Stonian p-Ortholattices: A new approach to the mereotopology RT_0

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Abstract

This paper gives an isomorphic representation of the subtheories RT^-, RT^-_{EC}, and RT of Asher and Vieu's first-order ontology of mereotopology RT_0. It corrects and extends previous work on the representation of these mereotopologies. We develop the theory of p-ortholattices – lattices that are both orthocomplemented and pseudocomplemented – and show that the identity $(x \cdot y)^* = x^* + y^*$ defines the natural class of Stonian p-ortholattices. Equivalent conditions for a p-ortholattice to be Stonian are given. The main contribution of the paper consists of a representation theorem for RT^- as Stonian p-ortholattices. Moreover, it is shown that the class of models of RT^-_{EC} is isomorphic to the non-distributive Stonian p-ortholattices and a representation of RT is given by a set of four algebras of which one need to be a subalgebra of the present model. As corollary we obtain that Axiom (A11) – existence of two externally connected regions – is in fact a theorem of the remaining axioms of RT.

Key words: mereotopology, Stonian p-ortholattice, non-distributive pseudocomplemented lattice, qualitative spatial reasoning, region-based space

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1. Introduction

Within AI and in particular Knowledge Representation (KR), region-based theories of space have been a prominent area of research in the recent years. Traditionally, space has been considered in mathematics as point-based theories such as geometric (e.g. Euclidean geometry) or topological representations (point-set topology) of space. Points are somewhat tricky to define and are far from intuitive in real-world applications. Instead, point-free theories of space such as region-based theories can be used to represent space in the context of (qualitative) spatial reasoning. Using regions instead of points as smallest units accounts more naturally for how humans conceptualize our physical world. Such commonsense spatial reasoning reflects rigid bodies or spatial regions more naturally than conventional, point-based models [19, 27]. Since the earliest work of de Laguna [12] and Whitehead [30], mereotopology has been considered for building point-free theories of space. In AI, these theories are of importance for qualitative spatial reasoning (QSR): they focus on simple properties that abstract from quantitative measurements while still being powerful enough to reason about spatial configurations and extract useful spatial knowledge, e.g. about bordering regions, intersecting regions, or the composition of regions. For an overview of mereotopology within QSR we refer to [10].

Broadly speaking, mereotopology is a composition of topological (from Greek *topos*, "place") notions of connectedness with mereological (from Greek *méros*, "part") notions of parthood. Neither topology nor mereology are by themselves powerful enough to express part-whole relations.

Topology can also be seen as a theory of wholeness, but has no means of expressing parthood relations. Connection does not imply a parthood relation between two individuals, as well as disconnection does not prevent parthood. Just consider the example of countries - there exist many countries, e.g. the United States, that are not self-connected. Alaska should be considered part of the United States but is by no intuitive means connected to the other states. The same applies for Hawaii, although the kind of separation is different here: Alaska is separated by Canada from the continental US, whereas Hawaii is solely separated by the Pacific ocean. If we consider landmass only, then Alaska and the continental US are part of a self-connected individual, namely continental North America, whereas Hawaii is separated from this landmass. On the other hand, mereology is not powerful enough to reason about connectedness. As the previous example shows, two individuals being part of a common individual does not imply that this sum is self-connected. Hence, parthood is not sufficient to model connectedness.

Consequently, to be able to reason about self-connected individuals, ways to combine mereology with topology are necessary. Previously, Casati and Varzi [6] classified mereotopologies by how the two independent theories are merged. Other systematic treatments of mereotopology can be found in [11, 16].

One of the ways of building mereotopology studied in [30] takes topology as basis and defines mereology on top of it reusing the topological primitive, thereby assuming a greater generality of topology than mereology. Clarke choose this approach for his seminal work in [7, 8], and many later works in AI used Clarke's work as starting point, e.g. the system RT_0 of Asher and Vieu [1], the *Region Connection Calculus (RCC)* [2, 9, 19, 25], Gotts theory [18], and Pratt and Schoop's polygonal mereotopology [24]. Due to the same origin all of these theories use a single primitive of connectedness (or contact) and express parthood in terms of connection, thus limiting the mereotopology to the expressiveness of the connection primitive.

Most mereotopologies are described in terms of first-order axioms. However, many of them lack soundness and completeness proofs. But even soundness and completeness proofs are insufficient, instead we aim for representation theorems up to isomorphism ("full duality" in the tradition of Stone's representation theorem of Boolean algebras [28], see also e.g. [13, 14, 26, 29]) that describe the models in a uniform, mathematically well-understood formalism. Among others, for the RCC [9, 25] and the framework of Pratt and Schoop [24], which is limited to planar polygonal mereotopology, there exist formal proofs that actually give insight into the possible models. But to better understand the relation between different mereotopologies, we need to identify the models of each mereotopology and compare them to each other. Algebraic concepts and relation algebras in particular provide a mathematical sound foundation for comparing various mereotopological theories. Most previous work in this direction focused on the RCC, generalizations and algebraic and topological representations thereof. Clarke's theory has also been characterized in terms of algebras, see [3]. Another approach relates mereotopologies with certain lattice structures. In particular, Stell shows in [27] that models of the RCC are isomorphic to so-called Boolean connection algebras (or Boolean contact algebras), i.e. Boolean algebras together with a binary contact relation C satisfying certain axioms. Since lattices and Boolean algebras in particular are well-known mathematical structures, this approach led to an intensive study of the properties of the RCC including several topological representation theorems [13, 14, 15, 26, 29]. In this paper we want to apply a similar method to the mereotopology RT_0 of Asher and Vieu [1]. We will show that the subtheory RT^- can be expressed by a certain class of lattices. Subsequently, we investigate the additional axioms of RT^-_{EC} and RT in terms of algebraic properties. This relationship between models of RT_0 and certain lattices is the main contribution of this paper. It can be seen as the start of a lattice-theoretic treatment of RT_0 in a similar way as [27]. The next step in this endeavor can be found in [31]. Another interesting result is Corollary 7.3 showing that the original axiom system in [1] is not independent.

Compared to the RCC, the system of Asher and Vieu [1] focuses on a larger set of regions. The standard models of RCC are made of regular closed sets only whereas the standard models of RT_0 contain regions with regular closed closures and regular open interiors. Therefore, the system RT_0 can be seen as a more general approach in the following sense. The closed elements in Asher and Vieu's theory correspond to the elements in RCC. It is, therefore, not very surprising that RT_0 does not provide the same algebraic structure as RCC models, i.e. Boolean algebras. Even though we will consider distributivity in Section 6 this is a very particular case. By requiring this property one basically forces the more general elements of Asher and Vieu, i.e. open, closed and other sets, into the framework of regular closed regions. It turns out that in this - and just in this - case the contact relation collapses to overlap similar to Clarke's original system. A more detailed study of the relationship between RCC models and the current framework via the skeleton can be found in [31].

2. The Mereotopology RT_0

The mereotopology RT_0 proposed by Asher and Vieu [1] evolved from Clarke's theory, addressing some of its shortcomings. RT_0 follows the strategy "Topology as Basis for Mereology" for defining mereotopology and hence does not contain an explicit mereology. Consequently, the parthood relation P is sufficiently defined by the extension of the primitive relation C, which limits the expressiveness of the whole theory to that of C. As a indirect consequence of our work, it will turn out that we could express the whole theory also only in terms of the partial order of the lattice representation which amounts to specifying the relations P and O to describe a unique model.

2.1. The first-order theory

The first-order theory RT_0 of Asher and Vieu [1] is based on a binary contact relation C as primitive. The following axioms (and definitions) define the theory RT_0:

(A1) $\forall x[C(x,x)]$ (C reflexive)

(A2) $\forall x,y[C(x,y) \rightarrow C(y,x)]$ (C symmetric)

(A3) $\forall x,y[\forall z(C(z,x) \leftrightarrow C(z,y)) \rightarrow x = y]$ (C extensional)

(A4) $\exists x \forall y[C(x,y)]$ (Existence of a unique universally connected element 1)

(A5) $\forall x,y \exists z \forall u[C(u,z) \leftrightarrow (C(u,x) \vee C(u,y))]$ (Existence of a unique sum $x \cup y$ for every x and y)

(D1) $P(x,y) \equiv_{\text{def}} \forall z[C(z,x) \rightarrow C(z,y)]$ (Parthood)

(D3) $O(x,y) \equiv_{\text{def}} \exists z[P(z,x) \wedge P(z,y)]$ (Overlap)

(A6) $\forall x,y[O(x,y) \rightarrow \exists z \forall u[C(u,z) \leftrightarrow \exists v(P(v,x) \wedge P(v,y) \wedge C(v,u))]]$ (Existence of a unique intersection $x \cap y$ for overlapping elements x and y)

(A7) $\forall x[\exists y[\neg C(y,x)] \rightarrow \exists z \forall u[C(u,z) \leftrightarrow \exists v(\neg C(v,x) \wedge C(v,u))]]$ (Existence of a unique complements $-x$ for elements $x \neq 1$)

(D4) $EC(x,y) \equiv_{\text{def}} C(x,y) \wedge \neg O(x,y)$ (External connection)

$$\begin{tikzcd} RT_0 \arrow[d, "-A13"] \\ RT \arrow[d, "-A12"] \\ RT^-_{EC} \arrow[d, "-A11"] \\ RT^- \end{tikzcd}$$

Figure 1: RT_0 and its subtheories.

