolean algebra. Since RT_0 heavily relies on the work of Clarke, it is natural to analyze the models of RT_0 and compare them to those of Clarke's system. We want to clarify how the changes proposed by Asher & Vieu alter the class of associated models, particularly in a lattice-theoretic representation. For the *RCC* similar characterizations have been pursued, but these focus on more generic structures such as the *weak contact structures* (Düntsch & Winter 2006) and the *Boolean Contact Algebras* (Düntsch & Winter 2005) and which are also restricted to representations of models in terms of topological spaces. Others (Stell 2000; Stell & Worboys 1997) give classes of lattices that are models of the *RCC*, but do not give a full representation up to isomorphism.

First, we show that in contrast to the exclusively atomless models of the *RCC* (Cohn *et al.* 1997a), the theory RT_0 allows finite and infinite models. The proof of lemma 1 constructs finite models; the existence of infinite models follows from the Compactness Theorem and the possibility to construct arbitrarily large finite models through products.

Lemma 1. *There exist finite, non-trivial models of RT^-, RT^-_{EC}, and RT_0.*

Proof. The model $\mathscr{M}$ defined by $\langle a^*, b\rangle, \langle a^*, c\rangle \in C^{\mathscr{M}}$ (with all reflexive and symmetric tuples also contained in $C^{\mathscr{M}}$) satisfies all axioms of RT^- and is of finite domain $\{a^*, b, c\}$ and hence is a finite model of RT^-. The model defined by $\langle a^*, b\rangle, \langle a^*, ib\rangle, \langle a^*, c\rangle, \langle a^*, ic\rangle, \langle b, ib\rangle, \langle c, ic\rangle, \langle b, c\rangle \in C^{\mathscr{M}}$ (again with all reflexive and symmetric tuples also contained in $C^{\mathscr{M}}$) with $\langle b, c\rangle, \langle c, b\rangle \in EC^{\mathscr{M}}$ satisfies all axioms of RT^-_{EC} and has a finite domain $\{a^*, b, c, ib, ic\}$ and thus is a finite model of RT^-_{EC}. In (Hahmann 2008) we proved that the Cartesian product of a finite model of RT^- and a finite model of RT^-_{EC}, which both must be extended by additional closures of their respective suprema, is always a finite model of RT_0. Hence, the product of the presented models is a finite model of RT_0. □

3.1 Topological spaces

Attempting to characterize the models of RT_0 using topological spaces and the common tool of separation axioms is natural since the intended models of the theory are defined over topological spaces. Here we only present the major result, see (Hahmann 2008) for details. A full characterization using separation axioms fails, but our results exhibit parallels with the topological characterizations of the *RCC* and *BCA*s in general. (Düntsch & Winter 2005) characterized the models of the *RCC* as weakly regular (a stronger form of semi-regularity) but also showed that the separation axioms T_0 and T_1 are not forced by the axioms. For any model of RT_0 there always exists an embedding topological space (X,T) over the set $X = \Sigma_U =_{def} \bigcup \{\Omega_{[c_n]} | c_n \in \Sigma_C\}$[3] and the topology $T = \Sigma_U^T = \{\emptyset\} \cup \{\Omega_{[c_n]} | c_n \in \Sigma_C \wedge \Sigma \vdash OP(c_n)\}$ $\cup \{\bigcup Z | Z \subseteq \{\Omega_{[c_n]} | c_n \in \Sigma_C \wedge \Sigma \vdash OP(c_n)\}\}$ that satisfies T_0, but T_0 cannot generally be assumed for topological spaces constructed from models of RT_0. Hence though all regions are regular by (ii) and (iii), the underlying space itself is not regular. For the finite (atomic) models the embedding space is always reducible to discrete topologies and hence uninteresting. The infinite models of RT_0 are embeddable in semi-regular spaces which are T_1 but not necessarily Hausdorff or regular. This follows from the *smooth boundaries* condition of $\mathfrak{RT}$ forcing all open sets to be regular open. An equivalent topological property to capture the *full interiors* condition was not found (local connectedness fails).

Theorem 1. *A model of RT_0 with infinite number of individuals can be embedded in a semi-regular topological space.*

Proof. See (Hahmann 2008). □

3.2 Lattices

The similarity between (a) posets that underlie lattices and (b) parthood structures as found in mereology suggests a representation of the models of RT_0 as lattices using the operations $+$ and $\cdot$ as join and meet. Since $\emptyset \notin Y$ for any mereotopological structure in $\mathfrak{RT}$, we add the empty set $\emptyset$ as zero element to form bounded lattices.

[3] For a model represented by set of saturated sentences Σ consistent with RT_0, Σ_C contains all constants c_n occurring in Σ, and each equivalence class $[c_n]$ of a constant c_n is associated with a set of points, denoted by $\Omega_{[c_n]}$

Proposition 1. *A model $\mathscr{M}$ of RT_0 and any subset thereof has a unique representation as lattice (algebraic structure) $(Y \cup \{\emptyset\}, \cdot, +, \emptyset, a^*)$ over the partial order $P^{\mathscr{M}}$: $x \leq_{\mathscr{L}^{\mathscr{M}}} y$ if $\langle x, y\rangle \in P^{\mathscr{M}}$.*

By the soundness and completeness proofs from (Asher & Vieu 1995) for any $\mathscr{M}$ the lattice $(Y \cup \{\emptyset\}, \cdot, +, \emptyset, a^*)$ has an isomorphic lattice $\mathscr{L}^{\mathscr{M}} = (Y \cup \{\emptyset\}, \cap^*, \cup^*, \emptyset, a^*)$, denoted by $\mathscr{L}^{\mathscr{M}}$, that is defined through the structure of the intended model in $\mathfrak{RT}$. The lattice is uniquely defined for any model of RT^-, RT^-_{EC}, or RT_0 because it only relies on a model's parthood extension. However, we do not claim the reverse: a particular lattice does not necessarily represent a unique model, e.g. there can be lattices that represent different models of RT^- with distinct definitions of $EC^{\mathscr{M}}$. Now we give a representation theorem for the models of RT^- using standard lattice concepts (e.g. unicomplementation and pseudocomplementation) from (Grätzer 1998), supplemented by semimodularity (Stern 1999), and orthocomplementation and orthomodularity (Kalmbach 1983) properties for the characterization; see (Hahmann 2008) for details.

An initial important observation is captured by lemma 2 (follows from A11, D4) which results in a special 6-element sublattice $\mathscr{L}_6$ ("benzene") for every model in $\mathfrak{RT}$ (lemma 3). Lemma 2 is a direct consequence of axiom A11.

