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On the algebra of regular sets

Properties of representable Stonian p-Ortholattices

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Abstract The mereotopology RT^- has in Stonian p-ortholattices its algebraic counterpart. We study representability of these lattices and show that not all Stonian p-ortholattices can be represented by the set of regular sets of a topological space. We identify five conditions that hold in algebras of regular sets and which can be used to eliminate non-representable Stonian p-ortholattices. This shows not only that the original completeness theorem for RT^- is incorrect, but is also an important step towards an algebraic representation (up to isomorphism) of the regular sets of topological spaces.

Keywords regular sets · Stonian p-ortholattice · representation · region-based topology · mereotopology · interior operation · localized distributivity

Mathematics Subject Classification (2000) Primary 54H10, Secondary 06B15, 06E75, 68T27, 68T30, 06B20, 06E15

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1 Introduction

In his seminal work, Stone [14] proved that every Boolean algebra has an isomorphic representation as the open sets of a topological space. Similarly, the regular open sets (and dually the regular closed sets) of a topological space also form Boolean algebras, cf. [11]. It is only natural to ask what algebraic structures represent the regular sets of a topological space. This is still an open question. It is motivated by the theory of qualitative space proposed in [1] whose intended models contain only regular sets of a topological space. Generally speaking, Qualitative Spatial Reasoning (QSR) aims to capture relations in space qualitatively, e.g. without applying a specific metric. Topological and mereological relations are fundamental amongst such relations and are often captured in logical theories referred to as mereotopologies. For many practical representation and reasoning problems it is appropriate to assume that all regions are of the same dimension, we refer to such theories as equidimensional mereotopologies. Therein all regions must be regular, i.e. isolated points or other lower-dimensional artifacts cannot occur in order to ensure closure under sums and intersections amongst regions. This commonsensical notion of regularity corresponds to the definition of regularity known from topology. Several alternative axiomatizations of mereotopology have been proposed, for a comprehensive overview of the different ontological assumptions and sets of axioms see e.g. [6, 9]. One way to classify these axiomatizations of equidimensional mereotopology is by their intended topological models. Either it is assumed that all regions are regular closed (or dually regular open) or that all regions are regular, but not necessarily open or closed [6].

The approach assuming that all regions are regular closed makes the ontological commitment that two regions with identical closures must be identical. Such an assumption is made for example by the most prominent mereotopology, the Region-Connection Calculus (RCC, [5]). Topological interpretations thereof include only regular closed sets and consequently each model forms a Boolean algebra. If we superimpose a contact relation on such a Boolean algebra we obtain so-called Boolean contact algebras [7, 13].

The alternative approach, first axiomatized as a first-order theory by Asher and Vieu [1] which we refer to as RT_0, can distinguish regions with identical closures. For example, a region that is neither open nor closed is distinct from both its interior and its closure. This approach is maybe less popular, but the arising models are both algebraically and topologically interesting. Most importantly, the theory RT_0 captures the same set of mereotopological relations as the RCC. Though regions with identical closures can be distinct, interiors and closures of all regions are still required to exist. Moreover, all regions must again be regular which leads to definitions of regular union and intersection that differ from standard set-theoretic union and intersection. Hence not every set of subsets of a topological space can be extended to a model of RT_0. For studying the mereotopology RT_0, it is convenient to study a weaker theory thereof which has been introduced in [10] and is referred to as RT^-. The models of RT^- are structurally equivalent to the models of RT_0. However, 'trivial' models of the RT^-, in which the extensions of external contact (EC) or weak contact (WCont) are empty, are not models of RT_0. Recall that two regions are externally connected if they share a point, i.e. are in contact, but do not share an equidimensional part, i.e. do not overlap. Two regions are in weak contact if their closure are not in contact but any open region containing one of them is in contact to the other.

In [10] Stonian p-ortholattices were introduced to characterize the models of RT^- algebraically. Moreover, it has been shown that the models of RT_0 are the Stonian p-ortholattices that contain at least one of C_{14}, C_{16}, C_{18}, or C_{20}[1] as a sublattice. Subsequent work [15] demonstrated that the skeleton, i.e. the set of all closed sets, of a Stonian p-ortholattice is a model of RCC and every model of RCC can be extended to a Stonian p-ortholattice while preserving the contact relation. This verified the close relationship between the RCC and RT^-. However, one important question has been left open: the topological, i.e. the point-set, representability of Stonian p-ortholattices. If every model of the theory RT_0 would indeed capture the regular sets of some topological space as claimed in [1], we could represent all Stonian p-ortholattices topologically by the regular sets of some topological space. An algebraic representation of the regular closed sets of a topological space would immediately follow. In this paper, we prove that this is not the case: not all Stonian p-ortholattice can be represented as regular point sets of a topological space. Therefore, the completeness theorem for RT_0 with respect to the intended models of regular subsets of a topological space from the original paper [1] is incorrect; we will show why it fails.

In this work, we make significant progress towards a full topological representation of Stonian p-ortholattices in two ways. We give examples of Stonian p-ortholattices that are not (topologically) representable and identify five necessary properties for Stonian p-ortholattices to be representable. These properties, amongst them a conditional form of distributivity, are topologically motivated and can be used to eliminate unintended models of the theories RT^- and RT_0. Though the identified properties eliminate all unintended, i.e. non-representable, models up to a domain size of 24, a proof whether the properties suffice to eliminate all non-representable Stonian p-ortholattices remains outstanding.

The paper is structured as following. First we review the mereotopology RT_0 and its subtheory RT^-, followed by a a review of Stonian p-ortholattices and their algebraic properties and a compact equational axiomatization of Stonian p-ortholattices. The equational theory enabled us in the first place to generate sufficiently large Stonian p-ortholattices that guided our inquiry into representable ones. We subsequently give examples of Stonian p-ortholattices that are models of the full mereotopology RT_0 and give an equational theory thereof. In Section 4 we formally define what we mean by representability of Stonian p-ortholattices and how this relates to the structures we can obtain from the regular sets of a topological space. Section 5 introduces relative notions of interior and closure; those are necessary for Section 6, in which we introduce the properties (RP_1), (RP_2), (M), (S), as well as a localized version of distributivity (D). These properties are always satisfied by lattices constructed from the regular sets of a topological space. All five properties can again be expressed as quasiidentities, thus preserving the equational character if we extend the theory of Stonian p-ortholattices by those properties. Section 6 concludes by showing that our localized version of distributivity implies the conditions (RP_1) and (RP_2). In Section 7 we look at

[1] C_{14}, C_{16}, C_{18}, or C_{20} are all Stonian p-ortholattices themselves, we will formally define them later in the paper.

small (with up to 24 elements) Stonian p-ortholattices and show that the only two representable ones amongst them, C_{18} and C_{24}, satisfy all five conditions while all non-representable ones amongst them violate at least one of the conditions. We also show how to construct a counterexample to the completeness proof of [1] and finally show why the original completeness proof is incorrect.