(D6) $NTP(x,y) \equiv_{\text{def}} P(x,y) \wedge \neg\exists z[EC(z,x) \wedge EC(z,y)]$) (Non-tangential parthood)

(A8) $\forall x \exists z \forall u[C(u,z) \leftrightarrow \exists v(NTP(v,x) \wedge C(v,u))]$ (Existence of a unique interior $i(x)$ for every x)

(D7) $c(x) \equiv_{\text{def}} -i(-x)$ (Closure operation)

(A9) $c(1) = 1$ (Closure as a total function)

(D8) $OP(x) \equiv_{\text{def}} x = i(x)$ (Open individuals)

(A10) $\forall x, y[(OP(x) \wedge OP(y) \wedge O(x,y)) \rightarrow OP(x \cap y)]$ (Intersection of open individuals is open)

(A11) $\exists x, y[EC(x,y)]$ (Existence of two externally connected elements)

(D11) $WCont(x,y) \equiv_{\text{def}} \neg C(c(x), c(y)) \wedge \forall z[(P(x,z) \wedge OP(z)) \rightarrow C(c(z), y)]$ (Weak contact)

(A12) $\exists x, y[WCont(x,y)]$ (Existence of two weakly connected elements)

(A13) $\forall x \exists y[P(x,y) \wedge OP(y) \wedge \forall z((P(x,z) \wedge OP(z)) \rightarrow P(y,z))]$ (Existence of a smallest open neighborhood $n(x)$ for every x)

Notice that the elements implied by the axioms (A4)-(A8), (A13) are indeed unique which follows immediately from (A3). In this paper we have chosen a different notation than Asher and Vieu [1] for those elements since the original notations may be confused with operations of p-ortholattices. However, we adapted the number system for definitions from the original paper but just listed those that are needed to define the theory.

In the following we will consider subtheories of RT_0 as illustrated in Figure 1. RT will denote theory $RT_0 \setminus \{(A13)\}$, RT^- the theory $RT_0 \setminus \{(A11), (A12), (A13)\}$ and RT^-_{EC} the theory $RT_0 \setminus \{(A12), (A13)\}$. Notice that this is a change to the previous naming of the subtheories as used in [22]. We exclude now axiom A13 from all of the subtheories, since it prevents dense models.

In the following lemma we have summarized some basic properties of models of RT^- which are theorems of the theory RT^-.

Lemma 2.1. *The theory RT^- entails the following theorems.*

1. $\forall x[x \neq 1 \rightarrow \neg C(x, -x)]$.
2. $\forall x[x \neq 1 \rightarrow -x \neq 1]$.
3. $\forall x[x \neq 1 \rightarrow x \cup -x = 1]$.
4. $\forall x, y[O(x, y) \rightarrow C(x, y)]$.
5. $\forall x, y[P(x, y) \rightarrow O(x, y)]$.
6. $\forall x[P(i(x), x)]$.
7. $\forall x, y[(NTP(x, y) \wedge P(y, z)) \rightarrow NTP(x, z)]$.
8. $\forall x, y[P(x, y) \rightarrow P(i(x), i(y))]$.
9. $\forall x, y[O(x, y) \leftrightarrow O(i(x), i(y))]$.
10. $\forall x[x \neq 1 \rightarrow \neg O(c(x), -x)]$.

Proof. 1. Assume $C(x, -x)$. Then there is a v with $\neg C(v, x)$ and $C(v, x)$ by (A7), a contradiction.

2. $-x = 1$ implies $C(x, -x)$ by (A4), a contradiction to 1.
3. Suppose $\neg C(u, x)$. Then $C(u, -x)$ by (A7) since $C(u, u)$ by (A1). (A5) implies that $x \cup -x$ is in contact to every element, and, hence, $x \cup -x = 1$ by (A3).
4. Suppose $O(x, y)$, i.e. there is an elements v with $P(v, x)$ and $P(v, y)$. By (A1) and (D1) (for $P(v, x)$) we conclude $C(v, x)$. Applying (D1) (now for $P(v, y)$) again we obtain $C(x, y)$.
5. Suppose $P(x, y)$. $P(x, x)$ always holds from (D1). Then there exists a z so that $P(z, x)$ and $P(z, y)$, namely $z = x$. Then by (D3) $O(x, y)$.
6. Suppose $C(u, i(x))$. Then there is v with $NTP(v, x)$ and $C(v, u)$. By (D6) we get $P(v, x)$, and, hence, $C(u, x)$. This shows $P(i(x), x)$.
7. Suppose $NTP(x, y)$ and $P(y, z)$. Then we have $P(x, y)$ and there is no u with $EC(u, x)$ and $EC(u, y)$. We obtain $P(x, z)$. Assume there is a v with $EC(v, x)$ and $EC(v, z)$. Then we have $C(v, x)$ which implies $C(v, y)$ since $P(x, y)$. Furthermore, we have $\neg O(v, z)$ which implies $\neg O(v, y)$ since $P(y, z)$. This shows that $EC(v, y)$, a contradiction.
8. Suppose $C(u, i(x))$. Then there is a v with $NTP(v, x)$ and $C(v, u)$. By 7. we obtain $NTP(v, y)$ so that $C(u, i(y))$, and, hence, $P(i(x), i(y))$ follows.
9. Suppose $O(x, y)$. Then there is a v with $P(v, x)$ and $P(v, y)$. From 8. we obtain $P(i(v), i(x))$ and $P(i(v), i(y)$, and, hence, $O(i(x), i(y))$. Conversely, suppose $O(i(x), i(y))$. Then there is a v with $P(v, i(x))$ and $P(v, i(y))$. By 6. we get $P(v, x)$ and $P(v, y)$, i.e. $O(x, y)$.
10. Assume $O(c(x), -x)$. Then there is a v with $P(v, c(x))$ and $P(v, -x)$. The first property implies $P(i(v), -i(-x))$ by 6. and the definition of the closure operation. From the second we conclude $P(i(v), i(-x))$ by 8. Together we obtain $O(-i(-x), i(-x))$. 4. gives $C(-i(-x), i(-x))$ which is a contradiction to 1. $\square$

2.2. Representation

In this paper, we will use the phrase representation in a very general way. For a representation we do not require that the elements in question are described by a different kind of elements. In our sense, an equivalent description by a different structure (possibly same universe) is regarded as a representation.

The main goal of our work is to provide a sound algebraic theory of the mereotopology RT^-. In earlier work [21, 22], we compared different mathematical representations of mereotopology: topological spaces, lattices, and graphs. It turned out that – at least for the class of mereotopological theories with a single primitive and a reflexive, anti-symmetric, transitive parthood relation – lattices and algebraic structures are most appropriate for a representation. Notice, that the original paper by Asher and Vieu [1] already provided a soundness and completeness proof with respect to arbitrary topological spaces, of which a subset of sets satisfies a set of conditions. However, this result does not establish equivalence up to isomorphism. That is exactly what we now provide. Moreover, the chosen structure here is applicable in a more general context: the work of Düntsch and Winter [14, 15] used contact lattices to represent the models of the well-known RCC. Some generalizations have been proposed in [23]. This work continues this tradition and shows close relation of RT^- to these structures. Our algebraic representation implies that *Stonian p-ortholattices* give rise to a class of contact algebras, which allows us to unveil the exact algebraic relationships between the mereotopologies RCC and RT^- in [31]. Since lattice theory is well-explored, many properties and characteristics of the classes of lattices can be applied to the mereotopologies they represent. Eventually, we hope that a topological representation of the Stonian p-ortholattices exhibits the exact topological nature of the models of RT^-.

3. p-Ortholattices

This section develops the theory of *Stonian p-ortholattices* from basic and well-known lattice concepts. The section develops the mathematical theory for the representation theorem of RT_0. For standard lattice-theoretic concepts not explained here, we refer to [4, 5, 20]. We first introduce pseudocomplemented, quasicomplemented, and orthocomplemented lattices and show how their properties restrict the class of p-ortholattices. Then, we demonstrate that every pseudocomplemented ortholattice must be also quasicomplemented. In the style of Glivenko and Frink, we define the skeleton (or center) of p-ortholattices and show how the pseudocomplementation and quasicomplementation operators can be used to define an interior and closure mapping with p-ortholattices.

We pay attention to the regularity conditions defined by Asher in Vieu in their "intended models" and show how they relate to properties that are satisfied in all p-ortholattices. Finally we state an additional condition on p-ortholattices which is required to give an isomorphic representation of the models of RT^-.

We show equivalent versions of this condition and demonstrate that the class of p-ortholattices satisfying this additional condition is a natural class of lattices, satisfying the Stone identities despite not being distributive in general.

3.1. Pseudo- and quasicomplemented lattices

Definition 3.1. *A pseudocomplemented lattice (or p-algebra) is an algebraic structure* $\langle L, +, \cdot, ^*, 0, 1\rangle$ *of type* $\langle 2, 2, 1, 0, 0\rangle$ *such that*

P0. $\langle L, +, \cdot, 0, 1\rangle$ *is a bounded lattice,*
P1. a^* *is the pseudocomplement of* a*, i.e.* $a \cdot x = 0 \iff x \leq a^*$.

Lemma 3.2. *Let* $\langle L, +, \cdot, ^*, 0, 1\rangle$ *be a p-algebra. Then we have*

1. $0^* = 1, 1^* = 0$,
2. $a \cdot a^* = a^* \cdot a^{**} = 0$,
3. $a \leq a^{**}$,
4. $a^{***} = a^*$,
5. $a \leq b$ *implies* $b^* \leq a^*$,
6. $a \cdot b = 0$ *and* $a \cdot c = 0$ *iff* $a \cdot (b + c) = 0$,
7. $(a + b)^* = a^* \cdot b^*$,
8. $(a \cdot b)^{**} = a^{**} \cdot b^{**}$.