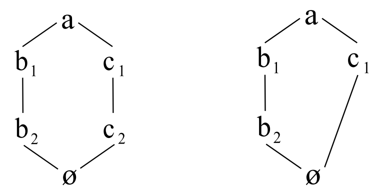
Lemma 2. *Any model of RT^-_{EC} or RT_0 contains at least two non-open, non-intersecting but connected individuals.*

Proof. Condition (vii) of $\mathfrak{RT}$ requires two elements $x, y \in Y$ to share a point, but no interior point ((note that $int(x \cap y) = int(x) \cap int(y)$)): $x \cap y \neq \emptyset \wedge int(x \cap y) = \emptyset$. Thus x and y share only boundary points. If w.l.g. x is open, i.e. $x = int(x)$, it cannot contain any boundary points to share in an external connection. Thus for some x, y to be externally connected, x and y must be non-open (but not necessarily closed). Then x and y cannot intersection in a common part, since this common part would have a non-empty interior (by condition (ii) of $\mathfrak{RT}$) and thus violate A11 or D4 in the equivalent model of RT_0 or RT^-_{EC}. □

Lemma 3. *Every model $\mathscr{M}$ of RT^-_{EC} or RT_0 entails the existence of a 6-element sublattice $\mathscr{L}_6$ of $\mathscr{L}^{\mathscr{M}} = (Y \cup \{\emptyset\}, \cap^*, \cup^*, \emptyset, a^*)$ with following properties:*

(1) $\mathscr{L}_6$ *has set* $Y' = \{\mathbf{a}, \mathbf{b_1}, \mathbf{b_2}, \mathbf{c_1}, \mathbf{c_2}, \emptyset\} \subseteq Y^{\mathscr{M}} \cup \{\emptyset\}$;
(2) *for* $n, m \in \{1, 2\}$, $\mathbf{a} = \mathbf{b_n} \cup^* \mathbf{c_m}$ *is the supremum of* $\mathscr{L}_6$;
(3) *for* $n, m \in \{1, 2\}$, $\emptyset = \mathbf{b_n} \cap^* \mathbf{c_m}$ *is the infimum of* $\mathscr{L}_6$;
(4) $\mathbf{b_1} \cap^* \mathbf{b_2} = \mathbf{b_2}$ *and* $\mathbf{c_1} \cap^* \mathbf{c_2} = \mathbf{c_2}$;
(5) $\mathbf{b_1} \cup^* \mathbf{b_2} = \mathbf{b_1}$ *and* $\mathbf{c_1} \cup^* \mathbf{c_2} = \mathbf{c_1}$;
(6) $\mathbf{a} \cup^* x = \mathbf{a}$ *and* $\mathbf{a} \cap^* x = x$ *for all* $x \in Y'$;
(7) $\emptyset \cup^* x = x$ *and* $\emptyset \cap^* x = \emptyset$ *for all* $x \in Y'$.

Proof. Since the axioms force the existence of a pair of externally connected individuals which are non-open, let us call these $\mathbf{b_1}$ and $\mathbf{c_1}$. Because of their non-openness, two open regions $\mathbf{b_2} = int(\mathbf{b_1})$ and $\mathbf{c_2} = int(\mathbf{c_1})$ must exist as interiors according to (ii) of $\mathfrak{RT}$. These regions $\mathbf{b_2}$ and $\mathbf{c_2}$ are part of and connected to the element they are interior of, $\mathbf{b_1}$ and $\mathbf{c_1}$, respectively. $\mathbf{b_2}$ and $\mathbf{c_2}$ are not connected to each other in order to satisfy the condition of external connection for $\mathbf{b_1}$ and $\mathbf{c_1}$ (see D4 or (vii) of $\mathfrak{RT}$). This set of regions Y' with $\mathbf{a} = \mathbf{b_1} \cup^* \mathbf{c_1}$ (for $\mathbf{a} = a^*$ it is actually the smallest model allowed by RT^-_{EC}) together with the empty set forms a sublattice with $\mathbf{a}$ as supremum, two branches consisting of $\mathbf{b_1}$ and $\mathbf{b_2} = int(\mathbf{b_1})$ respectively $\mathbf{c_1}$ and $\mathbf{c_2} = int(\mathbf{c_1})$, and the zero element $\emptyset$. Any model of $\mathfrak{RT}$ contains at least these elements. If the lattice contains additional elements, $\mathscr{L}_6$ always forms a sublattice of it, since the elements $\mathbf{a}, \mathbf{b_1}, \mathbf{b_2}, \mathbf{c_1}, \mathbf{c_2}, \emptyset$ are closed under $\cup^*$ and $\cap^*$. Hence the axioms force any model of RT_0 or RT^-_{EC} to have $\mathscr{L}_6$ as sublattice. □



(a) lattice $\mathscr{L}_6$ (b) pentagon sublattice N_5

Figure 1: Six element sublattice contained in every lattice $\mathscr{L}^{\mathscr{M}}$ and one possible pentagon sublattice

By removing an arbitrary element from $\{\mathbf{b_1}, \mathbf{c_1}, \mathbf{b_2}, \mathbf{c_2}\}$ of $\mathscr{L}_6$ we obtain a sublattice $\mathscr{L}_5$ that is still closed under join and meet and is a pentagon N_5, compare figure 1. With distributivity requiring modularity which again is equivalent to the absence of pentagons as sublattices (compare (Grätzer 1998)), we derive following significant corollary.

Corollary 1. *No lattice associated with a model of RT_0 or RT^-_{EC} is distributive.*

This result strictly separates the models of RT_0 from those of the *RCC* and Clarke's system. (Stell 2000; Stell & Worboys 1997) found models of the *RCC* representable as inexhaustible (atomless) pseudocomplemented distributive lattices and models of the (Clarke 1985) were in (Biacino & Gerla 1991) shown to be isomorphic to complete atomless Boolean algebras that are also distributive lattices. Notice that this corollary does not apply to models of RT^-.

For the models of RT^- we can prove join- and meet-pseudocomplementedness as well as orthocomplementedness (using the topological complement as orthocomplement) and that the intersection of these classes of lattices, the so-called class of p-ortholattices exactly represents the class of models of RT^-.

Definition 1. (Grätzer 1998; Stern 1999) Let L be a lattice with infimum 0 and supremum 1.
An element a' is a meet-pseudocomplement of $a \in L$ iff $a \wedge a' = 0$ and $\forall x (a \wedge x = 0 \Rightarrow x \leq a')$; a' is a join-pseudocomplement of $a \in L$ if and only if $a \vee a' = 1$ and $\forall x (a \vee x = 1 \Rightarrow x \geq a')$.