2 The Mereotopology RT_0

The mereotopology RT_0 proposed by Asher and Vieu [1] evolved from Clarke's theory, addressing some of its shortcomings. RT_0 is a first-order theory based on a binary contact relation C as primitive. We use the following definitions, we include only the ones necessary for the subsequent axioms or later in the paper:

(D1) $P(x,y) \equiv_{\text{def}} \forall z[C(z,x) \to C(z,y)]$ (Parthood)
(D3) $O(x,y) \equiv_{\text{def}} \exists z[P(z,x) \land P(z,y)]$ (Overlap)
(D4) $EC(x,y) \equiv_{\text{def}} C(x,y) \land \neg O(x,y)$ (External connection)
(D6) $NTP(x,y) \equiv_{\text{def}} P(x,y) \land \neg \exists z[EC(z,x) \land EC(z,y)])$
(Non-tangential parthood)
(D7) $c(x) \equiv_{\text{def}} -i(-x)$ (Closure operation)
(D8) $OP(x) \equiv_{\text{def}} x = i(x)$ (Open individuals)
(D11) $WCont(x,y) \equiv_{\text{def}} \neg C(c(x), c(y)) \land \forall z[(P(x,z) \land OP(z)) \to C(c(z), y)]$
(Weak contact)

The operations interior, i, and complement, $-$, necessary in (D7) and (D8) are defined by the axioms (A7) and (A8). Notice that the elements implied by the axioms (A4)-(A8), (A13) are indeed unique which follows immediately from (A3).

(A1) $\forall x[C(x,x)]$ (C reflexive)
(A2) $\forall x,y[C(x,y) \to C(y,x)]$ (C symmetric)
(A3) $\forall x,y[\forall z(C(z,x) \leftrightarrow C(z,y)) \to x = y]$ (C extensional)
(A4) $\exists x \forall y[C(x,y)]$ (Existence of a unique universally connected element 1)
(A5) $\forall x,y \exists z \forall u[C(u,z) \leftrightarrow (C(u,x) \lor C(u,y))]$
(Existence of a unique sum $x \cup y$ for every x and y)
(A6) $\forall x,y[O(x,y) \to \exists z \forall u[C(u,z) \leftrightarrow \exists v(P(v,x) \land P(v,y) \land C(v,u))]]$
(Existence of a unique intersection $x \cap y$ for overlapping elements x and y)
(A7) $\forall x[\exists y[\neg C(y,x)] \to \exists z \forall u[C(u,z) \leftrightarrow \exists v(\neg C(v,x) \land C(v,u))]]$
(Existence of a unique complement $-x$ for elements $x \neq 1$)
(A8) $\forall x \exists z \forall u[C(u,z) \leftrightarrow \exists v(NTP(v,x) \land C(v,u))]$
(Existence of a unique interior $i(x)$ for every x)
(A9) $c(1) = 1$ (Closure as a total function)
(A10) $\forall x,y[(OP(x) \land OP(y) \land O(x,y)) \to OP(x \cap y)]$
(Intersection of open individuals is open)
(A11) $\exists x,y[EC(x,y)]$ (Existence of two externally connected elements)
(A12) $\exists x,y[WCont(x,y)]$ (Existence of two weakly connected elements)
(A13) $\forall x \exists y[P(x,y) \land OP(y) \land \forall z((P(x,z) \land OP(z)) \to P(y,z))]$
(Existence of a smallest open neighborhood $n(x)$ for every x)

The axioms (A1) to (A13) together with the previous definitions axiomatize the theory RT_0. We will also use the subtheory $RT^- = RT_0 \setminus \{(A11), (A12), (A13)\}$ introduced in [10]. We have maintained the notation and numbering from [10]; the notation slightly differs from that in [1] to avoid confusion with the lattice operations we use in our work.

3 Stonian p-ortholattices

Stonian p-ortholattices have been introduced in [10] as algebraic counterparts of the mereotopology RT^-. Here we first review the definitions and properties of Stonian p-ortholattices and then give an equational axiomatization thereof. Subsequently, we give examples of Stonian p-ortholattices and show how to define the RT^- and the original theory RT_0 algebraically as extension thereof.

Definition 1 A Stonian p-ortholattice is a structure $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ of type $\langle 2, 2, 1, 1, 0, 0\rangle$ such that

1. $\langle L, +, \cdot, 0, 1\rangle$ is a bounded lattice;
2. x^* is the pseudocomplement of x, i.e. $x \cdot y = 0 \iff y \leq x^*$ for all $x, y \in L$;
3. $x^\perp$ is an orthocomplement of x, i.e. for all $x, y \in L$ we have
   (a) $x^{\perp\perp} = x$,
   (b) $x \cdot x^\perp = 0$,
   (c) $x \leq y$ implies $y^\perp \leq x^\perp$;
4. the Stone identity $(x \cdot y)^* = x^* + y^*$ holds for all $x, y \in L$.

In a Stonian p-ortholattice the structure $\langle L, +, \cdot, ^*, 0, 1\rangle$ is a pseudocomplemented lattice while the structure $\langle L, +, \cdot, ^\perp, 0, 1\rangle$ is an ortholattice. Moreover, with $x^+ = x^{\perp * \perp}$ the structure $\langle L, +, \cdot, ^+, 0, 1\rangle$ is a quasicomplemented lattice, also known as a dually pseudocomplemented lattice. Stonian p-ortholattices are a natural generalization of the Stone lattices, i.e. the distributive lattices that satisfy the Stone identity, to non-distributive lattices. Stonian p-ortholattices satisfy the De Morgan laws but are in general not even modular [10], which is a weaker notion than distributivity. The next theorem from [10] shows that modularity, like many other conditions, forces a Stonian p-ortholattice to be Boolean.

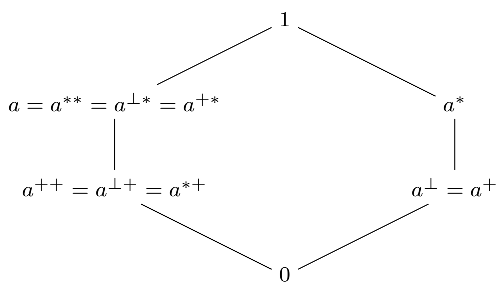


Fig. 1: The smallest non-Boolean Stonian p-ortholattice C_6

Theorem 1 *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a Stonian p-ortholattice. Then the following statements are equivalent:*

1. $x^* = x^\perp$ *for all* $x \in L$.
2. $x^* = x^+$ *for all* $x \in L$.
3. L *is modular, i.e.* $x \leq y \to x + (y \cdot z) = y \cdot (x + z)$ *for all* $x, y, z \in L$.
4. L *is distributive, i.e.* $x + (y \cdot z) = (x + y) \cdot (x + z)$ *for all* $x, y, z \in L$.
5. L *is uniquely complemented, i.e.* $x + y = 1 \wedge x \cdot y = 0 \wedge x + z = 1 \wedge x \cdot z = 0 \to y = z$ *for all* $x, y, z \in L$.
6. L *is a Boolean algebra.*
7. L *does not have* C_6 *(see Fig. 1) as a subalgebra.*

Basic properties of Stonian p-ortholattices have been proven in [10,15]. The properties of interest here are summarized by the following lemma.

Lemma 1 *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a Stonian p-ortholattice. Then we have for all $x, y \in L$:*

1. $x \cdot x^* = 0$, $x + x^* = 1$, $x \cdot x^+ = 0$, *and* $x + x^+ = 1$.
2. $x^+ \leq x^\perp \leq x^*$ *and* $x^{++} \leq x \leq x^{**}$.
3. $x^{+++} = x^+$ *and* $x^{***} = x^*$.
4. $x \leq y$ *implies* $y^* \leq x^*$ *and* $y^+ \leq x^+$.
5. $(x + y)^* = x^* \cdot y^*$, $(x \cdot y)^+ = x^+ + y^+$, *and* $(x + y)^+ = x^+ \cdot y^+$.
6. $(x + y)^\perp = x^\perp \cdot y^\perp$ *and* $(x \cdot y)^\perp = x^\perp + y^\perp$.
7. $x^{*\perp} = x^{\perp +} = x^{*+} = x^{++}$ *and* $x^{+\perp} = x^{\perp *} = x^{+*} = x^{**}$.