PROOF. 1. Since $0 \cdot 1 = 0$ we get $1 \leq 0^*$. From $1^* \leq 1^*$ follows $1^* = 1^* \cdot 1 = 0$.
2. This follows immediately from $a^* \leq a^*$ and $a^{**} \leq a^{**}$.
3. By 2. we have $a^* \cdot a = 0$ which implies $a \leq a^{**}$.
4. By 3. $a^* \leq a^{***}$. From $a \cdot a^{***} \leq a^{**} \cdot a^{***} = 0$ using 2. and 3. we conclude $a^{***} \leq a^*$.
5. We have $a \cdot b^* \leq b \cdot b^* = 0$ which implies $b^* \leq a^*$.
6. Assume $a \cdot b = 0$ and $a \cdot c = 0$. Then we have $b \leq a^*$ and $c \leq a^*$, and, hence, $b + c \leq a^*$. We conclude $a \cdot (b + c) = 0$. The converse implication is trivial.
7. From $a \leq a + b$ and 5. we get $(a + b)^* \leq a^*$. Analogously, we obtain $(a + b)^* \leq b^*$ so that $(a + b)^* \leq a^* \cdot b^*$ follows. Since $a^* \cdot b^* \cdot a = 0$ and $a^* \cdot b^* \cdot b = 0$ we obtain $a^* \cdot b^* \cdot (a + b) = 0$ using 6. which is equivalent to $a^* \cdot b^* \leq (a + b)^*$.
8. We have $a \cdot b \leq a$ so that $(a \cdot b)^{**} \leq a^{**}$ follows from 5. Analogously we get $(a \cdot b)^{**} \leq b^{**}$ so that we obtain $(a \cdot b)^{**} \leq a^{**} \cdot b^{**}$. For the converse inclusion we have $(a \cdot b) \cdot (a \cdot b)^* = 0$ by 2., which implies

$$\begin{aligned}
(a \cdot b) \cdot (a \cdot b)^* = 0 &\Leftrightarrow a \cdot (a \cdot b)^* \leq b^* \\
&\Leftrightarrow a \cdot (a \cdot b)^* \leq b^{***} && \text{by (4)} \\
&\Leftrightarrow a \cdot b^{**} \cdot (a \cdot b)^* = 0 \\
&\Leftrightarrow b^{**} \cdot (a \cdot b)^* \leq a^* \\
&\Leftrightarrow b^{**} \cdot (a \cdot b)^* \leq a^{***} && \text{by (4)} \\
&\Leftrightarrow a^{**} \cdot b^{**} \cdot (a \cdot b)^* = 0 \\
&\Leftrightarrow a^{**} \cdot b^{**} \leq (a \cdot b)^{**}.
\end{aligned}$$

This completes the proof. □

Throughout the paper we will use the properties of the previous lemma without mentioning.

The notion of a quasicomplement a^+ of a is dual to the notion of a pseudocomplement, i.e. it is characterized by $a^+ \leq x \iff a+x = 1$. A quasicomplemented lattice is a lattice in which every element has a quasicomplement, i.e. the dual of a pseudocomplemented lattice. The following properties of quasicomplements simply follow from this duality.

Corollary 3.3. *Let $\langle L, +, \cdot, ^+, 0, 1\rangle$ be a quasicomplemented lattice. Then we have*

1. $0^+ = 1, 1^+ = 0$,
2. $a + a^+ = a^+ + a^{++} = 1$,
3. $a^{++} \leq a$,
4. $a^{+++} = a^+$,
5. $a \leq b$ *implies* $b^+ \leq a^+$,
6. $a + b = 1$ *and* $a + c = 1$ *iff* $a + b \cdot c = 1$,
7. $(a \cdot b)^+ = a^+ + b^+$,
8. $(a + b)^{++} = a^{++} + b^{++}$.

To emphasize the dual nature of pseudocomplemented and quasicomplemented lattices, the naming as meet-pseudocomplemented and join-pseudocomplemented are also common. A lattice that is both pseudo- and quasicomplemented (or meet- and join-pseudocomplemented) is called double pseudocomplemented or double p-algebra.

3.2. Ortholattices

Definition 3.4. *An ortholattice (or orthocomplemented lattice) is a structure $\langle L, +, \cdot, ^\perp, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ such that*

- O0. $\langle L, +, \cdot, 0, 1\rangle$ *is a bounded lattice,*
- O1. $a^\perp$ *is an orthocomplement of a, i.e. for all $a, b \in L$ we have*
  - (a) $a^{\perp\perp} = a$,
  - (b) $a \cdot a^\perp = 0$,
  - (c) $a \leq b$ *implies* $b^\perp \leq a^\perp$.

Lemma 3.5. *Let $\langle L, +, \cdot, ^\perp, 0, 1\rangle$ be an ortholattice. Then we have*

1. $0^\perp = 1$ *and* $1^\perp = 0$
2. $(a + b)^\perp = a^\perp \cdot b^\perp$ *and* $(a \cdot b)^\perp = a^\perp + b^\perp$,
3. $a + a^\perp = 1$.

Proof. 1. From $0 \leq 1^\perp$ we conclude $1 = 1^{\perp\perp} \leq 0^\perp$. Analogously, $0^\perp \leq 1$ implies $1^\perp \leq 0^{\perp\perp} = 0$.

2. From $a \leq a+b$ we conclude $(a+b)^{\perp} \leq a^{\perp}$. Analogously, we get $(a+b)^{\perp} \leq b^{\perp}$. Together we obtain (*) $(a+b)^{\perp} \leq a^{\perp} \cdot b^{\perp}$. Similarly, from $a \cdot b \leq a$ we conclude $a^{\perp} \leq (a \cdot b)^{\perp}$ and $b^{\perp} \leq (a \cdot b)^{\perp}$ analogously. We obtain (**) $a^{\perp} + b^{\perp} \leq (a \cdot b)^{\perp}$. The remaining inclusions can be derived as follows

$$\begin{aligned} a^{\perp} \cdot b^{\perp} &= (a^{\perp} \cdot b^{\perp})^{\perp\perp} \\ &\leq (a^{\perp\perp} + b^{\perp\perp})^{\perp} && \text{by (**)} \\ &= (a+b)^{\perp}, \\ (a \cdot b)^{\perp} &= (a^{\perp\perp} \cdot b^{\perp\perp})^{\perp} \\ &\leq (a^{\perp} + b^{\perp})^{\perp\perp} && \text{by (*)} \\ &= a^{\perp} + b^{\perp}. \end{aligned}$$

3. Consider the following computation

$$\begin{aligned} 1 &= 0^{\perp} && \text{by (1)} \\ &= (a \cdot a^{\perp})^{\perp} \\ &= a^{\perp} + a^{\perp\perp} && \text{by (2)} \\ &= a + a^{\perp}. \end{aligned}$$

This completes the proof. □

Again, we will use the properties of the previous lemma throughout the paper without mentioning.

3.3. p-Ortholattices

Definition 3.6. *A pseudocomplemented ortholattice (or p-ortholattice) is a structure $\langle L, +, \cdot, ^*, ^{\perp}, 0, 1\rangle$ of type $\langle 2, 2, 1, 1, 0, 0\rangle$ such that*

PO0. *$\langle L, +, \cdot, ^*, 0, 1\rangle$ is a p-algebra,*
PO1. *$\langle L, +, \cdot, ^{\perp}, 0, 1\rangle$ is an ortholattice.*

The following computation

$$\begin{aligned} a^{\perp*\perp} \leq x &\iff x^{\perp} \leq a^{\perp*} \\ &\iff a^{\perp} \cdot x^{\perp} = 0 \\ &\iff (a+x)^{\perp} = 0 \\ &\iff a + x = 1. \end{aligned}$$

verifies that the construction $a^{+} = a^{\perp*\perp}$ in an arbitrary p-ortholattice is a quasicomplement. Hence every p-ortholattice is quasicomplemented and thus a double p-algebra. Dually, any quasicomplemented ortholattice must be a double p-algebra.

In the following, we show some equivalences that are satisfied in any p-ortholattice.

Lemma 3.7. *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a p-ortholattice. Then we have*

1. $a^+ \leq a^\perp \leq a^*$,
2. $a + a^* = 1$ *and* $a \cdot a^+ = 0$,
3. $a^{*\perp} = a^{\perp +}$ *and* $a^{+\perp} = a^{\perp *}$,
4. $a^{**} + a^{+**} = 1$ *and* $a^{++} \cdot a^{*++} = 0$,
5. $a^{**} = a^{\perp *}$ *and* $a^{++} = a^{\perp +}$,
6. $a^{+*} = a^{\perp *}$ *and* $a^{*+} = a^{\perp +}$.

PROOF. 1. follows immediately from $a \cdot a^\perp = 0$ and $a + a^\perp = 1$.

2. Using 1. we conclude $a \cdot a^+ \leq a \cdot a^\perp = 0$ and $a + a^* \geq a + a^\perp = 1$.
3. follows immediately from the definition of $^+$.
4. We have $1 = a^+ + a \leq a^{+**} + a^{**}$ and $0 = a^* \cdot a \geq a^{*++} \cdot a^{++}$.
5. From $a^\perp \leq a^*$ we conclude $a^{**} \leq a^{\perp *}$. For the converse inclusion we have $(a^* \cdot a^{\perp *}) \cdot a^{**} \leq a^* \cdot a^{**} = 0$ and $(a^* \cdot a^{\perp *}) \cdot a^{\perp **} \leq a^{\perp *} \cdot a^{\perp **} = 0$ which implies $(a^* \cdot a^{\perp *}) \cdot (a^{**} + a^{\perp **}) = 0$. From 4. and $a^{**} + a^{+**} \leq a^{**} + a^{\perp **}$ we conclude $a^* \cdot a^{\perp *} = 0$, and, hence, $a^{\perp *} \leq a^{**}$. The second equation follows from

$$
\begin{aligned}
a^{++} &= a^{\perp * \perp \perp * \perp} && \text{definition } ^+ \\
&= a^{\perp ** \perp} && \\
&= a^{\perp \perp * \perp} && \text{first equation} \\
&= a^{\perp +}. && \text{definition } ^+
\end{aligned}
$$

6. From $a^+ \leq a^\perp$ we conclude $a^{\perp *} \leq a^{+*}$. For the converse inclusion we have $(a^\perp \cdot a^{+*}) \cdot a^{**} \leq a^\perp \cdot a^{**} \leq a^* \cdot a^{**} = 0$ and $(a^\perp \cdot a^{+*}) \cdot a^{+**} \leq a^{+*} \cdot a^{+**} = 0$ which implies $(a^\perp \cdot a^{+*}) \cdot (a^{**} + a^{+**}) = 0$. From 4. we conclude $a^\perp \cdot a^{+*} = 0$, and, hence, $a^{+*} \leq a^{\perp *}$. The second equation follows from

$$
\begin{aligned}
a^{*+} &= a^{*\perp * \perp} && \text{definition } ^+ \\
&= a^{\perp \perp * \perp * \perp} && \\
&= a^{\perp + * \perp} && \\
&= a^{\perp \perp * \perp} && \text{first equation} \\
&= a^{\perp +}. && \text{definition } ^+
\end{aligned}
$$

This completes the proof. □

Again, we will use the properties of the previous lemma throughout the paper without mentioning.