Definition 2. (Kalmbach 1983) A bounded lattice is an ortholattice (orthocomplemented lattice) iff there exists a unary operation $^{\perp} : L \rightarrow L$ so that:
(1) $\forall x \left[x = x^{\perp\perp}\right]$ (involution law)
(2) $\forall x \left[x \wedge x^{\perp} = \perp\right]$ (complement law; or $\forall x \left[x \vee x^{\perp} = \top\right]$)
(3) $\forall x, y \left[x \leq y \equiv x^{\perp} \geq y^{\perp}\right]$ (order-reversing law).

Theorem 2. (Representation Theorem for RT^-). *The lattices arising from models of RT^- are isomorphic to doubly pseudocomplemented ortholattices (p-ortholattices).*

Proof. See appendix. □

The lattice representations of the models of RT_0 and RT_{EC}^- are then propers subsets of the p-ortholattices: they are not atomistic, not semimodular, not orthomodular, nor uniquely complemented; all because of the existence of a sublattice $\mathscr{L}_6$ (and thus N_5) in their lattice representations. External connection relations are not expressed in the lattices, therefore lattices alone fail to characterize the models of RT_{EC}^- and of the full theory RT_0. Nevertheless, the above representation is already helpful, since only the trivial models are not yet excluded. All known properties of join- and meet-pseudocomplemented and orthocomplemented lattices can be directly applied to the models of the full mereotopology.

3.3 Graphs

To characterize the extension of $EC^{\mathscr{M}}$, we can represent a model $\mathscr{M}$ of RT_0 as graph $G(\mathscr{M})$ where the individuals of the model are vertices and the dyadic primitive relation C is the adjacency relation of the graph.

Proposition 2. *A model $\mathscr{M}$ of (a subtheory of) RT_0 has a graph representation $G(\mathscr{M}) = (V,E)$ where $V_G = Y^{\mathscr{M}}$ and $\mathbf{xy} \in E_G \iff \langle \mathbf{x},\mathbf{y}\rangle \in C^{\mathscr{M}} \iff [\![x]\!]_g \cap [\![y]\!]_g \neq \emptyset$.*

If we consider the subtheory RT_C, the models can be captured by the absence of true twins in their graphs. This characterization of the topological core of RT_0 as graphs without true twins generalizes to *connection structures* and *weak contact algebras* (compare (Biacino & Gerla 1991) and (Düntsch & Winter 2006)). Notice although theorem 3 is not restricted to finite (or atomic) models of RT_C, only the finite models of RT_C result in finite, undirected graphs without multiple edges between pairs of vertices.

Definition 3. Two vertices $x,y \in V(G)$ are true (false) twins in a graph G iff $N[x] = N[y]$ $(N(x) = N(y))$.

Theorem 3. (Representation Theorem for RT_C). *The graph representations $G(\mathscr{M})$ of models $\mathscr{M}$ of RT_C are isomorphic to the graphs free of true twins.*

Proof. If a graph $G(\mathscr{M})$ has two vertices $x,y \in V(G(\mathscr{M}))$ with $N[x] = N[y]$, then A3 is violated unless $x = y$. On the reverse, a graph without true twins directly satisfies A3. □

In (Hahmann 2008) a more restricted graph class is defined through vertex orderings called maximum neighborhood inclusion orderings (*mnio*s) which are a stronger variant of maximum neighborhood orderings (compare the definition of dually chordal graphs in (Brandstädt *et al.* 1998). Graphs with *mnio* are free of true and false twins (twin-free). Most important, every graph $G(\mathscr{M})$ of a model $\mathscr{M}$ of RT_0 yields an *mnio* and hence is both dually chordal and twin-free. Although *mnio*s capture important properties of parthood hierarchies, not all graphs that yield *mnio*s are models of RT^- or RT_0.

3.4 Lattices as Graphs

Pure lattice representations do not account for external connections, but graph representations only capture models of RT_C up to isomorphism, not of RT_{EC}^- and RT_0. Most problematic: we cannot describe the resulting graphs through the absence of some subgraphs (Hahmann 2008). For a representation theorem for RT_{EC}^-, we thus combine the advantages of the lattice and graph representations to define graphs over lattice structures. Remember that the lattices nicely capture parthood structures and complementation whereas the graphs are able to represent the full extension of C for any model of RT_0 or RT_{EC}^-. We already know that for any p-ortholattice there exists a model of RT^-; the representation of such lattice as graphs allows to explicitly model external connection.

Proposition 3. *Every p-ortholattice $\mathscr{L}$ over a set of elements Y, has a representation as undirected graph $G^{\mathscr{L}} = (V,E)$ with $V \cong Y \setminus \{\emptyset\}$ and $xy \in E(G^{\mathscr{L}}) \iff \forall z \in Y\,[z \leq x \wedge z \leq y]$. $G^{\mathscr{L}}$ is finite if $\mathscr{L}$ is finite.*

A correlation between orthocomplements in the lattices and connectedness in the models leads to a representation theorem for the finite models of RT_{EC}^-. We must restrict the theorem to the finite models because the proof of claim 6 for theorem 4 relies on the lattices being atomic. If this claim can be proved unconditionally, the theorem extends to the infinite models as well.

Theorem 4. (Representation Theorem for finite models of RT_{EC}^-). *Let $G^{\mathscr{L}}$ be the graph of a finite, not unicomplemented p-ortholattice $\mathscr{L}$. Then the graph $(V_{G^{\mathscr{L}}}, E_{G^{\mathscr{L}}} \cup E_{EC})$ with the non-empty extension $E_{EC} = \{xy | y \not\leq x^{\perp}\} \setminus E_{G^{\mathscr{L}}}$ is isomorphic to the graph $G(\mathscr{M})$ of a finite model $\mathscr{M}$ of RT_{EC}^-.*

Proof. See appendix. □

As a corollary we see that the finite models are a proper subset of Clarke's *connection structures* characterized as complete ortholattice in (Biacino & Gerla 1991) (figure 2). The models of RT_{EC}^- can be alternatively captured by a representation referring only to the lattices. The extension $C^{\mathscr{M}}$ (which uniquely identifies the model itself) of any model $\mathscr{M}$ can be captured by the ordering of the associated p-ortholattice together with its unique defined orthocomplementation operation: $\langle x,y\rangle \in C^{\mathscr{M}} \iff y \not\leq x^{\perp}$. Hence we derive the following alternative representation:

Corollary 2. *A finite model $\mathscr{M}$ of RT_{EC}^- is isomorphic to a finite p-ortholattice $\mathscr{L}^{\mathscr{M}} = (Y \cup \{\emptyset\}, \cap^*, \cup^*, \emptyset, a^*)$ where $\{\langle x,y\rangle | x \cap^* y \neq \emptyset\} \subset \{\langle x,y\rangle | y \not\leq_{\mathscr{L}^{\mathscr{M}}} x^{\perp}\}$ holds.*

4 Discussion

We have used three kinds of mathematical structure to characterize the models of (subtheories of) RT_0. Our findings using topological spaces are sparse, especially these fail to characterize the finite models beyond discrete topologies. If we represent finite models by infinite point sets, the resulting spaces are not even T_0, and hence from a topological stance uninteresting. If we model the finite models by finite point sets, we reduce them to trivial discrete topology.