We will use these properties as well as the following equivalences throughout the paper without further mentioning.

Lemma 2 *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a Stonian p-ortholattice. Then the following statements are equivalent:*

1. $x^{++} \leq y$,
2. $x^{++} \leq y^{++}$,
3. $x^{**} \leq y^{**}$,
4. $x \leq y^{**}$.

Proof $(1) \Rightarrow (2):$ We have $x^{++} = x^{++++} \leq y^{++}$.
$(2) \Rightarrow (3):$ From $x^{++} \leq y^{++}$ we directly obtain $x^{**} = x^{++**} \leq y^{++**} = y^{**}$.
$(3) \Rightarrow (4):$ This follows from $x \leq x^{**}$.
$(4) \Rightarrow (1):$ From $x \leq y^{**}$ we obtain $x^{++} \leq y^{**++} = y^{++} \leq y$. $\square$

3.1 An Equational Theory of Stonian p-ortholattices

The Stonian p-ortholattices form a variety, i.e. they can be axiomatized by a set of equations. The equational theory can be constructed as the union of the axioms of the equational theories of pseudocomplemented lattices and ortholattices, together with the Stone identity (S1).

The theory of pseudocomplemented lattices consists of the axioms for bounded lattices $(L1^\wedge)$-$(L6^\wedge)$, $(L1^\vee)$-$(L6^\vee)$, and the definitions $(D1^\wedge)$ and $(D1^\vee)$ from [12] extended by the axioms (PC1)-(PC3) governing the pseudocomplement operation * [2]. The axioms (OC1)-(OC3) are a compact axiomatization of the theory of ortholattices [3]. The definition(QC1) defines quasicomplementation $^+$. Together with the Stone identity we obtain the equational theory of Stonian p-ortholattices consisting of all of the above axioms, and (S1).

$(L1^\wedge)$	$x = x \cdot x$	$(L1^\vee)$	$x = x + x$
$(L2^\wedge)$	$x \cdot y = y \cdot x$	$(L2^\vee)$	$x + y = y + x$
$(L3^\wedge)$	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$	$(L3^\vee)$	$x + (y + z) = (x + y) + z$
$(L4^\wedge)$	$x \cdot (x + y) = x$	$(L4^\vee)$	$x + (x \cdot y) = x$
$(L5^\wedge)$	$0 \cdot x = 0$	$(L5^\vee)$	$0 + x = x$
$(L6^\wedge)$	$1 \cdot x = x$	$(L6^\vee)$	$1 + x = 1$
$(D1^\wedge)$	$x \leq y \leftrightarrow x \cdot y = x$	$(D1^\vee)$	$x \leq y \leftrightarrow x + y = y$
(PC1)	$x \cdot (x \cdot y)^* = x \cdot y^*$	(OC1)	$(x + y) + z = (z^\perp \cdot y^\perp)^\perp + x$
(PC2)	$x \cdot 0^* = x$	(OC2)	$x \cdot (x + y) = x$
(PC3)	$0^{**} = 0$	(OC3)	$x + (y \cdot y^\perp) = x$
(QC1)	$x^+ = x^{\perp * \perp}$	(S1)	$(x \cdot y)^* = x^* + y^*$

The set of axioms $\{L2^\wedge, L3^\wedge, L4'^\wedge, L6^\wedge, PC1, PC2', PC2'', O1', O2', S1'\}$ is a more compact axiomatization of Stonian p-ortholattices[2] in which $+$ is definable in terms of $\cdot$ and $^\perp$ $(OC3')$. We verified that all the above properties are theorems of this reduced set of axioms using the automated theorem prover Prover9. See `www.cs.toronto.edu/~torsten/RegularSets/` for the proof output.

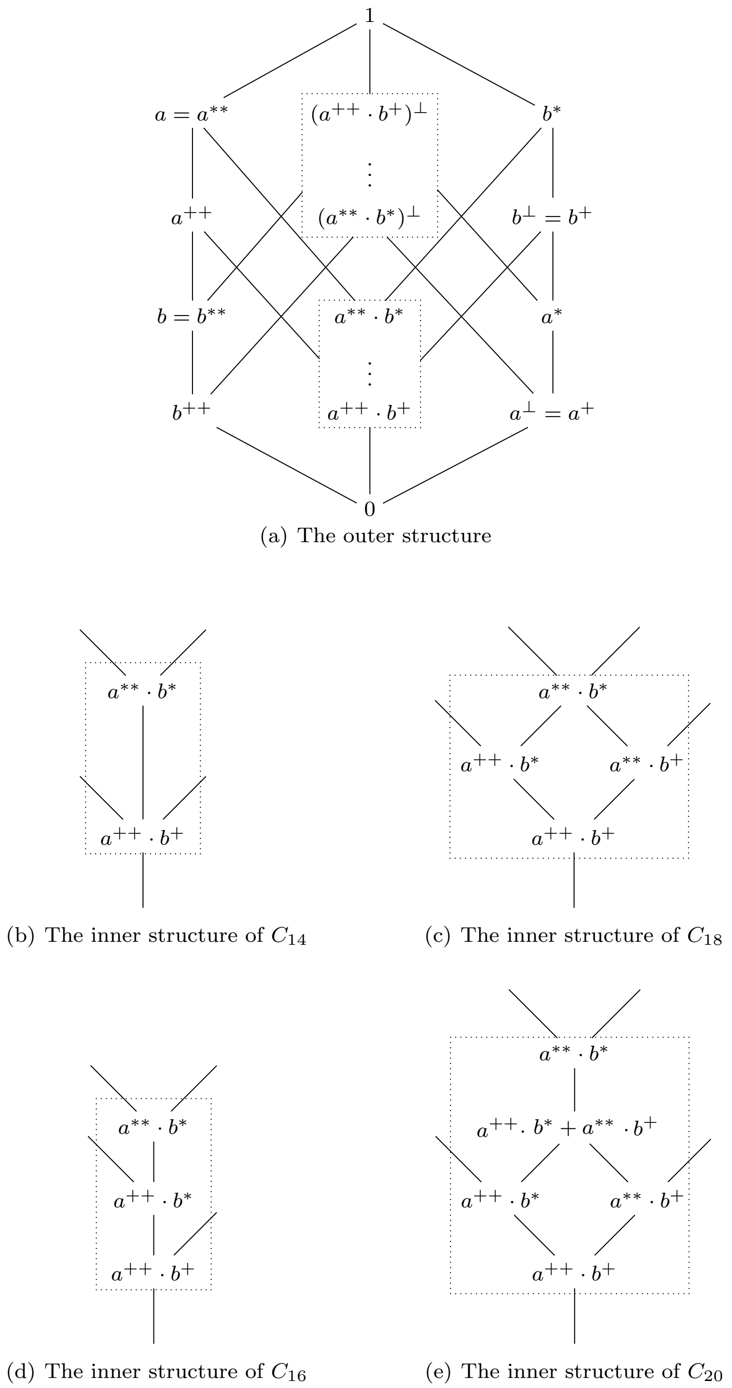
$(L2^\wedge)$	$x \cdot y = y \cdot x$	$(L3^\wedge)$	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
$(L4'^\wedge)$	$x \cdot (x^\perp \cdot y^\perp)^\perp = x$	$(L6^\wedge)$	$1 \cdot x = x$
$(PC1)$	$x \cdot (x \cdot y)^* = x \cdot y^*$	$(OC1')$	$x^{\perp\perp} = x$
$(PC2')$	$0^* = 1$	$(OC2')$	$x \cdot x^\perp = 0$
$(PC2'')$	$1^* = 0$	$(S1')$	$(x \cdot y)^* = (x^{*\perp} \cdot y^{*\perp})^\perp$
$(OC3')$	$(x + y) = (x^\perp \cdot y^\perp)^\perp$	$(QC1)$	$x^+ = x^{\perp * \perp}$

Throughout the paper we are interested in equational versions of all newly introduced properties. Then the extension of the theory of Stonian p-ortholattices by those conditions results again in a variety.