In [22], we gave a representation theorem for the theory of RT_0 in terms of p-ortholattices. Later it will turn out that p-ortholattices arising from mereotopology satisfy the additional property

$$(x \cdot y)^* = x^* + y^*.$$

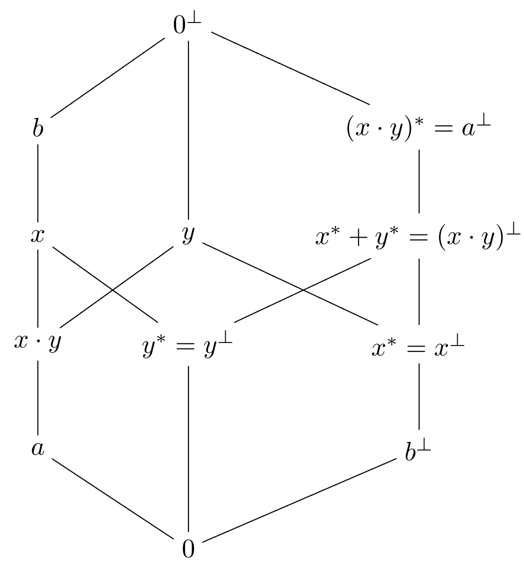


Figure 2: A p-ortholattice not satisfying $(x \cdot y)^* = x^* + y^*$.

In the following section, we show that the topological "regularity" properties are maintained in all p-ortholattices.

To conclude this section we want to provide an example of a p-ortholattice that does not satisfy the property above. Consider the p-ortholattice in Figure 2. In this lattice we have $x^* + y^* \lneq (x \cdot y)^*$. Hence, the original representation theorem from [22] needs to be corrected by an additional condition. In Section 3.5, we introduce the notion of a *skeleton* of pseudocomplemented lattices and use it in Section 3.6 to define the additional condition required to represent the models of RT^-. We show that this results in a rather natural class of lattices, which we call *Stonian p-ortholattices*. Afterwards, a new representation theorem for the models of RT_0 is given. It also shows the homomorphism from p-ortholattices to models of RT_0 directly, instead of relying on the representation of the intended models from [1]. It turns out that p-ortholattices where the additional property $(x \cdot y)^* = x^* + y^*$ holds, actually satisfy all the Stone identities.

3.4. Regularity

Now we are in a position to prove some quintessential properties of p-ortholattices that capture conditions imposed on the regions in the intended models of RT_0. We show that the closure and interior mappings, $a \rightarrow a^{**}$ and $a \rightarrow a^{++}$, are both regular in the sense of [1]: $cl(x) = cl(int(x))$ and $int(x) = int(cl(x))$ hold for all regions in the topological interpretation of the models of RT^-. We show that the algebraic counterparts of these properties hold in any p-ortholattice.

Lemma 3.8. *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a p-ortholattice. Then we have*

1. $a^{**} = (a^{++})^{**}$
2. $a^{++} = (a^{**})^{++}$

Proof. Consider the following computation as proof for (1):

$$\begin{aligned} x^{**} &= x^{\perp\perp**} \\ &= x^{\perp***} \\ &= x^{+\perp**} \\ &= x^{++\perp*} \\ &= x^{++**} \\ &= (x^{++})^{**} \end{aligned}$$

Analogously we can prove (2). □

3.5. Skeleton

Skeletons (also called centers) have been first defined in 1929 by Glivenko in the context of distributive lattices. Later, Frink [17] generalized this result to arbitrary pseudocomplemented meet-semilattices. Since p-ortholattices and their duals are a subclass of the meet-semilattices, we can define skeletons and dual skeletons on p-ortholattices using pseudo- and quasicomplementation. Then, both the skeletons and dual skeletons are always Boolean.

Definition 3.9. *Let $\langle L, \cdot, ^*, 0\rangle$ be a pseudocomplemented semilattice. Let $S(L) = \{a^* | a \in L\}$ be the skeleton of L, maintaining the order relation of L and with meet $a \wedge b = a \cdot b$ and union $a \vee b = (a^* \cdot b^*)^*$.*

Theorem 3.10 (Glivenko-Frink theorem). *[17] Let L be a pseudocomplemented semilattice. Then $S(L)$ is a Boolean algebra. The (unique) complement of an element $a \in S(L)$ is its pseudocomplement $a^* \in L$.*

Theorem 3.11 (Glivenko's theorem). *[17] Let L be a pseudocomplemented semilattice. Then the mapping $a \to a^{**}$ from L to $S(L)$ is a closure operation. The mapping is a homomorphism preserving meets, pseudocomplements, the 0 element, and joins when they exist. $S(L)$ is complete if L is complete.*

More details, the proof, and a list of properties of the skeleton of pseudocomplemented lattices can be found in [5]. We immediately derive the following corollary for skeletons of pseudocomplements in p-ortholattices.

Corollary 3.12. *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a p-ortholattice. Then $S(L) = \{a^* | a \in L\} = \{a^{**} | a \in L\}$ forms with meet $a \wedge b = a \cdot b$ and union $x \vee y = (x^* \cdot y^*)^*$ a Boolean algebra. The operation $a \to a^{**}$ is a closure mapping, with $S(L)$ containing all closed elements of L.*

Dually, we obtain the following corollary for the dual skeleton $\bar{S}(L)$ of quasi-complements in a p-ortholattice L.

Corollary 3.13. *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a p-ortholattice. Then $\bar{S}(L) = \{a^+ | a \in L\} = \{a^{++} | a \in L\}$ forms with meet $x \wedge y = (x^+ + y^+)^+$ and union $+$ a Boolean algebra. The operation $a \to a^{++}$ is an interior mapping, where $\bar{S}(L)$ contains all open elements of L.*

The equivalences for p-ortholattices from Lemma 3.7 define a set of equivalent combinations of the operators $^*, ^+, ^\perp$ for the closure and interior mappings, $a \to a^{**}$ and $a \to a^{++}$, respectively. The following corollary gives alternative, equivalent closure and interior mappings for p-ortholattices.

Corollary 3.14. *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a p-ortholattice. Then we have*

1. $a^{**} = a^{\perp *} = a^{+*} = a^{+\perp}$
2. $a^{++} = a^{\perp +} = a^{*+} = a^{*\perp}$

Proof. Follows directly from Lemma 3.7. □

3.6. Stonian p-ortholattices

Here we introduce an additional condition for p-ortholattices that do not hold for all p-ortholattices as demonstrated by Figure 2. We show that for all p-ortholattices that satisfy this additional condition, the skeleton as introduced in the previous section is in fact Boolean. This suffices to define the Stonian p-ortholattices which will be used for the representation of models of RT^- in the following sections.

Definition 3.15. *A p-ortholattice $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ is called Stonian iff $(x \cdot y)^* = x^* + y^*$ for all $x, y \in L$.*

The next lemma shows that the skeleton $S(L)$ for Stonian p-ortholattices is not only a Boolean algebra (as stated in corollary 3.12) but in fact a Boolean subalgebra of L, i.e. $x \vee y = x + y$.

Lemma 3.16. *If $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ is a Stonian p-ortholattice, then $S(L)$ is a Boolean subalgebra of L.*

PROOF. By [17] it remains to show that $x \vee y = x + y$ for all elements $x, y \in S(L)$. This follows for Stonian p-ortholattices immediately from

$$\begin{aligned} x \vee y &= (x^* \cdot y^*)^* && \\ &= x^{**} + y^{**} && L \text{ Stonian} \\ &= x + y. && {}^* \text{ complement in } S(L) \end{aligned}$$

This completes the proof. □

Dually, we can show that for Stonian p-ortholattices the dual skeleton $\bar{S}(L)$ is also a subalgebra of L, i.e. $x \wedge y = x \cdot y$.

Corollary 3.17. *If $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ is a Stonian p-ortholattice, then $\bar{S}(L)$ is a Boolean subalgebra of L.*

Now, we can justify the naming of these p-ortholattices as Stonian in the tradition of pseudocomplemented distributive lattices that satisfy the Stone identities. A Stone lattice is defined to be a pseudocomplemented distributive lattice that satisfies any (one) of the equivalent conditions (1), (3), and (5) of Theorem 3.18 or $(\forall x, y \in L)\ x^* + x^{**} = 1$. However, this condition is true for all p-ortholattices (compare Lemma 3.7(2)), so it is not sufficient to prove any of the other equivalent properties for p-ortholattices. The following theorem shows the applicability of the remaining Stone identities for p-ortholattices (see [5] for more details). Moreover, it shows that every Stonian p-ortholattice is indeed a double Stonian p-ortholattice.

Theorem 3.18. *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a p-ortholattice. Then the following statements are equivalent:*

1. $(x \cdot y)^* = x^* + y^*$ *for all* $x, y \in L$;
2. $(x + y)^+ = x^+ \cdot y^+$ *for all* $x, y \in L$;
3. $(x \cdot y)^{++} = x^{++} \cdot y^{++}$ *for all* $x, y \in L$;
4. $(x + y)^{**} = x^{**} + y^{**}$ *for all* $x, y \in L$;
5. $S(L)$ *is a Boolean subalgebra of* L.
6. $\bar{S}(L)$ *is a Boolean subalgebra of* L.