The lattice-theoretic approach is more fruitful: characteristic properties of the models of RT^- are captured solely through orthocomplementation and pseudocomplementation which together give an isomorphic description of the models of RT^- as p-ortholattices. Nevertheless, there is no room for the distinctive mereotopological concepts of external connection and weak contact; lattices alone cannot account for A12 and A13. The existence of external connections prohibits unicomplemented and any kind of modular lattices from representing models of RT^-_{EC} and RT_0. By this property, the models are delimited from those of (Clarke 1985) and of the *RCC*. The former are characterized as Boolean lattices which are equivalent to the unicomplemented distributive pseudocomplemented lattices[4] and from inexhaustible (corresponds to atomless) distributive pseudocomplemented lattices models of the *RCC* can be constructed.[5] Hence both theories have models that are distributive unicomplemented lattices. This arises in Clarke's system by the error in the definition of external connection that maps it to overlap and in *RCC* by the lack of any distinction between open and closed elements in the models. This simplification in the *RCC* sacrifices a higher expressiveness offered by the system RT_0. Empirical approaches will be necessary to evaluate when such simplifications are acceptable and which applications or domains require the higher expressiveness of Asher and Vieu's theory.

Our third approach represents the models uniquely as undirected graphs based on the single dyadic primitive C. We characterized RT_C and the more generic *connection structures* as graphs free of true twins; however, for this kind of twin-freeness no characteristic properties are known in graph-theory. In (Hahmann 2008) we further defined a new vertex ordering called *maximum neighborhood inclusion order* (mnio) and demonstrated that this ordering defines a class of graphs that includes all graphs of RT^-_{EC}, and itself is a proper subset of the dually chordal graphs. These orderings are characteristic for the graphs of RT^-_{EC} but not all properties defined by the axioms of RT^-_{EC} are captured by them, especially the existence of sums, intersections, and interiors is not properly translated to graphs withs *mnio*s. We know graphs with *mnio* exist that do not represent models of RT^-_{EC}. Therefore, *mnio*s fail to characterize the models of RT^-_{EC} and RT_0 up to isomorphism. Nevertheless, the graph-theoretic characterization exhibits an alternative approach to ontological evaluation of other mereotopological theories (compare (Hahmann 2008) for details).

Bounded lattices naturally capture the existence of sums and intersections of pairs of elements as well as the essential parthood order of mereological theories, while graphs are capable of fully representing models of RT_0. This leads to a full characterization of the finite models of RT^-_{EC} in terms of graphs of lattices: every finite not unicomplemented p-

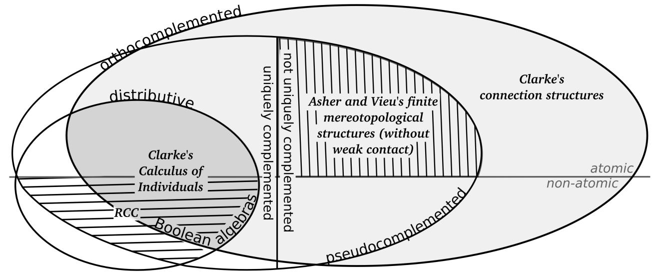


Figure 2: Asher and Vieu's mereotopology, Clarke's *Calculus of Individuals*, and the *RCC* as subclasses of lattices. All known models of *RCC* are representable as subsets of atomless distributive pseudocomplemented lattices (horizontally shaded) but no full representation theorem for RCC exists yet. The models of Clarke's full system are the distributive ortholattices (dark) and the finite models of RT^-_{EC} are the atomic not-unicomplemented, pseudocomplemented ortholattices (vertically shaded).

ortholattice $\mathscr{L}$ uniquely defines a graph $G^{\mathscr{L}}$ that is equivalent to a finite model $\mathscr{M}$ of RT^-_{EC} where $\langle x,y\rangle \in O^{\mathscr{M}} \iff \exists z[z \leq x \wedge z \leq y \wedge z \neq \emptyset] \iff xy \in E(G^{\mathscr{L}})$ and $\langle x,y\rangle \in EC^{\mathscr{M}} \iff \{xy \in \left(E(G^{\mathscr{L}}_{EC}) \setminus E(G^{\mathscr{L}})\right) \wedge y \not\leq x^{\perp}\}$. Considered as lattices, these structures maintain ortho- and pseudocomplementation while they uniquely extend to graphs free of true twins with non-empty extensions $EC^{\mathscr{M}}$.

In a final step, see (Hahmann 2008) for details, finite models of RT_0 with weak contacts can be constructed as direct products of finite p-ortholattices (see the proof of lemma 1 for an example). The product of two finite p-ortholattices of which at least one is not unicomplemented, each extended by separate closures of their suprema, is a (finite) model of RT_0. The problem of whether any model of RT_0 can be obtained in a similar fashion is still open; this would lead to a representation theorem of the models of full RT_0. Although such a theorem is desired, we think that it can give little extra insight into the models of RT_0. The representation theorem for RT^- is more important and characteristic for the mereotopology. Through the given characterization it is now easy to construct p-ortholattices that correspond to models. Even more important, we can identify the extensions of all mereotopological relations from the lattice alone. Orthocomplements in the lattices map to complements in the models, the join and meet of pairs in the lattice represent the unique sum and intersection in the corresponding model. The closure and interior are equivalent to the meet- and join-pseudocomplements of the orthocomplement. Overlap relations produce meets distinct from $\emptyset$; all elements externally connected to a given e are not connected to the orthocomplement of e and not to e in any other way.

Still open is the question whether the infinite models of Asher & Vieu's mereotopology always give complete lattices. If not, the theory RT_0 actually weakens Clarke's unrestricted fusion axiom. Otherwise, we obtain a proof that the unrestricted fusion can be replaced lossless by the sum axiom A5 without impacting the infinite models.

[4]Distributive pseudocomplemented is not enough, this class contains Heyting and Stone lattices as well.

[5]A full representation theorem for the models of *RCC* is still outstanding, but we expect all models of the *RCC* to be distributive.