3.2 Examples of Stonian p-ortholattices

Before we define the mereotopologies RT^- and RT_0 as extension of Stonian p-ortholattices, we review the four Stonian p-ortholattices C_{14}, C_{16}, C_{18} and C_{20} introduced in [10]. Those lattices have a common outer structure given in Figure 2(a), and they only differ in the intervals between $a^{++} \cdot b^+$ and $a^{**} \cdot b^*$

[2] Peter Jipsen pointed out that a reduction to a theory with only the binary operation $\cdot$ and the unary operations $'$ and * is possible.



(a) The outer structure

(b) The inner structure of C_{14}

(c) The inner structure of C_{18}

(d) The inner structure of C_{16}

(e) The inner structure of C_{20}

Fig. 2: The outer and inner structure of C_{14}, C_{16}, C_{18} and C_{20}

and $(a^{**} \cdot b^{*})^{\perp}$ and $(a^{++} \cdot b^{+})^{\perp}$. Notice that those two intervals must be dual due to the orthocomplement operation $^{\perp}$. The specific inner structure of those lattices is given by Fig. 2(b)–(e). All four structures as well as the examples in the remainder of the paper were checked to be Stonian p-ortholattices by a program written in the programming language Haskell, which is available from `www.cs.toronto.edu/~torsten/RegularSets/`. Furthermore, it was also checked that all lattices satisfy the additional axioms $(A11)-(A13)$, i.e. that all lattices are models of the full theory RT_0. As we will see shortly, C_{14} is in fact the only Stonian p-ortholattice satisfying $(A12)$ that contains only 14 elements and is thereby the smallest model of RT_0. However, several other Stonian p-ortholattices with 16, 18, or 20 elements exist.

3.3 RT_0 as extension of Stonian p-ortholattices

The definitional extensions defining parthood as $P(x, y) \Leftrightarrow x \leq y$ and contact as $C(x, y) \Leftrightarrow x \not\leq y^{\perp}$ reconstruct the mereotopology RT^{-} from Stonian p-ortholattices [10]. Other mereotopological relations such as external connection EC, overlap O, non-tangential parthood NTP, etc. can be defined subsequently using the definitions from RT^{-}. The models of the mereotopology RT^{-} are the class of all Stonian p-ortholattices [10, Theorems 4.3 and 5.1]:

Theorem 2 *Let L be a set, C a binary relation over L. Define $x \not\leq y^{\perp} \Leftrightarrow C(x, y)$. Then $\langle L, C\rangle$ is a model of RT^{-} iff the structure $\langle L \cup \{0\}, +, \cdot, ^{*}, ^{\perp}, 0, 1\rangle$ with $0 \notin L$ and $\leq$ as underlying partial order is a Stonian p-ortholattice.*

Recall that the full mereotopology RT_0 extends RT^{-} non-conservatively by the axioms $(A11)$, $(A12)$, and $(A13)$. $(A12)$ and $(A13)$ can be restated in algebraic form as $(A12')$ and $(A13')$, while we proved in [10] that $(A11)$ is a theorem of Stonian p-ortholattices satisfying $(A12)$; hence $(A11)$ can be omitted altogether.

Lemma 3 *Let $\langle L, +, \cdot, ^{*}, ^{\perp}, 0, 1\rangle$ be a Stonian p-ortholattice. Then*

1. *the following statements are equivalent:*
   (a) *L satisfies (A12)*
   (b) *L satisfies the property (A12′):*

$$(A12') \quad \exists x, y[x \neq 0 \wedge x^{**} \leq y^{++} \wedge \forall z[x \not\leq z^{++} \vee z^{**} \not\leq y]];$$

2. *(A12′) implies (A11);*
3. *the following statements are equivalent:*
   (a) *L satisfies (A13)*
   (b) *L satisfies the property (A13′):*

$$(A13') \quad \forall x \exists y[x \leq y^{++} \wedge \forall z[x \not\leq z^{++} \vee y^{++} \leq z]];$$

Proof The first part has been shown in [10, Lemma 7.1], while the second part immediately follows from the fact that $(A11)$ is a theorem of Stonian p-ortholattices satisfying $(A12)$, which we proved in [10, Corollary 7.3]. It remains to show that $(A13)$ and $(A13')$ are equivalent.

$(a) \Rightarrow (b):$ Suppose x, y satisfy $(A13)$. Then we want to show that x and y satisfy $(A13')$. We immediately have $x \leq y = y^{++}$. If we pick an arbitrary z, we also have $x \leq z = z^{++}$ if $P(x,z)$ and $OP(z)$. Then $y^{++} = y \leq z$, otherwise $x \not\leq z^{++}$.

$(b) \Rightarrow (a):$ Suppose x, y satisfy $(A13')$. Then we want to show that x and y^{++} satisfy $(A13)$. First notice that for any y, y^{++} must exist. From $x \leq y^{++}$ we get $P(x, y^{++})$. We also know $OP(y^{++})$. If for an arbitrary z, it holds $x \not\leq z^{++}$, then $P(x,z)$ would require that $x \leq z$. Then $x \neq x^{++}$ and thus $\neg OP(z)$ follow. If on the other hand $x \leq z^{++}$, then $y^{++} \leq z$ by $(A13')$ and we obtain $P(y^{++}, z)$. □

In [10] we also proved that a Stonian p-ortholattice satisfying $(A12')$ has at least one of C_{14}, C_{16}, C_{18} or C_{20} as a subalgebra. Consequently, all examples in this paper satisfy $(A12')$. By looking at the dual skeleton of only open elements $\{x \mid x = x^{++}, x \in L\}$ we can easily verify that all examples in this paper also satisfy $(A13')$ and thus are indeed models of the full mereotopology RT_0.

Many other small examples of Stonian p-ortholattices quite often do not provide elements in weak contact, i.e., Axiom $(A12')$ is not valid. Therefore, those lattices are not models of the full theory RT_0. However, the next lemma shows that such lattices can be extended into a lattice satisfying $(A12')$. Notice that the class of Stonian p-ortholattice is a variety, and, hence, closed under products.

Lemma 4 *Let L_1 and L_2 be Stonian p-ortholattices. If L_1 satisfies $(A12')$, then so does the Stonian p-ortholattice $L_1 \times L_2$.*

Proof Suppose $x, y \in L_1$ are the two elements required by $(A12')$. Then we have $(x,1) \neq (0,0)$ since $x \neq 0$ and $(x,1)^{**} = (x^{**},1) \leq (y^{++},1) = (y,1)^{++}$ because $x^{**} \leq y^{++}$. Now suppose $(x,1) \leq (z_1,z_2)^{++}$ and $(z_1,z_2)^{**} \leq (y,1)$. Then we have $x \leq z_1^{++}$ and $z_1^{**} \leq y$, a contraction. □

4 Topological and representable Stonian p-ortholattices

In this section, we show how the regular sets of a topological space form a Stonian p-ortholattice and define when a Stonian p-ortholattice is representable, i.e. has a representation by the regular sets of some topological space. Beforehand, we review some basic definitions and properties of topological spaces relevant for our work and prove properties of regular sets. For any notion not explained here, we invite the reader to consult [8].