PROOF. We only show (1) ⇔ (2), (1) ⇔ (3), (2) ⇔ (4), (5) ⇒ (1), and (6) ⇒ (2). (1) ⇒ (5) and (2) ⇒ (6) have already been established by Lemma 3.16 and Corollary 3.17.
(1) $\iff$ (2): From the computation

$$\begin{aligned} &(x \cdot y)^* = x^* + y^* \text{ for all } x, y \in L \\ &\Leftrightarrow (x^\perp \cdot y^\perp)^* = x^{\perp *} + y^{\perp *} \text{ for all } x, y \in L \\ &\Leftrightarrow (x^\perp \cdot y^\perp)^{*\perp} = (x^{\perp *} + y^{\perp *})^\perp \text{ for all } x, y \in L \\ &\Leftrightarrow (x + y)^{\perp * \perp} = x^{\perp * \perp} \cdot y^{\perp * \perp} \text{ for all } x, y \in L \\ &\Leftrightarrow (x + y)^+ = x^+ \cdot y^+ \text{ for all } x, y \in L. \end{aligned}$$

we conclude the assertion.

(1) $\iff$ (3): Consider the computation:

$$\begin{aligned}
&(x \cdot y)^* = x^* + y^* \text{ for all } x, y \in L \\
&\Leftrightarrow (x \cdot y)^{*\perp} = (x^* + y^*)^{\perp} \text{ for all } x, y \in L \\
&\Leftrightarrow (x \cdot y)^{\perp +} = (x^* + y^*)^{\perp} \text{ for all } x, y \in L \\
&\Leftrightarrow (x \cdot y)^{++} = x^{*\perp} \cdot y^{*\perp} \text{ for all } x, y \in L \\
&\Leftrightarrow (x \cdot y)^{++} = x^{++} \cdot y^{++} \text{ for all } x, y \in L.
\end{aligned}$$

(2) $\iff$ (4): Consider the computation:

$$\begin{aligned}
&(x + y)^+ = x^+ \cdot y^{+\perp} \text{ for all } x, y \in L \\
&\Leftrightarrow (x + y)^{+\perp} = (x^+ \cdot y^+)^{\perp} \text{ for all } x, y \in L \\
&\Leftrightarrow (x + y)^{\perp *} = (x^+ \cdot y^+)^{\perp} \text{ for all } x, y \in L \\
&\Leftrightarrow (x + y)^{**} = x^{+\perp} + y^{+\perp} \text{ for all } x, y \in L \\
&\Leftrightarrow (x + y)^{**} = x^{**} + y^{**} \text{ for all } x, y \in L.
\end{aligned}$$

(5) $\Rightarrow$ (1): Assume $x, y \in L$. Then $x^*, y^* \in S(L)$ and we obtain

$$\begin{aligned}
x^* + y^* &= x^* \vee y^* && \text{since } S(L) \text{ is a sublattice of } L \\
&= (x^{**} \cdot y^{**})^* \\
&= (x \cdot y)^{***} && \text{Lemma 3.2(8)} \\
&= (x \cdot y)^*.
\end{aligned}$$

(6) $\Rightarrow$ (2) works analogously. □

This completes our characterization of the Stonian p-ortholattices. In the next two sections, we show that the models of the theory RT_0 are isomorphic to the class of Stonian p-ortholattices.

4. From models of RT^- to Stonian p-ortholattices

A model U of RT^- consists of a set and a primitive relation C. In addition, we can define the relation P and - using the axioms (A3) and (A6)-(A9) - we can define the operations $\cup, \cap, -, i$, and c. In order to obtain a lattice from U we have to add an additional element 0 and define

$$
\begin{aligned}
x \leq y &\equiv_{\text{def}} x = 0 \vee (x, y \in U \wedge P(x,y)) \\
x \cdot y &\equiv_{\text{def}} \begin{cases} x \cap y & \text{iff } x, y \in U \wedge O(x,y) \\ 0 & \text{otherwise} \end{cases} \\
x + y &\equiv_{\text{def}} \begin{cases} x \cup y & \text{iff } x, y \in U \\ y & \text{iff } x = 0 \\ x & \text{iff } y = 0 \end{cases} \\
x^{\perp} &\equiv_{\text{def}} \begin{cases} -x & \text{iff } x \in U \wedge x \neq 1 \\ 1 & \text{iff } x = 0 \\ 0 & \text{iff } x = 1 \end{cases} \\
\text{int}(x) &\equiv_{\text{def}} \begin{cases} i(x) & \text{iff } x \in U \\ 0 & \text{iff } x = 0 \end{cases}
\end{aligned}
$$

Notice that a similar result to the lemma below has already been shown in [3] for Clarke's mereotopology. However, besides the different scope of that mereotopology the set of primitive or derived operations is different. In particular, Clarke's mereotopology contains an infinite fusion operation whereas Asher and Vieu's theory has an explicit complement.

Lemma 4.1. *Let U be a model of RT^-. Then $\langle U \cup \{0\}, +, \cdot, ^{\perp}, 0, 1\rangle$ is an ortholattice with $C(x,y) \iff x \not\leq y^{\perp}$ for all $x, y \in U$.*

Proof. In order to prove that $U \cup \{0\}$ is an ortholattice we just show that $x^{\perp}$ is an orthocomplement of x.

O1(a): If $x = 0$ or $x = 1$, then $x^{\perp\perp} = x$ follows immediately from the definition. Suppose $x \neq 0$ and $x \neq 1$. Then $x^{\perp} = -x$, i.e. $x^{\perp} \neq 0$ by definition and $x^{\perp} \neq 1$ by Lemma 2.1(2). This implies $x^{\perp\perp} = --x$. We want to show that $C(u, --x)$ iff $C(u,x)$ which implies by (A3) that $x = --x$. Therefore, suppose $C(u, --x)$ and $\neg C(u,x)$. Axiom (A7) implies that $C(--x, -x)$, a contradiction to Lemma 2.1(1). Conversely, suppose $C(u,x)$ and $\neg C(u, --x)$. The latter implies $C(v, -x)$ or $\neg C(v,u)$ for all v. In particular, we get $C(x, -x)$ or $\neg C(x,u)$. The first property is a contradiction to Lemma 2.1(1) and the second to the assumption $C(u,x)$.

O1(b): If $x = 0$ or $x = 1$, then $x \cdot x^{\perp} = 0$ by the definition of $\cdot$ and $^{\perp}$. Suppose $x \neq 0$ and $x \neq 1$ and assume that $O(x, -x)$. Then Lemma 2.1(4) implies $C(x, -x)$, a contradiction to Lemma 2.1(1). We conclude $\neg O(x, -x)$, and, hence, $x \cdot x^{\perp} = 0$.

O1(c): Suppose $x \leq y$. If $x = 0$, then $y^{\perp} \leq 1 = x^{\perp}$ follows immediately. If $x = 1$, then $y = 1$ and we obtain $y^{\perp} = 0 \leq x^{\perp}$. Now suppose $x \neq 0$ and $x \neq 1$. In this case $y \neq 0$, and the case $y = 1$ follows as above so that we assume $y \neq 0$ and $y \neq 1$. Notice that in this case $x^{\perp} = -x$, $y^{\perp} = -y$ and

$x \leq y$ is equivalent to $P(x, y)$. Let be $C(u, -y)$. Then there is a v with $\neg C(v, y)$ and $C(v, u)$. By the definition of P we conclude $\neg C(v, x)$, and, hence, $C(u, -x)$. This implies $P(-y, -x)$, and, hence, $y^\perp \leq x^\perp$.

Suppose $x, y \in U$ and $C(x, y)$, and assume $x \leq y^\perp$. The latter implies that $y^\perp = -y \in U$ so that we obtain $P(x, -y)$. We conclude $C(-y, y)$, a contradiction to Lemma 2.1(1). Suppose $\neg C(x, y)$, and let be $C(u, x)$. Notice that $y \neq 1$, i.e. $-y$ exists and is equal to $y^\perp$, since y is not universally connected. By (A7) we conclude $C(u, -y)$, and, hence $P(x, -y)$. From $x, -y \in U$ we conclude $x \leq y^\perp$. This completes the proof. $\square$

Now we may define pseudocomplementation and quasicomplementation by:

$$
\begin{aligned}
x^* &\equiv_{\text{def}} \text{int}(x)^\perp, \\
x^+ &\equiv_{\text{def}} \text{int}(x^\perp)
\end{aligned}
$$

The following two propositions do not have a corresponding result for Clarke's mereotopology.

Lemma 4.2. *Let U be a model of RT^-. Then $\langle U \cup \{0\}, +, \cdot, ^*, 0, 1\rangle$ is a p-algebra.*

Proof. We have to show that $x \cdot y = 0$ iff $y \leq x^*$. Instead we show that x^+ is a quasicomplement of x, i.e. $x + y = 1$ iff $x^+ \leq y$. This immediately implies

$$
\begin{aligned}
x \cdot y = 0 &\Leftrightarrow x^\perp + y^\perp = 1 \\
&\Leftrightarrow x^{\perp +} \leq y^\perp \\
&\Leftrightarrow y \leq x^*.
\end{aligned}
$$

Suppose $x + y = 1$. If $x = 0$, then $y = 1$, and, hence, $x^+ \leq 1$. If $y = 0$, then $x = 1$. If $x = 1$ we immediately conclude $x^+ = 0 \leq y$. Now suppose $x, y \in U$ with $x \neq 0, 1$ and $y \neq 0$ which implies $x + y = x \cup y$ and $x^+ = i(-x)$. Let be $C(u, i(-x))$. Then there is a $v \in U$ with $NTP(v, -x)$ and $C(v, u)$. Assume $\neg O(u, -x)$. Then $EC(u, -x)$ since $C(u, -x)$ which follows from $P(i(-x), -x)$ (Lemma 2.1(5)) and $C(u, i(-x))$. On the other hand, we conclude $\neg O(u, v)$ since otherwise $O(u, v)$ and $P(v, -x)$ obtained from $NTP(v, -x)$ implies $O(u, -x)$. Since $C(v, u)$ we get $EC(u, v)$. Together $EC(u, -x)$ and $EC(u, v)$ is a contradiction to $NTP(v, -x)$. Therefore, we must have $O(u, -x)$. Consequently, $u \cap -x$ exists, and we have $\neg C(u \cap -x, x)$ since otherwise $C(x, -x)$ would follow. Since $x + y = 1$ we conclude that $C(u \cap -x, y)$, and, hence, $C(u, y)$. We obtain $P(i(-x), y)$, and, hence, $x^+ \leq y$. Conversely, suppose $x^+ \leq y$. If $x = 0$, then $y = 1$, and, hence, $x + y = 1$. If $x = 1$, $x + y = 1$ follows immediately. Now suppose $x, y \in U$ and $x \neq 0, 1$, i.e. $x^+ = i(-x)$. Let be $\neg C(u, x)$. Then $u \leq x^\perp = -x$ by Lemma 4.1, i.e. $P(u, -x)$. Since we have $P(i(u), u)$ by Lemma 2.1(5) and $P(i(u), i(-x))$ by Lemma 2.1(7) we obtain $O(u, i(-x))$.