Overall, this paper outlines a methodology for characterizing models of a mereotopology that enables a model-theoretic comparison of mereotopological axiomatizations, which allows to understand differences and commonalities between the axiomatizations. The lattice-based approach turned out most promising since it captures essential mereological and topological concepts such as parthood and complements. All mereotopological theories using a single primitive can be also represented as graphs of lattices as demonstrated. In future work, we will analyze the system of (Borgo, Guarino, & Masolo 1996) that explicitly distinguishes a topological (*simple region*) and a mereological primitive (*parthood*) and comprises a notion of convexity. Other ontologies not yet fully treated in a model-theoretic include the mereotopology of (Smith 1996). Conversely, one can choose a promising class of lattices and show whether it yields useful mereotopological systems – either generic or limited to a certain application domain. The set of potential candidates include semimodular lattices, geometric lattices, the full class of p-ortholattices, Stone lattices, Heyting lattices (compare (Stell & Worboys 1997)), the full class of pseudocomplemented distributive lattices, and – more generic – pseudocomplemented or orthocomplemented (see the connection structures of (Biacino & Gerla 1991)) lattices.

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Appendix

Proof outline for theorem 2

We need proposition 4, in particular point (4), for the upcoming proof. For details, see (Hahmann 2008).

Proposition 4. *In a model $\mathscr{M}$ of RT_0, RT^-_{EC} and RT^- it holds for an individual x and its complement $-x$:*
(1) $\forall x\,[\neg C(x,-x)]$
(2) $\forall x,y\,[C(x,y) \vee C(-x,y)]$
(3) $\forall x,z\,[PP(x,z) \rightarrow C(-x,z)]$
(4) $\forall x\,[PP(z,x) \equiv \neg PP(z,-x)]$ and
$\forall x,z \neq \mathbf{a}^*\,[PP(x,z) \equiv \neg PP(-x,z)]$.

We first prove the direction $\mathfrak{L}_{RT} \rightarrow$*p-ortholattices*, splitted up into two claims: (1) each lattice $\mathfrak{L}_{RT} \rightarrow$ is meet- and join-pseudocomplemented and (2) each lattice $\mathfrak{L}_{RT} \rightarrow$ is orthocomplemented. Afterwards the direction *p-ortholattices*$\rightarrow \mathfrak{L}_{RT}$ of the theorem is proved.

$\mathfrak{L}_{RT} \rightarrow$ *p-ortholattices*:

Claim 1. Any lattice $\mathscr{L}$ constructed from a model of RT_0 is meet- and join-pseudocomplemented: $cl(a')$ and $int(a')$ define uniquely the meet- and join-pseudocomplement, respectively, of any element $a \in \mathscr{L}$ where a' is any complement of a.

Proof. We know every such lattice $\mathscr{L}$ is complemented: for every $a \in Y \cup \emptyset$ there exists a complement a' so that $a \wedge a' = 0$ and $a \vee a' = 1$. In (Hahmann 2008), we proved that for any complement a' of a, $cl(a')$ and $int(a')$ are also complements of a. Now we only need to show that $cl(a')$ and $int(a')$ are the unique meet- and join-pseudocomplements. For that we claim: (i) every element b with $b > cl(a')$ has a non-zero meet with a and thus cannot be meet-pseudocomplement of a, and every element c with $c < int(a')$ has a join with a that is not the supremum; (ii) every element b with $a \wedge b = 0$ or $a \vee b = 1$ satisfies the condition $b \leq cl(a')$ or $b \geq int(a')$, respectively. I.e. all other complements are $\leq cl(a')$ or $\geq int(a')$, respectively.
(i) Assume b with $b > cl(a')$ and $b \wedge a = 0$ exists. Then the extension of C in which b participates must subsume the extension of C in which $cl(a')$ participates. If the extensions of O where b or $cl(a')$ participate are the same then either $cl(a')$ is not closed (b has an additional external connection) or b and $cl(a')$ have the same extensions of C and are by A3 identical. If the extension of O in which b participates is strictly greater than that of $cl(a')$, then b must overlap with some part of a and $b \wedge a = 0$ does no longer hold. In both cases we derive a contradiction.
(ii) From (i) we know there exists no such b with $b > cl(a')$ so that $b \wedge a = 0$. Now we prove that no other element b exists with $b \wedge a = 0$ that is incomparable to $cl(a')$[6]. Then $b \leq cl(a')$ follows immediately. Notice that every element $b \in Y^{\mathscr{M}}$ is either in a proper part relation to a or its (topological) complement $-a$, see proposition 4. In the lattice representation, b is either comparable to a or $-a$. Assume a' to be the orthocomplement of a in the corresponding lattice (we later prove that such orthocomplement always exists). If b is comparable to a then obviously $a \wedge b = 0$ does not hold. Hence b must be comparable to the element representing $-a$ in the lattice. The trivial case is $cl(a') = -a$. Otherwise the sum $b \cup^* cl(a')$ overlaps in some part(s) with a (because $cl(a') \geq -a$), which in turn requires one part (either of b or $cl(a')$, or of a third element) to overlap with a. That would mean either b or $cl(a')$ overlaps with a: obviously a and $cl(a')$ cannot overlap, so a and b must overlap. But in this case $a \wedge b = 0$ does not hold. Hence no such b can exist.
From (i) and (ii) together with the fact that $cl(a')$ is also a complement of a, $cl(a')$ must be the meet-pseudocomplement of a.
The proof for the join-pseudocomplements is analogous. □

Claim 2. Any lattice $\mathscr{L}$ constructed from a model of RT_0 is an ortholattice with the topological (set-theoretic) complement $\sim$ as orthocomplementation operation.

Proof. We check conditions (1) to (3) for the operation $\sim$, choosing $\sim a^* = \emptyset$ and $\sim \emptyset = a^*$ to ensure that the operation $\sim$ is a complete function on the set $Y \cup \{\emptyset\}$. Property (1) and (2) ($x \cap^* \sim x = \emptyset$) hold from the set-theoretic definition of topological complements. To prove (3), consider x and y as sets of points: $x \leq y$ (in the lattice) iff $x \subseteq y$. If $x = y$ then $\sim x = \sim y$ and (3) holds trivially. Hence assume $x \subset y$, then all the points in $y \setminus x$ (non-empty) must be part of the complement of x, i.e. $y \setminus x \subseteq \sim x$. Since all points that are both in x and y are in neither complement and all points in neither set are in both complements, $\sim y$ must be a proper subset of $\sim x$, i.e. $\sim x = a^* \setminus (x \cap y)$ and $\sim y = a^* \setminus (x \cap y) \setminus (y \setminus x)$. $a^* \setminus (x \cap y) \setminus (y \setminus x) \subseteq a^* \setminus (x \cap y)$ follows and with $y \setminus x$ distinct from $x \cap y$ and assumed to be non-empty: $a^* \setminus (x \cap y) \setminus (y \setminus x) \subset a^* \setminus (x \cap y)$. Thus $\sim y < \sim x$, satisfying the order-reversing law (3). □

p-ortholattices$\rightarrow \mathfrak{L}_{RT}$:

Now we show the reverse direction of the main theorem: every p-ortholattice can be associated to a structure $\langle Y, f, [\![]\!] \rangle$ satisfying the conditions (i)-(vi) of the definition of the intended models in $\mathfrak{RT}$. We use the following notation throughout the rest of the proof:

Definition 4. Let $\mathscr{L}$ be a p-ortholattice. For any lattice element p, its unique orthocomplement is denoted by $^\perp : Y \rightarrow Y$ and $jpc : Y \rightarrow Y$ and $mpc : Y \rightarrow Y$ identify the unique the join- and meet-pseudocomplement, respectively.