4.1 Topological spaces

In the following we will denote union and intersection of sets by $\cup$ and $\cap$, respectively. If $a \subseteq X$, then we write $X \setminus a$ for the complement of a with respect to X. If X is understood, we will simply write $\overline{a}$ instead.

We will denote topological spaces by $\langle X, \tau\rangle$, where τ is the topology on X, i.e., τ is a collection of sets containing $\emptyset$ and X, and being closed under arbitrary unions and finite intersections. The elements of τ are called *open*, and a subset $a \subseteq X$ is called *closed* if $\overline{a} \in \tau$, i.e., if its complement is open. We let $\mathrm{cl}_\tau(a) = \bigcap\{b \mid b \text{ is closed}, a \subseteq b \subseteq X\}$ be the τ-closure of a, and $\mathrm{int}_\tau(a) = \bigcup\{b \mid b \text{ is open}, b \subseteq a\}$ its τ-interior. If τ is understood, we will just speak of X as a topological space, and drop the subscripts from the operators. The interior and the closure operator are monotone, and they satisfy the following properties which we will use throughout the paper without mentioning:

$$
\begin{aligned}
\mathrm{int}(\mathrm{int}(a)) &= \mathrm{int}(a), & \mathrm{cl}(\mathrm{cl}(a)) &= \mathrm{cl}(a),\\
\mathrm{int}(a \cap b) &= \mathrm{int}(a) \cap \mathrm{int}(b), & \mathrm{cl}(a \cup b) &= \mathrm{cl}(a) \cup \mathrm{cl}(b),\\
\mathrm{int}(a) \cup \mathrm{int}(b) &\subseteq \mathrm{int}(a \cup b), & \mathrm{cl}(a \cap b) &\subseteq \mathrm{cl}(a) \cap \mathrm{cl}(b),\\
\mathrm{int}(\overline{a}) &= \overline{\mathrm{cl}(a)}, & \mathrm{cl}(\overline{a}) &= \overline{\mathrm{int}(a)}.
\end{aligned}
$$

4.2 Regular sets

Given a topological space X, a subset $a \subseteq X$ is called *regular* iff $\mathrm{int}(a) = \mathrm{int}(\mathrm{cl}(a))$ and $\mathrm{cl}(a) = \mathrm{cl}(\mathrm{int}(a))$. Intuitively, regular sets are those sets that have full interior and no isolated points. We will denote the set of regular sets of a topological space X by $\mathrm{RT}(X)$, i.e., $\mathrm{RT}(X) = \{a \subseteq X \mid a \text{ is regular}\}$.

Lemma 5 *Suppose $\langle X, \tau\rangle$ is a topological space. Then we have*

1. *a is regular iff $\overline{a}$ is regular,*
2. $\mathrm{cl}(a) \cap b \subseteq \mathrm{cl}(a \cap b)$ *if b open,*
3. $\mathrm{cl}(\mathrm{int}(a) \cap b) = \mathrm{cl}(\mathrm{int}(a) \cap \mathrm{int}(b))$ *if b regular,*
4. $\mathrm{cl}(a \cup \mathrm{int}(\mathrm{cl}(a))) = \mathrm{cl}(a)$.
5. $\mathrm{int}(a \cup b) \subseteq \mathrm{int}(a) \cup b$ *if b is closed,*
6. $\mathrm{int}(\mathrm{cl}(a) \cup b) = \mathrm{int}(\mathrm{cl}(a) \cup \mathrm{cl}(b))$ *if b is regular,*
7. $\mathrm{int}(a \cap \mathrm{cl}(\mathrm{int}(a))) = \mathrm{int}(a)$.

Proof 1. Suppose a is regular, i.e., $\mathrm{int}(\mathrm{cl}(a)) = \mathrm{int}(a)$ and $\mathrm{cl}(\mathrm{int}(a)) = \mathrm{cl}(a)$. Then we have $\mathrm{int}(\mathrm{cl}(\overline{a})) = \mathrm{int}(\overline{\mathrm{int}(a)}) = \overline{\mathrm{cl}(\mathrm{int}(a))} = \overline{\mathrm{cl}(a)} = \mathrm{int}(\overline{a})$. The second equation as well as the converse implication follow analogously.
2. This follows immediately from

$$
\begin{aligned}
\mathrm{cl}(a) \cap b &\subseteq \mathrm{cl}(a \cup \overline{b}) \cap b\\
&= \mathrm{cl}((a \cap b) \cup \overline{b}) \cap b\\
&= (\mathrm{cl}(a \cap b) \cup \mathrm{cl}(\overline{b})) \cap b\\
&= (\mathrm{cl}(a \cap b) \cup \overline{b}) \cap b && b \text{ open}\\
&= \mathrm{cl}(a \cap b) \cap b\\
&\subseteq \mathrm{cl}(a \cap b).
\end{aligned}
$$

3. The inclusion '$\supseteq$' is trivial, and for the converse inclusion consider the following computation

$$\begin{aligned}
\mathrm{int}(a) \cap b &\subseteq \mathrm{int}(a) \cap \mathrm{cl}(b) \\
&= \mathrm{int}(a) \cap \mathrm{cl}(\mathrm{int}(b)) && b \text{ regular} \\
&\subseteq \mathrm{cl}(\mathrm{int}(a) \cap \mathrm{int}(b)) && \text{by 1.}
\end{aligned}$$

This implies $\mathrm{cl}(\mathrm{int}(a) \cap b) \subseteq \mathrm{cl}(\mathrm{cl}(\mathrm{int}(a) \cap \mathrm{int}(b))) = \mathrm{cl}(\mathrm{int}(a) \cap \mathrm{int}(b))$.
4. We have

$$\begin{aligned}
\mathrm{cl}(a \cup \mathrm{int}(\mathrm{cl}(a))) &= \mathrm{cl}(a) \cup \mathrm{cl}(\mathrm{int}(\mathrm{cl}(a))) \\
&= \mathrm{cl}(a)
\end{aligned}$$

since $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(a))) \subseteq \mathrm{cl}(cl(a)) = \mathrm{cl}(a)$.
5.-7. follow from 1.-4. and the fact that $\mathrm{int}(a) = \overline{\mathrm{cl}(\overline{a})}$. □

4.3 Topological Stonian p-ortholattices

Topological models of the theory of Stonian p-ortholattices are given by a suitable structure on the set $\mathrm{RT}(X)$ of regular sets. The notations $a \cap^* b$ and $a \cup^* b$ are maintained from [1] as follows.

$$\begin{aligned}
a \cap^* b &= a \cap b \cap \mathrm{cl}(\mathrm{int}(a \cap b)), \\
a \cup^* b &= a \cup b \cup \mathrm{int}(\mathrm{cl}(a \cup b)), \\
a^* &= \mathrm{cl}(X \setminus a), \\
a^\perp &= X \setminus a.
\end{aligned}$$

The next theorem was shown in [15, Theorem 4.3] and verifies that the set $\mathrm{RT}(X)$, the regular sets of an arbitrary topological space X, together with the operations $\cup^*, \cap^*, ^*$, and $^\perp$ always forms a Stonian p-ortholattice. It moreover verifies that the class of all structures $\mathrm{RT}(X)$ can indeed be seen as the class of standard topological models of the mereotopology RT^-.