Lemma 2.1(4) shows $C(u, i(-x))$. Since $i(-x) = x^+ \leq y$ we have $C(u, y)$. We have just shown that every element is either in contact to x or to y so that $x + y = 1$ follows. $\square$

Theorem 4.3. *Let U be a model of RT^-. Then $\langle U \cup \{0\}, +, \cdot, ^*, ^\perp, 0, 1\rangle$ is a Stonian p-ortholattice.*

Proof. It remains to show that $(x \cdot y)^* = x^* + y^*$. To this end we show

$$(*) \quad \mathrm{int}(x \cdot y) = \mathrm{int}(x) \cdot \mathrm{int}(y)$$

which immediately implies

$$\begin{aligned} (x \cdot y)^* &= \mathrm{int}(x \cdot y)^\perp \\ &= (\mathrm{int}(x) \cdot \mathrm{int}(y))^\perp && (*) \\ &= \mathrm{int}(x)^\perp + \mathrm{int}(y)^\perp \\ &= x^* + y^*. \end{aligned}$$

If $x = 0$ or $y = 0$, $(*)$ is true by definition. Suppose $x \neq 0$ and $y \neq 0$, i.e. $\mathrm{int}(x) = i(x)$ and $\mathrm{int}(y) = i(y)$. If $\neg O(x, y)$, then $i(x \cdot y) = 0$. From Lemma 2.1(9) we obtain $\neg O(i(x), i(y))$, and, hence, $i(x) \cdot i(y) = 0$. Suppose we have $O(x, y)$, i.e. $x \cdot y = x \cap y$. Then we have $O(i(x), i(y))$ by Lemma 2.1(9) so that $i(x) \cdot i(y) = i(x) \cap i(y)$ follows.

We have $P(i(x \cap y), i(x))$ and $P(i(x \cap y), i(y))$ by Lemma 2.1(8). This implies $P(i(x \cap y), i(x) \cap i(y))$, i.e. $i(x \cap y) \leq i(x) \cap i(y)$.

Conversely, $i(x)$ and $i(y)$ are open so that $i(x) \cap i(y)$ is open by (A10), i.e. $i(i(x) \cap i(y)) = i(x) \cap i(y)$. Furthermore, $i(x) \leq x$ and $i(y) \leq y$ (Lemma 2.1(5)) implies $i(x) \cap i(y) \leq x \cap y$, and, hence, $i(x) \cap i(y) = i(i(x) \cap i(y)) \leq i(x \cap y)$ using Lemma 2.1(8). $\square$

5. From Stonian p-ortholattices to models of RT^-

In this section we want to show the converse of Theorem 4.3. Therefore, we start with a Stonian p-ortholattice and construct a model of RT^-. This requires at least to remove the smallest element 0.

Theorem 5.1. *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a Stonian p-ortholattice. Then $L^+ = \{x \in L \mid x \neq 0\}$ together with the relation $xCy \iff x \nleq y^\perp$ is a model of RT^-.*

Proof. First, we prove the following three properties for all $x, y \in L^+$:

(a) $P(x, y)$ iff $x \leq y$,

(b) $O(x,y)$ iff $x \cdot y \neq 0$,
(c) $NTP(x,y)$ iff $\neg C(x,y^*)$.

(a) This follows immediately from

$$\begin{aligned} P(x,y) &\Leftrightarrow \forall u \neq 0[C(u,x) \to C(u,y)] \\ &\Leftrightarrow \forall u \neq 0[u \not\leq x^\perp \to u \not\leq y^\perp] \\ &\Leftrightarrow \forall u \neq 0[u \leq y^\perp \to u \leq x^\perp] \\ &\Leftrightarrow y^\perp \leq x^\perp \\ &\Leftrightarrow x \leq y && \text{O1(a,c)} \end{aligned}$$

(b) Using (a) we get

$$\begin{aligned} O(x,y) &\Leftrightarrow \exists u \neq 0[u \leq x \wedge u \leq y] \\ &\Leftrightarrow x \cdot y \neq 0. \end{aligned}$$

(c) Suppose $NTP(x,y)$ and assume $C(x,y^*)$. From $NTP(x,y)$ we get $P(x,y)$ and $x \leq y$. The first property implies $C(y,y^*)$. Since $y \cdot y^* = 0$ we have in fact $EC(y,y^*)$. The second property $x \leq y$ implies $x \cdot y^* \leq y \cdot y^* = 0$ so that $EC(x,y^*)$ follows. But $EC(y,y^*)$ and $EC(x,y^*)$ is a contradiction to $NTP(x,y)$. Conversely, suppose $\neg C(x,y^*)$. Then we have $x \leq y^{*\perp} = y^{\perp +} = y^{++} \leq y$. Assume there is a $z \neq 0$ with $EC(z,x)$ and $EC(z,y)$. Then $C(z,x)$ and $z \cdot y = 0$, i.e. $z \leq y^*$. The latter implies $P(z,y^*)$, and, hence, $C(y^*,x)$, a contradiction. Therefore, we have $NTP(x,y)$.

The ten axioms of RT^- are now shown as follows.

(A1). If $x \neq 0$, then $x \not\leq x^\perp$, and, hence $C(x,x)$.
(A2). This follows immediately from O1(a,c).
(A3). Suppose $z \leq x^\perp$ iff $z \leq y^\perp$. Then we obtain $x^\perp \leq y^\perp$ and $y^\perp \leq x^\perp$, and, hence, $x^\perp = y^\perp$. O1(a) implies $x = y$.
(A4). If $z \neq 0$, then $z \not\leq 0 = 1^\perp$ so that $C(z,1)$ follows.
(A5). The following computation

$$\begin{aligned} \neg C(u,x+y) &\Leftrightarrow u \leq (x+y)^\perp \\ &\Leftrightarrow u \leq x^\perp \cdot y^\perp \\ &\Leftrightarrow \neg C(u,x) \wedge \neg C(u,y) \end{aligned}$$

shows that $C(u,x+y)$ iff $C(u,x)$ or $C(u,y)$.
(A6). Suppose $x \cdot y \neq 0$. If $\neg C(u,x \cdot y)$, then $u \leq (x \cdot y)^\perp$, and, hence $x \cdot y \leq u^\perp$. If $v \leq x$ and $v \leq y$, then $v \leq x \cdot y \leq u^\perp$ so that $\neg C(v,u)$ follows. Conversely, suppose $v \leq x$ and $v \leq y$ implies $\neg C(v,u)$ for all $v \neq 0$. In particular, we obtain $\neg C(x \cdot y, u)$.

(A7). Notice that we have

$$\begin{aligned}\exists u \neq 0[\neg C(u,x)] &\Leftrightarrow \exists u \neq 0[u \leq x^{\perp}] \\ &\Leftrightarrow x^{\perp} \neq 0 \\ &\Leftrightarrow x \neq 1.\end{aligned}$$

Suppose $x \neq 1$ and compute

$$\begin{aligned}\neg C(u,x^{\perp}) &\Leftrightarrow u \leq x \\ &\Leftrightarrow P(u,x) && \text{by (a)} \\ &\Leftrightarrow \forall v \neq 0[C(v,u) \rightarrow C(v,x)] \\ &\Leftrightarrow \forall v \neq 0[\neg C(v,u) \vee C(v,x)] \\ &\Leftrightarrow \neg\exists v \neq 0[\neg C(v,x) \wedge C(v,u)]\end{aligned}$$

(A8). This axiom follows immediately from

$$\begin{aligned}\neg C(u,x^{\perp +}) &\Leftrightarrow u \leq x^{\perp + \perp} \\ &\Leftrightarrow u \leq x^{*} && \text{Lemma 3.7(3)} \\ &\Leftrightarrow P(u,x^{*}) && \text{by (a)} \\ &\Leftrightarrow \forall v \neq 0[C(v,u) \rightarrow C(v,x^{*})] \\ &\Leftrightarrow \forall v \neq 0[\neg C(v,u) \vee C(v,x^{*})] \\ &\Leftrightarrow \forall v \neq 0[\neg C(v,u) \vee \neg NTP(v,x)] && \text{by (c)} \\ &\Leftrightarrow \neg\exists v \neq 0[NTP(v,x) \wedge C(v,u)]\end{aligned}$$

(A9). We immediately conclude $c(1) = i(1^{\perp})^{\perp} = 1^{\perp\perp+\perp} = 1^{+\perp} = 0^{\perp} = 1$.