For the proof we split (ii) and (iii) into more manageable subconditions (ii.1), (ii.2), (ii.3), (iii.1), and (iii.2). Except for (ii.3) and (iii.2), which will be handled separately, the proofs are straightforward and thus only briefly outlined.

(i) $Y \subseteq \mathscr{P}(X)$ and $X \in Y$
Satisfied when the top element of the lattice (guaranteed by boundedness) is mapped to the set X.

(ii.1) $\forall x\,[int(x) \in Y]$
By the involution property of orthocomplementation each element $x^\perp$ is the orthocomplement of some element x. Choosing the unique join-pseudocomplement of $x^\perp$ as $int(x) = jpc(x^\perp)$ (equiv-

[6] a and b are called comparable if either $a \geq b$ or $b \geq a$. Otherwise a and b are incomparable.

alence proved in (Hahmann 2008)) gives an interior for each element.

(ii.2) $\forall x\, [int(x) \neq \emptyset]$
$jpc(x)$ cannot be equal to the $\emptyset$ unless $x = 1$ since $jpc(x) \vee x = 1$ always must hold. The top element $x = 1$ has itself as interior.

(ii.3) $\forall x\, [int(x) = int(cl(x))]$- *See separate proof.*

(iii.1) $\forall x\, [cl(x) \in Y]$
Each element $x^{\perp}$ is the orthocomplement of some element x (by involution property of orthocomplementation). Choosing $mpc(x^{\perp}) = cl(x)$ (equivalence proved in (Hahmann 2008)) gives us a closure for each element.

(iii.2) $\forall x\, [cl(x) = cl(int(x))]$ - *See separate proof.*

(iv) $\forall x \in Y\, (int(\sim x) \neq \emptyset \rightarrow \sim x \in Y)$
Each element in the lattice has a unique orthocomplement. If the orthocomplement is the empty set (can happen only when the interior is empty), then the element is not in Y.

(v) $\forall x, y \in Y\, (int(x \cap y) \neq \emptyset \rightarrow (x \cap^* y) \in Y)$
Map the element representing the greatest lower bound (meet) of x and y to $x \cap^* y$. In any lattice must by definition exist a unique greatest lower bound for any pair of elements x, y. If the meet is $\emptyset$, then $x \cap^* y$ is not an element of Y.

(vi) $\forall x, y \in Y\, ((x \cup^* y) \in Y)$
Map the element representing the least upper bound (join) of x and y to $x \cup^* y$. The lattices themselves again ensure the existence of this element for every pair x, y.

To prove (ii.3) and (iii.2), we restate these conditions in purely lattice-theoretic terms:

(E1) $mpc(x) = mpc\left[(jpc(x))^{\perp}\right]$

(E2) $jpc(x) = jpc\left[(mpc(x))^{\perp}\right]$

To proof that these two conditions hold in every p-ortholattice, we need the following theorem.

Theorem 5. *In every p-ortholattice $jpc(p) \leq p^{\perp} \leq mpc(p)$ holds for any element $p \in \mathscr{L}$.*

Proof. Assume the contrary: $p^{\perp} \not\geq jpc(p)$: since $p \vee p^{\perp} = 1$ this violates the definition of the join-pseudocomplement. Similarly for $p^{\perp} \not\leq mpc(p)$ where $p \wedge p^{\perp} = 0$. $\square$

Finishing the proof of the direction *p-ortholattices* $\rightarrow \mathfrak{L}_{RT}$ of theorem 2 by proving E1 and E2: *Proof.* We show that E1 and E2 are immediate consequences in any finite p-ortholattice where the orthocomplementation $^{\perp}$ is the topological complement. The proof is by contradiction: we show that if for some element p of the lattice, $m = mpc(p)$ and $m' = mpc\left[(jpc(p))^{\perp}\right]$ are satisfied, then $m = m'$. We distinct the following cases depending on the relative position of m and m' in the lattice:

(a) assume m incomparable to m',
(b) assume $m > m'$ and
(c) assume $m < m'$.

All three cases lead to a contradiction, thus the only valid solution being $m = m'$. Note hereby that the corollary **??** can be restricted to: $p < jpc(p)^{\perp}$ and $p > mpc(p)^{\perp}$, otherwise $m = m'$ would follow immediately from $p = jpc(p)^{\perp}$ and $p = mpc(p)^{\perp}$, respectively. In the following we do the proof for E1 only, for E2 it is analogous.
Case (a) assume that m and m' are incomparable. We know that $jpc(p)^{\perp} \geq p$: $jpc(p) \leq p'$ (the join-pseudocomplement is the smallest of all complements of p), in particular it then holds $jpc(p) \leq p^{\perp}$ and thus by the order-reversing law of ortholattices $jpc(p)^{\perp} \geq p$ follows. Then naturally it follows that $mpc(jpc(p)^{\perp}) \leq mpc(p)$. Hence $m' \leq m$ and m and m' are comparable - contrary to the assumption.
Case (b) assume that $m > m'$ holds. Assuming $m \wedge jpc(p)^{\perp} = \emptyset$ with $m > m'$ would lead to a contradiction because m' can no longer be the meet-pseudocomplement of $jpc(p)^{\perp}$. Hence, $m \wedge jpc(p)^{\perp} > \emptyset$ must hold. By completeness of the lattice, this intersection results in some element, let us denote it by z, s.t. $m \wedge jpc(p)^{\perp} = z$. Since (i) $m' \wedge jpc(p)^{\perp} = \emptyset$ and (ii) $m \wedge p = \emptyset$, the element z must further satisfy following properties: (iii) $z \wedge m = z$, (iv) $z \wedge jpc(p)^{\perp} = z$, (v) $z \wedge p = \emptyset$, and (vi) $z \wedge m' = \emptyset$ (because $jpc(p)^{\perp} \wedge m' = \emptyset$ and $z < jpc(p)^{\perp}$). Now let us consider the element $(z \vee m)$: From (ii) and (v) it follows $(z \vee m) \wedge p = \emptyset$ (by DeMorgan laws which apply for all orthocomplemented lattices, see (Kalmbach 1983)) with $(z \vee m) \geq m$, hence m cannot be the meet-pseudocomplement of p unless $z = m$ and we derive a contradiction again: no such z can exist that is distinct from m, hence $m = m'$.
Case (c) assume that $m < m'$ holds. Again $m' \wedge p = \emptyset$ would lead to a contradiction because m can no longer be the meet-pseudocomplement of p. Hence, $m' \wedge p > \emptyset$. We know $jpc(p)^{\perp} \geq p$ from corollary **??**, and therefore $m' \wedge jpc(p)^{\perp} \geq m' \wedge p$ (notice that $a \geq b \rightarrow c \wedge a \geq c \wedge b$ holds in any complete lattice, thus in the finite lattices). With $m' \wedge jpc(p)^{\perp} = \emptyset$ we obtain $m' \wedge p = \emptyset$, which is contradictory to our previous assumption $m' \wedge p > \emptyset$.
From the cases (a), (b), and (c) all resulting in a contradiction, $m = m'$ must hold. Thus $mpc(p) = mpc\left[jpc(p)^{\perp}\right]$ for any element p in the lattice. Analogous one can prove E2. Together, E1 and E2 finish the proof for (ii.3) and (iii.2) and thus for theorem 2. $\square$

