A Theory of Multidimensional Qualitative Space: Semantic Integration of Spatial Theories that Distinguish Interior from Boundary Contact
(Extended Abstract)

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Summary. We extend our multidimensional qualitative theory of space based on relative dimension and containment by closures under intersections and differences and by a new primitive relation of boundary containment, a special kind of incidence. We provide a descriptions of the intended structures which ensure that boundary and interior are always clearly distinguishable. The resulting theory can define all the mereotopological relations from previous accounts of areas, lines, and points in a general multidimensional setting and is thus capable of semantically integrating those accounts of space.

Qualitative descriptions of space can provide a more natural and human interface to spatial models used in Geographic Information Systems (GIS), Computer-Aided Design (CAD) or Computer-Aided Manufacturing (CAM) software and are helpful when precise quantitative spatial data is not available due to noise, incompleteness, or insufficient granularity. Many such theories, e.g. [1–3, 6], distinguish topological situations based on whether the interior or boundary is in contact and what the dimension of the contact is. Exchanging spatial information among systems using such theories requires some form of semantic integration. We pursue integration by developing a theory of multidimensional qualitative space in first-order logic [5, 4] with successively stronger extensions that are equivalent to the existing spatial theories. This results in a hierarchy of theories in which the strongest theory shared by two spatial theories indicates what sentences can be exchanged between the software systems using those theories.

In [5] we proposed a basic theory consisting of fairly weak axiomatizations of the core primitive relations relative dimension $x <_{dim} y$ (x is of a lower dimension than y) and spatial containment $Cont(x, y)$ (in their point set interpretations, all points contained in x are contained in y; a lower-dimensional entity can be spatially contained in a higher-dimensional entity though it cannot be part thereof) which suffice to define three jointly exhaustive, pairwise disjoint types of contact (Figure 1): Partial overlap, incidence, and superficial contact – only distinguished by the relative dimension of the entities and their shared entity.

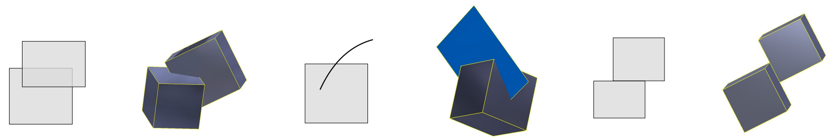

Fig. 1. Partial overlap, incidence, and superficial contact each in 2D, 3D (left to right).

Additional axioms in [4] define various extensions that are definably interpreted by existing equidimensional and multidimensional mereotopologies, incidence structures, or incidence geometries.

Our objective here is to semantically integrate the spatial theories which distinguish between interiors and boundaries being in contact, in particular the theories in [1–3, 6] that capture mereotopological relations between areas, lines, and points in two-dimensional space. Similar to our work in [5, 4], we approach this problem by extending our basic theory from [5] in a way that allows us to distinguish boundary from interior contact.

First, we restrict the intended models to well-behaved ones in which the boundary is clearly distinguished from the interior of an entity. In general, the structures we want to capture are n-dimensional spaces in which *simple* spatial entities are arranged arbitrarily. All simple entities are of uniform dimension $m \leq n$ and composed of *atomic*, i.e., self-connected entities. Simple entities cannot self-intersect, that is, no point can be both in the interior and boundary at the same time, and have no singularities or missing lower-dimensional entities. The atomic entities that constitute a simple entity may only be connected in their boundaries or may not connect at all. Atomic entities are m-manifolds with boundaries: They are locally Euclidean in $\mathbb{R}^m$, that is, each point has a

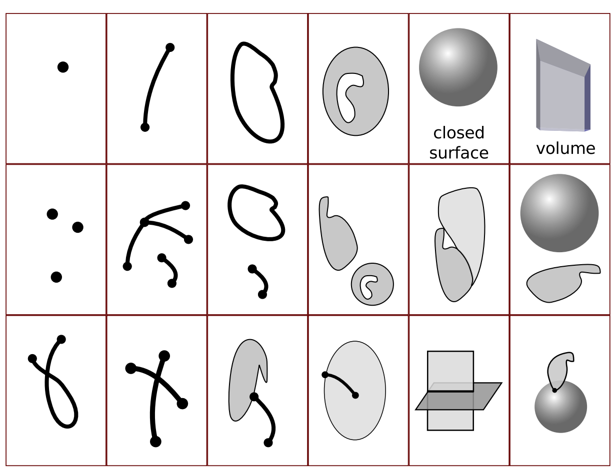

Fig. 2. Examples of simple atomic entities (top row), simple non-atomic entities (middle row), and non-simple entities (bottom row).