(A10). First of all, we have $OP(x)$ iff $x = i(x) = x^{\perp +} = x^{++}$. Now, assume $OP(x)$, $OP(y)$ and $O(x,y)$. Then $x \cdot y \neq 0$ by (b), i.e. $x \cdot y \in L^{+}$, and we have

$$\begin{aligned}(x \cdot y)^{++} &= (x^{+} + y^{+})^{+} && \text{Lemma 3.3(7)} \\ &= x^{++} \cdot y^{++} && L \text{ Stonian and Theorem 3.18} \\ &= x \cdot y, && OP(x) \text{ and } OP(y)\end{aligned}$$

and, hence, $OP(x \cdot y)$. $\square$

Due to Theorem 4.3 and Theorem 5.1 in the remainder of the paper we will always consider the standard contact relation $C(x,y)$ iff $x \nleq y^{\perp}$ on a Stonian p-ortholattice.

6. Strict non-distributivity for RT^{-}_{EC}

Previously, mereotopologies have been represented using Boolean Contact Algebras [13, 14, 26, 27], whose main structure constitutes a Boolean algebra

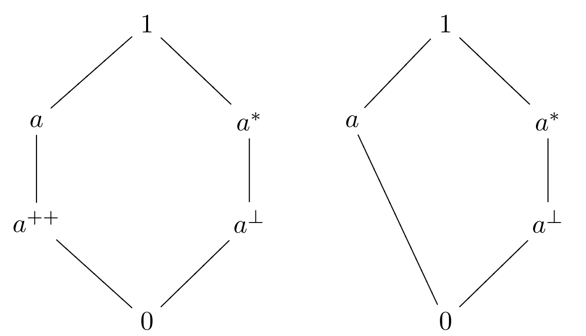


Figure 3: The Stonian p-ortholattice C_6 and the non-modular lattice N_5.

or more generally a pseudocomplemented distributive lattice. Both have distributivity as an important characteristic. Notice that the lattices representing Clarke's full theory [3] are also distributive. Although we have $a^* + a^{**} = 1$ and $a^+ \cdot a^{++} = 0$ so that any p-ortholattice satisfies the double Stone identities, the models of RT^- are far from being distributive. The next theorem will show that a model of RT^- is distributive if and only if it does not satisfy axiom (A11). In fact, all models satisfying (A11) are then non-modular, which is a more rigorous restriction than non-distributivity. This gives us a characterization of RT^-_{EC}. Moreover, a new condition when a p-ortholattice is distributive and thus Boolean follows from Theorem 6.1.

Theorem 6.1. *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a Stonian p-ortholattice. Then the following statements are equivalent:*

1. *L is modular.*
2. *$a^* = a^\perp$ for all $a \in L$.*
3. *$a^* = a^+$ for all $a \in L$.*
4. *L is a Boolean algebra*
5. *L is distributive.*
6. *L does not satisfy (A11).*
7. *L does not have C_6 as a subalgebra.*

Proof. We are going to show 1. $\Rightarrow$ 2. $\Rightarrow$ 3. $\Rightarrow$ 4. $\Rightarrow$ 5. $\Rightarrow$ 1. and 2. $\Leftrightarrow$ 6. and 3. $\Leftrightarrow$ 7. Notice that the implications 4. $\Rightarrow$ 5. and 5. $\Rightarrow$ 1. are trivial.

1. $\Rightarrow$ 2. : Suppose there is an element a with $a^\perp \lneq a^*$. Then the elements $0, a, a^\perp, a^*, 1$ form by Lemma 3.2(2), Corollary 3.3(2), and Lemma 3.7(2) a sublattice that is isomorphic to the pentagon N_5 (see Figure 3), i.e. L is not modular, a contradiction to 1.

2. ⇒ 3. : This follows immediately from

$$\begin{aligned} a^{\perp} &= a^{\perp\perp\perp} \\ &= a^{\perp*\perp} && \text{by 2.} \\ &= a^{+} \end{aligned}$$

3. ⇒ 4. : If $a^* = a^+$ for all $a \in L$, then L is a complemented lattice in which the complementation is simultaneously a quasicomplementation. Such a lattice is known to be a Boolean algebra [4].

2. ⇔ 6.**:** First of all, Axiom (A11) can be rewritten as follows:

$$\begin{aligned} (A11) &\iff \exists x, y[x \neq 0 \wedge y \neq 0 \wedge C(x,y) \wedge \neg O(x,y)] \\ &\iff \exists x, y[C(x,y) \wedge \neg O(x,y)] && \text{since } \forall z[\neg C(0,z)] \\ &\iff \exists x, y[x \not\leq y^{\perp} \wedge x \cdot y = 0] \\ &\iff \exists x, y[x \not\leq y^{\perp} \wedge y \leq x^*] \end{aligned}$$

It remains to show that the last property is equivalent to existence of an element $a \in L$ with $a^{\perp} \neq a^*$. Therefore, assume there are x, y with $x \not\leq y^{\perp} \wedge y \leq x^*$. Then $y \not\leq x^{\perp}$, and, hence, $x^{\perp} \neq x^*$. Conversely, suppose $a^{\perp} \neq a^*$, i.e. $a^* \not\leq a^{\perp}$ by Lemma 3.7(1), and choose $x = a^*$ and $y = a$.

3. ⇔ 7.**:** Assume L has C_6 as a subalgebra. With the notation in Figure 3 we have $a^+ = a^{\perp} \neq a^*$. Conversely, assume there is an element $a \in L$ with $a^+ \neq a^*$. Then we have $a^{++} = a^{*\perp} \neq a^{+\perp} = a^{**}$ so that the elements $0, a^+, a^*, a^{++}, a^{**}, 1$ form by Lemma 3.2(2), Corollary 3.3(2), and Lemma 3.7(2) a subalgebra of L that is isomorphic to C_6. □

7. Representation of *RT*

In the presence of Axiom (A12) four Stonian p-ortholattices are of interest. In particular, we will show in Theorem 7.2 that one of those lattices has to be a sublattice of the structure in question. We now introduce those lattices.

The four Stonian p-ortholattices of C_{14}, C_{16}, C_{18} and C_{20} have a common outer structure. They only differ in the intervals between $y^{++} \cdot x^+$ and $y^{**} \cdot x^*$ and between $(y^{**} \cdot x^*)^{\perp}$ and $(y^{++} \cdot x^+)^{\perp}$. Notice that those two intervals must be dual due to the orthocomplement operation $.^{\perp}$. The common outer structure of all four lattices is provided in Figure 4 and the specific inner structure in Figure 5. Even though the outer structure of all four lattices is the same, none of them is a sublattice of any of the others. They differ either in the meet of y^{++} and x^* and the meet of y^{**} and x^+ or the union of $y^{++} \cdot x^*$ and $y^{**} \cdot x^+$. In C_{14} we have $y^{++} \cdot x^* = y^{++} \cdot x^+ = y^{**} \cdot x^+$ whereas $y^{++} \cdot x^* \neq y^{++} \cdot x^+$ holds in $C_{16} - C_{20}$, and $y^{**} \cdot x^+ \neq y^{++} \cdot x^+$ holds in C_{18} and C_{20}. In C_{18} we have $y^{++} \cdot x^* + y^{**} \cdot x^+ = y^{**} \cdot x^*$ whereas the two elements are different in C_{20}.

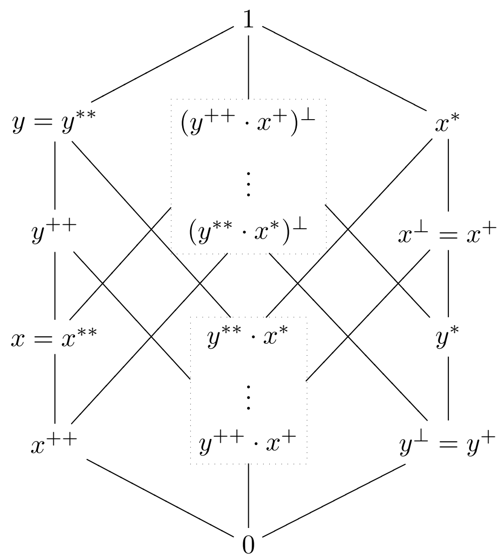


Figure 4: The outer structure of C_{14}, C_{16}, C_{18} and C_{20}.

We have $WCont(x, y^*)$ and $EC(x, x^*)$, i.e. the lattices satisfy $(A11)$ and $(A12)$. Since $(A13)$ holds in all finite Stonian p-ortholattices all four lattices are models of RT_0.

Lemma 7.1. *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a Stonian p-ortholattice. Then the following statements are equivalent:*

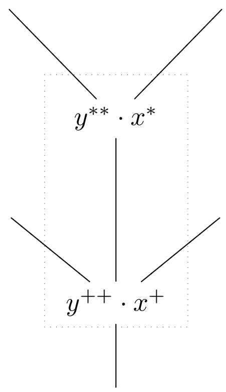
1. *L satisfies (A12).*
2. *L satisfies the property (A12'):*

$$(A12') \;\; \exists x, y[x \neq 0 \wedge x^{**} \leq y^{++} \wedge \forall z[x \not\leq z^{++} \vee z^{**} \not\leq y]].$$

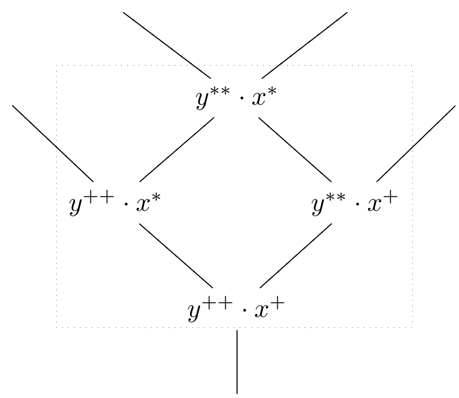
Proof. First of all, we have

$$\begin{aligned}(A12) \iff\; & \exists x, y[x \neq 0 \wedge y \neq 0 \wedge WCon(x, y)] \\ \iff\; & \exists x, y[x \neq 0 \wedge y \neq 0 \wedge \neg C(x^{**}, y^{**}) \\ & \quad \wedge \forall z[z \neq 0 \wedge x \leq z \wedge z^{++} = z \rightarrow C(z^{**}, y)]] \\ \iff\; & \exists x, y[x \neq 0 \wedge y \neq 0 \wedge x^{**} \leq y^{**\perp} \\ & \quad \wedge \forall z[z \neq 0 \wedge x \leq z \wedge z^{++} = z \rightarrow z^{**} \not\leq y^{\perp}]].\end{aligned}$$

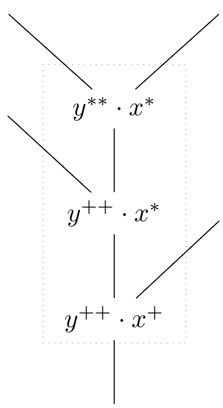
1. $\Rightarrow$ 2. : Suppose x, y satisfy (A12). Then we want to show that x and $y^\perp$ satisfy $(A12')$. We have $x \neq 0$ and $x^{**} \leq y^{**\perp} = y^{\perp++}$. Now, suppose $z \in L$ with $x \leq z^{++}$. Then we have to show that $z^{**} \not\leq y^\perp$. We have $z^{++} \neq 0$ since $x \neq 0$ and $(z^{++})^{++} = z^{++}$. From the last of the equivalent versions of (A12) above we conclude $z^{++**} \not\leq y^\perp$. This implies $z^{**} \not\leq y^\perp$.