Proof outline for theorem 4

We need proposition 5 which is a consequence of the definition of *PP* as irreflexive partial order. It can be proved by contradiction from D1, D2, together with proposition 2.

Proposition 5. *For $\mathbf{x}, \mathbf{y} \in Y^{\mathscr{M}}$ in a model $\mathscr{M}$ of RT_0, $\langle \mathbf{x}, \mathbf{y} \rangle \in PP^{\mathscr{M}}$ iff $N[\mathbf{x}] \subset N[\mathbf{y}]$ holds in the representing graph $G(\mathscr{M})$.*

Proof for theorem 4. Notice that for every p-ortholattice $\mathscr{L}$ the graph $G^{\mathscr{L}}$ is uniquely defined because of the unique definitions of the set of vertices $V(G^{\mathscr{L}})$ and the set of edges $E(G^{\mathscr{L}})$ (we refer to them as V and E if no confusion can arise about the graph). Thus the graph $G^{\mathscr{L}}$ is uniquely defined for every model of RT^{-}_{EC}. Moreover, the lattices representing models of RT^{-}_{EC} are not unicomplemented p-ortholattices, where a finite model $\mathscr{M}$ of RT_{EC}^{-} gives a finite not-unicomplemented p-ortholattice $\mathscr{L}$ which again gives a finite graph $G^{\mathscr{L}}$ with non-empty extension E_{EC}. Thus every finite model of RT_{EC}^{-} results in a graph $G^{\mathscr{L}}$ as required by the theorem. In the reverse direction, any graph $G_{EC}^{\mathscr{L}} = \left(V(G^{\mathscr{L}}), E(G^{\mathscr{L}}) \cup E_{EC}\right)$ constructed from a not unicomplemented p-ortholattice gives a model $\mathscr{M}$ of RT_{EC}^{-}. The extension $E_{EC} = \left\{xy|y \nleq x^{\perp}\right\} \setminus E(G^{\mathscr{L}})$ is non-empty (claim 1) and thus satisfies A11. Afterwards we show that $G_{EC}^{\mathscr{L}}$ satisfies the axioms A1 to A10 and A13 (A1, A2, A4, A7 and A9 are straightforward and omitted here).

Claim 1. $E_{EC} = \left\{xy|y \nleq x^{\perp}\right\} \setminus E(G^{\mathscr{L}})$ is non-empty.

Assume the contrary, i.e. that $E_{EC} = \{\}$ for a graph $G^{\mathscr{L}}$. Then it holds that $\left\{xy|y \nleq x^{\perp}\right\} \subseteq E(G^{\mathscr{L}})$. Additionally, $E(G^{\mathscr{L}}) \subseteq \left\{xy|y \nleq x^{\perp}\right\}$ because no individual can be connected to its complement or parts thereof. But then the graph representation of each element x has a unique neighborhood $N[x] = \left\{xy|y \nleq x^{\perp}\right\}$ just from the parthood relation. Hence the underlying lattice is unicomplemented.

Claim 2. $G_{EC}^{\mathscr{L}} = G^{\mathscr{L}} \cup \left\{xy|y \nleq x^{\perp}\right\}$ satisfies A3.

Assume there exist two elements $x, y \in V(G_{EC}^{\mathscr{L}})$ such that $N[x] = N[y]$. Since $xx^{\perp} \notin E(G_{EC}^{\mathscr{L}})$ and thus $x^{\perp} \notin N[x]$ it follows that $x^{\perp} \notin N[y]$. The same for $y^{\perp}$, i.e. $y^{\perp} \notin N[x]$. Then by the definition of $G_{EC}^{\mathscr{L}}$, a contradiction arises because both $y^{\perp} \leq x^{\perp}$ and $x^{\perp} \leq y^{\perp}$ must hold. Hence no two vertices $x, y \in V(G_{EC}^{\mathscr{L}})$ with $N[x] = N[y]$ can exist.

Claim 3. The extension $P^{\mathscr{M}}$ of the parthood relation in $\mathscr{M}$ is given by the lattice $\mathscr{L}$, i.e. $x \leq y \iff \langle x, y\rangle \in P^{\mathscr{M}}$.

Assume $x \leq y$ for some pair x, y. That means $N[x] \subseteq N[y]$. Whenever a third element z is connected to x, it will also be connected to y, since by the order-reversing law, $y^{\perp} \leq x^{\perp}$ holds and if $z \nleq x^{\perp}$ then $z \nleq y^{\perp}$. So $N[x] \subseteq N[y]$ is preserved in $G_{EC}^{\mathscr{L}}$ (when adding the edges in E_{EC} to $G^{\mathscr{L}}$) and thus $\langle x, y\rangle \in P^{\mathscr{M}}$. On the reverse, if $\langle x, y\rangle \in P^{\mathscr{M}}$ in a model of RT_{EC}^{-}, then $N[x] \subseteq N[y]$ in the graph $G_{EC}^{\mathscr{L}}$. If now $N[x] \nsubseteq N[y]$ in $G^{\mathscr{L}}$, then $x \nleq y$ and $y^{\perp} \nleq x^{\perp}$. I.e. some z exists with $\langle x, z\rangle \in E_{EC}$ but $\langle y, z\rangle \notin E_{EC}$. Then either $y^{\perp} > x^{\perp}$ or $y^{\perp}$ and $x^{\perp}$ are incomparable - with the consequence of $z \in N[x]$ but $z \notin N[y]$, or $y^{\perp} \in N[x]$ but $y^{\perp} \notin N[y]$, respectively.