Theorem 3 *Let be $\langle X, \tau\rangle$ a topological space. Then $\langle \mathrm{RT}(X), \cup^*, \cap^*, ^*, ^\perp, \emptyset, X\rangle$ is a Stonian p-ortholattice. We call such a structure a topological Stonian p-ortholattice.*

Notice that the order on $\mathrm{RT}(X)$ is regular set inclusion. As a consequence we have $a \subseteq b$ iff $a \cap^* b = a$ for all $a, b \in \mathrm{RT}(X)$. Furthermore, using the definition $a^+ = a^{\perp * \perp}$ we obtain $a^+ = \mathrm{int}(\overline{a})$.

Example 1 *Consider some topological space (X, τ) with $X = 1$ and the topology*

$$\tau = \{a^+, b^+ +, c^+, b^{++} \cup c^+, a^+ \cup c^+, b^{++} \cup c^+, 0, 1\}.$$

Suppose every set in τ is regular, i.e. is a regular open set. It is well known that this forms a Boolean algebra, depicted in Figure 3 with $c^+ = a^{++} \cdot b^+$, $a^{++} = b^{++} \cup c^+$,

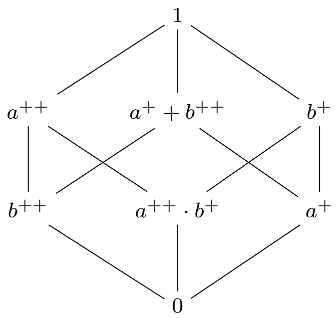


Fig. 3: The 8-element Boolean lattice of regular open sets that generates C_{18}.

and $b^+ = a^+ \cup c^+$. *If all elements in* τ *are also closed, these eight sets are in fact the only regular sets, hence the structure* $\langle \mathrm{RT}(X), \cup^*, \cap^*, ^*, ^\perp, 0, 1\rangle$ *is the Boolean algebra as shown in Figure 3 and thereby also a Stonian p-ortholattice. If none of the elements in* τ *except for* $0, 1$ *are closed, we must add their closures, resulting in additional six regular elements. Closure under sums and intersections amongst the regular sets result in an extra four elements, thereby leading to the 18-element lattice* $C_{18} = \langle \mathrm{RT}(X), \cup^*, \cap^*, ^*, ^\perp, 0, 1\rangle$*, which again is a Stonian p-ortholattice. Figure 7 instantiates* C_{18} *with concrete point sets: all sets with* $^{++}$ *denote open sets, together with* 0 *and* 1 *we have the underlying topological space* (X, τ) *with*

$$\tau = \{a^{++}, b^{++}, c^{++}, d^{++}, e^{++}, f^{++}, 0, 1\}.$$

C_{14}, on the other side, is not a topological Stonian p-ortholattice, that is, it cannot be generated from the regular sets of some topological space. We will show why towards the end of the paper. Since C_{14} is a model of RT_0, it will also mean that not every model of RT_0 can be generated from the regular sets of some topological space. To formalize this notion, we next look at the reverse of Theorem 3.

4.4 Representable Stonian p-ortholattices

Now we study when a Stonian p-ortholattice can be represented by $\mathrm{RT}(X)$ of some topological space X. For that we define a Stonian p-ortholattice to be representable (by some topological space X) if orthocomplementation in the lattice corresponds to set complementation in X and the space is minimal, i.e. the set of (regular) open sets forms a basis of X. This is captured as follows.

Definition 2 A Stonian p-ortholattice $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ is called *representable* iff there is a topological space $\langle X, \tau\rangle$ and an injective lattice homomorphism $h : L \to RT(X)$ satisfying:

(a) $h(x^\perp) = X \setminus h(x)$ for all $x \in L$.
(b) The set $\mathcal{B} = \{h(x) \mid x \in L, x^{++} = x\}$ is a basis of $\langle X, \tau\rangle$.

The next lemma verifies that a representable Stonian p-ortholattice is isomorphic to a subalgebra of $\mathrm{RT}(X)$.

Lemma 6 *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a Stonian p-ortholattice and $h : L \to RT(X)$ be a representation. Then we have:*

1. *$h(x) \subseteq h(y)$ iff $x \leq y$.*
2. *$h(x^{++}) = \mathrm{int}(h(x))$ and $h(x^+) = h(x)^+$.*
3. *$h(x^{**}) = \mathrm{cl}(h(x))$ and $h(x^*) = h(x)^*$.*

Proof 1. Consider the following equivalences:

$$\begin{aligned} h(x) \subseteq h(y) &\Leftrightarrow h(x) \cap^* h(y) = h(x) && \text{order on } \mathrm{RT}(X) \\ &\Leftrightarrow h(x \cdot y) = h(x) && h \text{ lattice homomorphisms} \\ &\Leftrightarrow x \cdot y = x && h \text{ injective homomorphism} \\ &\Leftrightarrow x \leq y. \end{aligned}$$

2. Since the set $\mathcal{B} = \{h(y) \mid y \in L, y^{++} = y\}$ is a basis of $\langle X, \tau\rangle$ the interior of $h(x)$ is the union of all those elements from $\mathcal{B}$ that are smaller or equal to $h(x)$. Obviously, the element $h(x^{++})$ is among those elements. Assume $h(y)$ is also such an element, i.e., $y^{++} = y$ and $h(y) \subseteq h(x)$. From 1. we conclude $y \leq x$. This implies $y = y^{++} \leq x^{++}$. We obtain $h(y) \subseteq h(x^{++})$ so that $h(x^{++}) = \mathrm{int}(h(x))$ follows immediately. We obtain the second assertion from the first by computing

$$\begin{aligned} h(x^+) &= h(x^{+++}) && \text{Lemma 1(3)} \\ &= h(x^{\perp++}) && \text{Lemma 1(7)} \\ &= \mathrm{int}(h(x^\perp)) \\ &= \mathrm{int}(\overline{h(x)}) && \text{Def. 2(a)} \\ &= h(x)^{\perp++} \\ &= h(x)^+. \end{aligned}$$

3. This property is analogously to 2. using that $\mathcal{C} = \{h(y) \mid y \in L, y^{**} = y\}$ is a basis of closed elements. □

In particular, this verifies that the interior and closure operations *int* and *cl* map to $^{++}$ and **, respectively, as expected. Moreover, we have verified that a representable Stonian p-ortholattice is isomorphic to a subalgebra of $\mathrm{RT}(X)$ for some topological space X. Therefore, any quasiidentity, i.e., any universally quantified implication of the form $s_1 = t_1 \wedge \ldots \wedge s_n = t_n \to s = t$ where $s_1, \ldots, s_n, s$ and $t_1, \ldots, t_n, t$ are terms in the language of Stonian p-ortholattices, that holds in any topological Stonian p-ortholattice also holds in any representable Stonian p-ortholattice. We will use this result to identify properties of topological Stonian p-ortholattices that must also hold for representable Stonian p-ortholattices. In other words, once we identify properties that hold for all Stonian p-ortholattices that arise from the regular sets of an arbitrary topological space, these properties must equally hold for representable Stonian p-ortholattices.