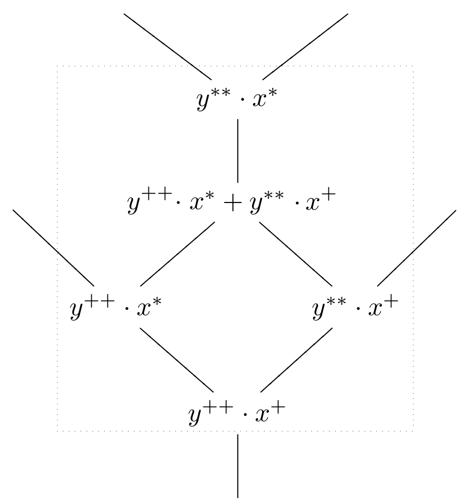
(a) The inner structure of C_{14}



(b) The inner structure of C_{18}



(c) The inner structure of C_{16}



(d) The inner structure of C_{20}

Figure 5: The inner structure of C_{14}, C_{16}, C_{18} and C_{20}.

2. $\Rightarrow$ 1. : Suppose x, y satisfy (A12'). Then we want to show that x and $y^{\perp}$ satisfy the last of the equivalent versions of ($A12$) above. We have $x \neq 0$. If $y^{\perp} = 0$, then $y = 1$ and we have $x \leq 1^{++}$ and $1^{**} \leq y$, a contradiction to (A12'). Furthermore, $x^{**} \leq y^{++} = y^{\perp\perp++} = y^{\perp**\perp} = (y^{\perp})^{**\perp}$. Now assume that there is a $z \in L$ with $z \neq 0$, $x \leq z$ and $z^{++} = z$. Then we have $x \leq z = z^{++}$ so that (A12') implies $z^{**} \not\leq y = (y^{\perp})^{\perp}$. $\square$

We are now ready to prove that in the context of Axiom (A12) one of the four lattices C_{14}, C_{16}, C_{18} or C_{20} is always included as a sublattice.

Theorem 7.2. *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a Stonian p-ortholattice satisfying (A12). Then L has C_{14}, C_{16}, C_{18} or C_{20} as a subalgebra.*

Proof. By Lemma 7.1 L satisfies (A12'). We want to show that given x, y satisfying ($A12'$) the elements $0, x^{++}, x^{**}, y^{++}, y^{**}, y^{+}, y^{*}, x^{+}, x^{*}, 1$ induce a subalgebra isomorphic to one of the structures C_{14}, C_{16}, C_{18} or C_{20}. Therefore, we first show that $x^{++} \lneq x^{**} \lneq y^{++} \lneq y^{**}$. Notice that this will imply that x^{+}, x^{*}, y^{+} and y^{*} are also different by applying the orthocomplementation. Assume $x^{++} = x^{**}$. Then we have $x \leq x^{**} = x^{++}$ and $x^{**} \leq y^{++} \leq y$, a contradiction to ($A12'$) (with $z = x$). Assume $y^{++} = y^{**}$. Then we have $x \leq x^{**} \leq y^{++}$ and $y^{**} = y^{++} \leq y$, again a contradiction to ($A12'$) (with $z = y$). Finally, assume $x^{**} = y^{++}$. Then we have $x^{**} = y^{++} = (y^{++})^{++} = (x^{**})^{++} = x^{++}$, which we have already shown is impossible. So far we have verified the left and the right chain of the outer structure of the four lattices. In the next step we concentrate on the rest of the outer structure. Obviously we have $x^{**} \cdot x^{*} = 0 = y^{**} \cdot y^{*}$. We want to show that $y^{++} \cdot x^{+} \neq 0$. Assume $y^{++} \cdot x^{+} = 0$. This implies $y^{++} \leq x^{+*} = x^{**}$, again a property which cannot hold. Finally, we want to show that $y^{++} \cdot x^{+} \neq y^{**} \cdot x^{*}$. Assume that $y^{++} \cdot x^{+} = y^{**} \cdot x^{*}$. From $y^{**} \cdot x^{*} \cdot y^{*} = 0$ and $y^{**} \cdot x^{*} \cdot x = 0$ we conclude $y^{**} \cdot x^{*} \cdot (y^{*} + x) = 0$ using Lemma 3.2. This implies

$$\begin{aligned}
y^{**} &\leq (x^{*} \cdot (y^{*} + x))^{*} && \text{see above} \\
&= x^{**} + (y^{**} \cdot x^{*}) && \text{Stone identities} \\
&= x^{**} + (y^{++} \cdot x^{+}) && \text{assumption} \\
&\leq x^{**} + y^{++} \\
&= y^{++},
\end{aligned}$$

again a contradiction. This completes the outer structure since all remaining properties follow from those shown using orthocomplementation. Depending on whether the elements $y^{++} \cdot x^{*}$ and $y^{++} \cdot x^{+}$, the elements $y^{**} \cdot x^{+}$ and $y^{++} \cdot x^{+}$ or the elements $y^{++} \cdot x^{*} + y^{**} \cdot x^{+}$ and $y^{**} \cdot x^{*}$ are equal or different we obtain the lattices C_{14}, C_{16}, C_{18} or C_{20}. Notice that in the case that $y^{++} \cdot x^{*} = y^{++} \cdot x^{+}$ and $y^{**} \cdot x^{+} \neq y^{++} \cdot x^{+}$ we obtain C_{16} by letting x be y^{*} and y be x^{*}. $\square$

As already mentioned all four lattices of Figure 4,5 satisfy (A11), i.e. the pair (x, x^*) always satisfies $EC(x, x^*)$, so that the previous theorem induces the following corollary.

Corollary 7.3. *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a Stonian p-ortholattice that satisfies (A12). Then L also satisfies (A11).*

Proof. By Theorem 7.2 one of the four lattices C_{14}, C_{16}, C_{18} or C_{20} is a subalgebra of L. In that subalgebra we have $x \not\leq x^{++} = x^{*\perp}$ and $x \cdot x^* = 0$. This is equivalent to $C(x, x^*)$ and $\neg O(x, x^*)$, i.e. $ECx, x^*)$. None of the properties does depend on any other element in L so that $EC(x, x^*)$ also holds in L. $\square$

This finishes the representation of the theory RT (RT_0 without (A13)). The last corollary shows that (A11) is captured by (A12) and thus unnecessary in the full theory. However, this is far from obvious in the original theory and in the topological models to which soundness and completeness has been proved.

8. Conclusion and Outlook

In this paper we have provided a representation of RT and its subtheories by Stonian p-ortholattices. This representation shows that the connection relation can be uniquely defined through the lattice structure alone. Since the lattices are only defined by their order and meet relation, this hints that the theory can also be based on parthood and overlap relations while having exactly the same models. An alternative axiomatization of RT^- can be based on the properties of Stonian p-ortholattices (orthocomplementation, pseudocomplementation, Stone identity) which can be defined solely in terms of the partial order underlying the lattice. Together with the operations of ortho- and pseudo-complementation, we are then able to uniquely define the contact relation $C(x, y) \iff x \not\leq y^\perp$.

The paper gives a full lattice-theoretic characterization of the models of RT and RT^-. It contributes to the understanding of different region-based (point-free) QSR frameworks. In particular, as a pure mathematical account it helps in understanding the models of the theory RT. The main part of the paper introduces Stonian p-ortholattices as generalization of the well-known (distributive) Stone lattices. The work exhibits the non-distributive character of Asher and Vieu's [1] spatial theory, which is so far unique amongst mereotopologies. All other characterizations [3, 27, 14] have identified Boolean or pseudocomplemented distributive lattices as models of other mereotopologies. This paper is a significant step towards a unified lattice-theoretic account of mereotopologies and, more generally, of qualitative region-based theories of space.

In Section 6 we have shown that distributivity forces Stonian p-ortholattices to be Boolean algebras, i.e. all three complement operations coincide. In that case the contact relation collapses to overlap similar to Clarke's original system.

As already discussed in the introduction this is not a defect since Stonian p-ortholattices describe a larger set of regions than distributive theories such as RCC and their algebraic counterpart, Boolean contact algebras (BCAs). On the contrary, this fact actually shows that distributivity is not a desired property in Asher and Vieu's framework. Even though Stonian p-ortholattices are, in general, not distributive, their skeleton (and its dual) is. For a detailed study of the relationship between BCAs and Stonian p-ortholattices via their skeleton we refer to [31].

Further work will concentrate on topological representation theorems of Stonian p-ortholattices similar to those already developed for Boolean contact algebras. This will also clarify the exact nature of the topological models of RT_0.

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