Claim 4. $G_{EC}^{\mathscr{L}} = G^{\mathscr{L}} \cup \left\{xy|y \nleq x^{\perp}\right\}$ satisfies A5.

Let $z \in V(G^{\mathscr{L}})$ be the sum element $z = x \cup^* y$ for some pair $x, y \in V(G^{\mathscr{L}})$; hence $z = x \cup^* y \geq x, y$. We prove each direction of the equivalence in A5 individually.

(a) $\exists v\left[(C(v,x) \vee C(v,y)) \rightarrow C(v,z)\right]$
Since $z \geq x$, either $z = x$ and $zv \in E \iff xv \in E$ or $z > x$ and by proposition 5: $xv \in E \Rightarrow zv \in E$; analogous for y.

(b) $\exists v\left[(C(v,x) \vee C(v,y)) \leftarrow C(v,z)\right]$
Assume there exists an element v s.t. $zv \in E$ but $xv, yv \notin E$. Let v be comparable to z but not to x and y. This can only occur if $v < z$ and v is disjoint with both x and y (otherwise by transitivity, $v \geq x, y$ would follow). If there is a common proper part u, i.e. w.l.g. $u < v, x$ then v and x are connected. If no such u exists, there exists at least three atoms in this subbranch of the lattice. But then the lattice is not pseudocomplemented, since dual-atoms not comparable to these atoms would not have unique join-pseudocomplements. Otherwise if v is comparable to z, it is comparable to at least one of x and y.
If v is not comparable to z, then v is comparable to $z^{\perp}$, i.e. either $v \leq z^{\perp}$ or $v > z^{\perp}$. In the first case v cannot be connected to z by definition but contrary to the assumption. In the latter case v is comparable to one of $x^{\perp}$ and $y^{\perp}$. If v would be incomparable to both, there must exist three distinct dual-atoms in this subbranch of the lattice and the lattice is not meet-pseudocomplemented. If v is comparable to only one of them, i.e. w.l.g. to x then $yv \in E$ since $v \nleq y^{\perp}$. If v is comparable to x and y and $v < x^{\perp}, y^{\perp}$ then $v = x \cap^* y$ and thus $v^{\perp} = x \cup^* y$ by the order-reversing law. Hence z is not the sum of x and y. If $v > x^{\perp}, y^{\perp}$ (note that if x and y are comparable with each other, they are ordered and z is not the sum of x and y) then $v > z^{\perp}$ and $xv, yv, zv \notin E$ would follow contrary to our assumption that $zv \in E$.

Claim 5. $G_{EC}^{\mathscr{L}} = G^{\mathscr{L}} \cup \left\{xy|y \nleq x^{\perp}\right\}$ satisfies A6.

By claim 3 the parthood and hence the overlap relation is predefined by the lattice. We show that if the intersection $z = x \cap^* y$ given by the lattice $\mathscr{L}$ with $z < x, y$ has an additional element $v \in N(z)$, then $v \in N(x), N(y)$: assume v with $zv \in E$, then $v \nleq z^{\perp}$. Since $z^{\perp} \geq x^{\perp}, y^{\perp}$ it follows that $v > x^{\perp}, y^{\perp}$ or v is incomparable to $x^{\perp}, y^{\perp}$. The latter case also implies $v \nleq x^{\perp}$ and $v \nleq x^{\perp}$. Thus in any case, $vx, vy \in E$.

Claim 6. $G_{EC}^{\mathscr{L}} = G^{\mathscr{L}} \cup \left\{xy|y \nleq x^{\perp}\right\}$ satisfies A8.

For any x take the greatest open element $y \leq x$ with the same overlap extension as x. Such an element must exist if the underlying lattices are atomic: any atom $y < x$ satisfies A8 because it is not externally connected: its orthocomplement $y^{\perp}$ is a dual-atom (by orthocomplementation) and $\forall z\left[z \ngeq y \rightarrow y^{\perp} \geq z\right]$ and for all such z, $yz \notin E$ follows.

Claim 7. $G_{EC}^{\mathscr{L}} = G^{\mathscr{L}} \cup \left\{xy|y \nleq x^{\perp}\right\}$ satisfies A10.

By D8 $\langle x\rangle \in OP^{\mathscr{M}}$ for a model $\mathscr{M}$ associated to $G_{EC}^{\mathscr{L}}$ iff $\left\{xv|v \in V, v \nleq x^{\perp}\right\} = \{\}$ (similar for y). Then $\neg\exists v\left[v \leq x^{\perp}, y^{\perp}|xv \in E_{EC} \textit{ or } yv \in E_{EC}\right]$ and with $z = x \cap^* y \leq x, y$, $\{\langle z, v\rangle \in E_{EC}\} \subseteq \{\langle x, v\rangle \in E_{EC}\}, \{\langle y, v\rangle \in E_{EC}\} \subseteq \{\}$ follows for all $v \in Y$, i.e. z is not externally connected. Then $\neg\exists v\left[v \leq z^{\perp}|zv \in E\right]$ and $\langle z\rangle \in OP^{\mathscr{M}}$ because $zz^{\perp} \notin E$ and $z^{\perp} \geq x^{\perp}, y^{\perp}$. If $z = x$ (or $z = y$) then $y < x$ (or $x < y$) and again $\langle z\rangle \in OP^{\mathscr{M}}$.

Claim 8. $G_{EC}^{\mathscr{L}} = G^{\mathscr{L}} \cup \left\{xy|y \nleq x^{\perp}\right\}$ satisfies A13.

For all x, a^* is an (not the smallest) open neighborhood for any other element (including itself). A13 is only violated for some x if and only if there exist two open elements y_1 and y_2 with $y_1, y_2 \geq x$. If y_1 and y_2 are comparable, then either $y_1 \geq y_2$ or vice versa. So let us assume that y_1, y_2 are incomparable. Notice that all of x, y_1, y_2 are related to the same set of elements by an overlap relation, otherwise this branch of the lattice contains two atoms and the lattice would not be pseudocomplemented. Then, since y_1 and y_1 are not externally connected, $N[y_1] = N[y_2]$ follows which contradicts twin-freeness of $G_{EC}^{\mathscr{L}}$. Hence, A13 is satisfied in every graph $G_{EC}^{\mathscr{L}}$.

The claims together prove the representation theorem. □