5 The operations $\mathrm{int}_y(x)$ and $\mathrm{cl}_y(x)$

Some of the properties of topological Stonian p-ortholattices that we are about to discuss require a new kind of interior and closure operation. In this section, we introduce these operations and prove some useful properties about them.

In Stonian p-ortholattices we define the operation $\mathrm{int}_y(x) = (x \cdot y + y^{\perp}) \cdot x$. We call $\mathrm{int}_y(x)$ the interior of x with respect to y. In the next lemma we will show some basic properties of this operation. Only later, we will justify the notation we have chosen.

Lemma 7 *Let $\langle L, +, \cdot, ^*, ^{\perp}, 0, 1\rangle$ be a Stonian p-ortholattice. Then the following properties hold for all $x, y \in L$:*

1. $\mathrm{int}_y(x) \leq x$.
2. int_y *is monotone, i.e.,* $x \leq x'$ *implies* $\mathrm{int}_y(x) \leq \mathrm{int}_y(x')$.
3. $\mathrm{int}_y(x) \cdot y = x \cdot y$.
4. $\mathrm{int}_y(\mathrm{int}_y(x)) = \mathrm{int}_y(x)$.
5. $\mathrm{int}_y(x) \geq x^{++}$ *and* $\mathrm{int}_{x^*}(x) = x^{++}$.
6. $(\mathrm{int}_y(x))^{++} = x^{++}$.
7. *If* $y^{++} \leq x^{++}$, *then* $\mathrm{int}_y(x) = x$.
8. *If* $x \leq y^{\perp}$, *then* $\mathrm{int}_y(x) = x$.
9. $\mathrm{int}_y(\mathrm{int}_y(x) \cdot \mathrm{int}_z(x)) = \mathrm{int}_y(\mathrm{int}_z(x))$.

Proof 1. is obvious.
2. follows immediately from the monotonicity of $\cdot$ and $+$.
3. We immediately conclude

$$\begin{aligned} \mathrm{int}_y(x) \cdot y &= (x \cdot y + y^{\perp}) \cdot x \cdot y \\ &= x \cdot y. && \text{absorption} \end{aligned}$$

4. This follows from

$$\begin{aligned} \mathrm{int}_y(\mathrm{int}_y(x)) &= (\mathrm{int}_y(x) \cdot y + y^{\perp}) \cdot \mathrm{int}_y(x) \\ &= (x \cdot y + y^{\perp}) \cdot \mathrm{int}_y(x) && \text{by 3.} \\ &= (x \cdot y + y^{\perp}) \cdot x \cdot \mathrm{int}_y(x) && \text{by 1.} \\ &= \mathrm{int}_y(x) \cdot \mathrm{int}_y(x) \\ &= \mathrm{int}_y(x). \end{aligned}$$

5. First, we have

$$\begin{aligned} x \cdot y + y^{\perp} + x^{+} &\geq x \cdot y + y^{+} + x^{+} && \text{Lemma 1(2)} \\ &= x \cdot y + (x \cdot y)^{+} && \text{Lemma 1(5)} \\ &= 1, && \text{Lemma 1(1)} \end{aligned}$$

which implies $x^{++} \leq x \cdot y + y^{\perp}$. Since $x^{++} \leq x$ by Lemma 1(2) we obtain $x^{++} \leq (x \cdot y + y^{\perp}) \cdot x = \mathrm{int}_y(x)$. For the second assertion consider the following computation

$$\begin{aligned}
\mathrm{int}_{x^*}(x) &= (x \cdot x^* + x^{*\perp}) \cdot x && \\
&= x^{*\perp} \cdot x && \text{pseudocomplement} \\
&= x^{++} \cdot x && \text{Lemma 1(7)} \\
&= x^{++}. && \text{Lemma 1(2)}
\end{aligned}$$

6. follows immediately from 1., 4., the monotonicity of $(.)^{++}$, and Lemma 1(3).
7. We have $y^{++} \leq x^{++} \leq \mathrm{int}_y(x) \leq x \cdot y + y^{\perp}$ by 5. This implies $x \cdot y + y^{\perp} + y^{+} = 1$, and, hence, $x \cdot y + y^{\perp} = 1$ since $y^{+} \leq y^{\perp}$. We conclude $\mathrm{int}_y(x) = (x \cdot y + y^{\perp}) \cdot x = x$.
8. From $x \leq y^{\perp}$ we get $x \cdot y = 0$ so that $\mathrm{int}_y(x) = (x \cdot y + y^{\perp}) \cdot x = y^{\perp} \cdot x = x$ follows.
9. We compute

$$\begin{aligned}
&\mathrm{int}_y(\mathrm{int}_y(x) \cdot \mathrm{int}_z(x)) && \\
&= (\mathrm{int}_y(x) \cdot \mathrm{int}_z(x) \cdot y + y^{\perp}) \cdot \mathrm{int}_y(x) \cdot \mathrm{int}_z(x) && \\
&= (x \cdot y \cdot \mathrm{int}_z(x) + y^{\perp}) \cdot \mathrm{int}_y(x) \cdot \mathrm{int}_z(x) && \text{by 3.} \\
&= (y \cdot \mathrm{int}_z(x) + y^{\perp}) \cdot \mathrm{int}_y(x) \cdot \mathrm{int}_z(x) && \text{by 1.} \\
&= \mathrm{int}_y(\mathrm{int}_z(x)) \cdot \mathrm{int}_y(x) && \\
&= \mathrm{int}_y(\mathrm{int}_z(x)). && \text{by 1. and 2.}
\end{aligned}$$

This completes the proof. □

Notice that 1., 2. and 4. of the previous lemma shows that int_y is indeed an interior operation. Properties 5. and 6. will – in just a moment – justify why we we call $\mathrm{int}_y(x)$ the interior of x with respect to y.

Now, we define dually the closure of x with respect to y as $\mathrm{cl}_y(x) = (x+y) \cdot y^{\perp} + x = \mathrm{int}_{y^{\perp}}(x^{\perp})^{\perp}$. From the previous lemma we immediately obtain:

Corollary 1 *Let $\langle L, +, \cdot, ^*, ^{\perp}, 0, 1\rangle$ be a Stonian p-ortholattice. Then the following properties hold for all $x, y \in L$:*

1. $x \leq \mathrm{cl}_y(x)$.
2. cl_y *is monotone, i.e.,* $x \leq x'$ *implies* $\mathrm{cl}_y(x) \leq \mathrm{cl}_y(x')$.
3. $\mathrm{cl}_y(x) + y = x + y$.
4. $\mathrm{cl}_y(\mathrm{cl}_y(x)) = \mathrm{cl}_y(x)$.
5. $\mathrm{cl}_y(x) \leq x^{**}$ *and* $\mathrm{cl}_{x^+}(x) = x^{**}$.
6. $\mathrm{cl}_y(x)^{**} = x^{**}$.
7. *If* $x^{**} \leq y^{**}$*, then* $\mathrm{cl}_y(x) = x$.
8. *If* $x^{\perp} \leq y$*, then* $\mathrm{cl}_y(x) = x$.
9. $\mathrm{cl}_y(\mathrm{cl}_y(x) + \mathrm{cl}_z(x)) = \mathrm{cl}_y(\mathrm{cl}_z(x))$.