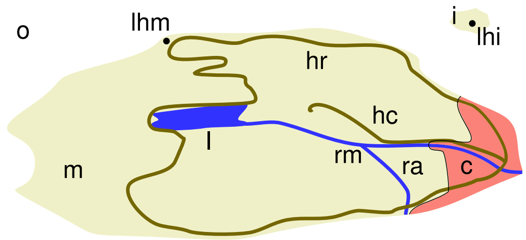

Fig. 3. A sketch map containing 2D entities (**o**cean, **m**ain island, small **i**sland, **c**ity, **l**ake); 1D entities (**r**iver **m**ain, **r**iver **a**rm, **h**ighway **r**ing, **h**ighway **c**entral); and point entities (**l**ight**h**ouse **m**ain, **l**ight**h**ouse **i**sland).

neighbourhood homeomorphic to an open subset of the m-dimensional upper half of space $\mathbb{H}^m = \{(x_1, ..., x_m) \in \mathbb{R}^n : x_n \geq 0\}$. They may have an empty boundary such as a closed line (loop) or the surface of a solid object. Complex entities are not objects of the domain, but can be captured as sets of simple entities or introduced as a separate class using a new unary relation (sortal).

Exemplary classes of models are: (1) pieces of curves and curved planes randomly placed in a three-dimensional space, (2) sketch maps containing entities of various dimensions (cf. Figure 3), (3) 3D fire escape maps, or (4) 3D models of buildings, cars, or airplanes which may include wires (electrical, communication) or utility pipes (water, gas).

To capture those models, we first extend our basic theory of containment and linear dimension [5] by two types of 'downward mereological closures': (1) Intersections, the shared entities of highest dimension, and (2) differences between entities and their proper parts of identical dimension. Sums are unnecessary for our purpose. In the resulting theory $CODI_\downarrow$ entities are decomposed by intersections and differences into atomic entities, compare Figure 4. Secondly, we axiomatize the relation of 'boundary containment' $BCont(x, y)$, a specialization of incidence, with the intended meaning of 'x is contained in the boundary of y' (x can be of any dimension lower than y). Boundaries are guaranteed to exist between entities in superficial contact of which one has a local codimension of zero (they are contained in a common entity of no greater dimension). Other axioms state the conditions under which entities cannot be contained in the boundary. However, in many cases boundary containment is neither forced nor

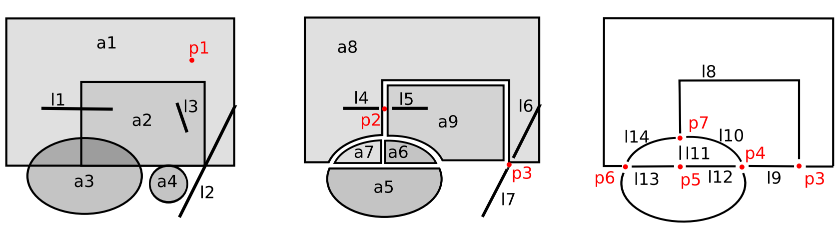

Fig. 4. A model of $CODI_\downarrow$ decomposed by intersections and differences into simple atomic entities (points p1 – p7, lines l3 – l14, and areas a4 – a9).

ruled out by the axiomatization, see e.g. Figure 5. Evidently, $BCont$ is not definable in $CODI_{\downarrow}$ and necessitates a primitive relation. But interior and tangential containment, $ICont$ and $TCont$, are subsequently definable.

The resulting theory[3] can define three symmetric overlap relations (interior, boundary, and exterior overlap) and three non-symmetric contact relations (interior-boundary, interior-exterior, and boundary-exterior contact); exterior contact is implicitly defined as contact without containment. Thus, the theory can distinguish the nine resulting combinations of contact as well as different strengths of contact (the relative dimension of the shared entity). If $BCont$ correctly captures all shared entities in the boundaries of an intended structure, our extension distinguishes all the mereotopological 'cases' from [1–3, 6] in the corresponding model. Hence, those theories can be semantically integrated by our more general multidimensional qualitative theory which is not restricted to a fixed set of dimensions. We can still axiomatically constrain our theory to only allow the models of [1–3, 6].

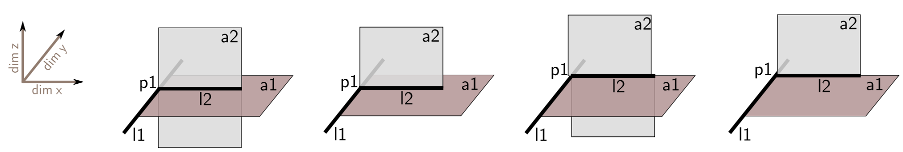

Fig. 5. Four models equivalent in $CODI_{\downarrow}$ with different extensions of $BCont$: in the leftmost model neither $BCont(l2, a1)$ nor $BCont(l2, a2)$ while in the two models in the middle one of them hold and in the rightmost model both hold. They are non-equivalent models of $CODI_{\downarrow} \cup$ {BC-A1 – BC-A5} with $BCont$ as primitive relation.

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[3] All axioms are available at `www.cs.toronto.edu/~torsten/DCT-BCont`