Example 2 *Consider the interior of b^* with respect to a^{++}, i.e. $int_{a^{++}}(b^*)$, in C_{14} and C_{18}. In C_{14} it is not the interior of b^*, i.e.*

$$int_{a^{++}}(b^*) = (b^* \cdot a^{++} + a^{++\perp}) \cdot b^* = (a^{++} \cdot b^+ + a^*) \cdot b^* = b^+ \cdot b^* = b^+;$$

while in C_{18} the interior of b^* with respect to a^{++} is the interior b^+ of b^*

$$int_{a^{++}}(b^*) = (b^* \cdot a^{++} + a^{++\perp}) \cdot b^* = (a^{++} \cdot b^* + a^*) \cdot b^* = b^* \cdot b^* = b^*.$$

As another example consider C_{18} in Fig. 7. For $int_a(f)$ we obtain

$$int_a(f) = (a \cdot f + a^\perp) \cdot f = (0 + f^{++}) \cdot f = f^{++}$$

while for $int_{a^{++}}(f)$ we obtain

$$int_{a^{++}}(f) = (a^{++} \cdot f + a^{++\perp}) \cdot f = (0 + f) \cdot f = f.$$

Now, we want to investigate the operation $\mathrm{int}_b(a)$ for regular sets a and b. In the Euclidean plane $\mathrm{int}_b(a)$ removes elements from a that are isolated points in $a \cap b$. Since $\mathrm{int}(a) \subseteq \mathrm{int}_b(a)$ by Lemma 7(5) and Lemma 6(2) those elements are always elements of the border of a. This together with the fact that $\mathrm{int}_{a^*}(a) = \mathrm{int}(a)$ by Lemma 7(5) was our motivation to call $\mathrm{int}_b(a)$ the interior of a with respect to b. Generally, we obtain the following characterization.

Lemma 8 *Let a, b, c be regular. Then we have $c \subseteq \mathrm{int}_b(a)$ iff $c \subseteq a$ and $c \cap b \subseteq \mathrm{cl}(\mathrm{int}(a \cap b))$.*

Proof '⇒': It is sufficient to show that $\mathrm{int}_b(a) \cap b \subseteq \mathrm{cl}(\mathrm{int}(a \cap b))$. We have

$$\begin{aligned}
&\mathrm{cl}(\mathrm{int}(\mathrm{cl}((a \cap^* b) \cup \overline{b})) \cap b) \\
&= \mathrm{cl}(\mathrm{int}(\mathrm{cl}((a \cap^* b) \cup \overline{b})) \cap \mathrm{int}(b)) && \text{Lemma 5(3)} \\
&= \mathrm{cl}(\mathrm{int}(\mathrm{cl}((a \cap^* b) \cup \overline{b})) \cap \mathrm{int}(\mathrm{int}(b))) \\
&= \mathrm{cl}(\mathrm{int}(\mathrm{cl}((a \cap^* b) \cup \overline{b}) \cap \mathrm{int}(b))) \\
&= \mathrm{cl}(\mathrm{int}((\mathrm{cl}(a \cap^* b) \cup \mathrm{cl}(\overline{b})) \cap \mathrm{int}(b))) \\
&= \mathrm{cl}(\mathrm{int}((\mathrm{cl}(a \cap^* b) \cup \overline{\mathrm{int}(b)}) \cap \mathrm{int}(b))) \\
&= \mathrm{cl}(\mathrm{int}(\mathrm{cl}(a \cap^* b) \cap \mathrm{int}(b))) \\
&= \mathrm{cl}(\mathrm{int}(\mathrm{cl}(a \cap^* b)) \cap \mathrm{int}(\mathrm{int}(b))) \\
&= \mathrm{cl}(\mathrm{int}(\mathrm{cl}(a \cap^* b)) \cap \mathrm{int}(b)) \\
&= \mathrm{cl}(\mathrm{int}(a \cap^* b) \cap \mathrm{int}(b)) && a \cap^* b \text{ regular} \\
&= \mathrm{cl}(\mathrm{int}(a \cap b \cap \mathrm{cl}(\mathrm{int}(a \cap b))) \cap \mathrm{int}(b)) \\
&= \mathrm{cl}(\mathrm{int}(a \cap b) \cap \mathrm{int}(b)) && \text{Lemma 5(7)} \\
&= \mathrm{cl}(\mathrm{int}(a \cap b)).
\end{aligned}$$

This implies

$$
\begin{aligned}
&\mathrm{int}_b(a) \cap b \\
&= (((a \cap^* b) \cup^* \overline{b}) \cap^* a) \cap b \\
&\subseteq ((a \cap^* b) \cup^* \overline{b}) \cap a \cap b \\
&= ((a \cap^* b) \cup \overline{b} \cup \mathrm{int}(\mathrm{cl}((a \cap^* b) \cup \overline{b}))) \cap a \cap b \\
&= ((a \cap^* b) \cap a \cap b) \cup (\mathrm{int}(\mathrm{cl}((a \cap^* b) \cup \overline{b})) \cap a \cap b) \\
&= (a \cap^* b) \cup (\mathrm{int}(\mathrm{cl}((a \cap^* b) \cup \overline{b})) \cap a \cap b) \\
&\subseteq (a \cap^* b) \cup (\mathrm{cl}(\mathrm{int}(\mathrm{cl}((a \cap^* b) \cup \overline{b})) \cap b) \cap a \cap b) \\
&= (a \cap^* b) \cup (\mathrm{cl}(\mathrm{int}(a \cap b)) \cap a \cap b) && \text{see above} \\
&= a \cap^* b.
\end{aligned}
$$

'$\Leftarrow$': Suppose $c \subseteq a$ and $c \cap b \subseteq a \cap^* b$. Then $c \subseteq (a \cap^* b) \cup \overline{b} \subseteq (a \cap^* b) \cup^* \overline{b}$, and, hence, $c \subseteq ((a \cap^* b) \cup^* \overline{b}) \cap^* a = \mathrm{int}_b(a)$ since c is regular. □

The previous lemma implies the following important characterization of when the meet of regular sets ($\cap^*$) coincides with set intersection.

Lemma 9 *Let a and b be regular. Then $\mathrm{int}_b(a) = a$ iff $a \cap b = a \cap^* b$.*

Proof '$\Rightarrow$': From $a \subseteq \mathrm{int}_b(a)$ we get $a \cap b \subseteq \mathrm{cl}(\mathrm{int}(a \cap b))$ from Lemma 8, and, hence, $a \cap b \subseteq a \cap b \cap \mathrm{cl}(\mathrm{int}(a \cap b)) = a \cap^* b$. The converse inclusion is trivial.
'$\Leftarrow$': Suppose $a \cap b = a \cap^* b$. Then we have $a \cap b \subseteq \mathrm{cl}(\mathrm{int}(a \cap b))$ so that Lemma 8 implies $a \subseteq \mathrm{int}_b(a)$. The converse inclusion is again trivial. □

Since $\mathrm{cl}_b(a)$ is defined dually we obtain the following two results as corollaries.

Corollary 2 *Let a, b, c be regular. Then we have:*

1. $\mathrm{cl}_b(a) \subseteq c$ *iff* $a \subseteq c$ *and* $\mathrm{int}(\mathrm{cl}(a \cup b)) \subseteq c \cup b$.
2. $\mathrm{cl}_b(a) = a$ *iff* $a \cup b = a \cup^* b$.

6 Necessary Properties of Representable Stonian p-ortholattices

In this section we want to investigate five properties of representable Stonian p-ortholattices. We do so by proving them for topological Stonian p-ortholattices, which by Lemma 6 implies that they are valid in representable Stonian p-ortholattices as well. Again, we are also interested in equational versions of each property.

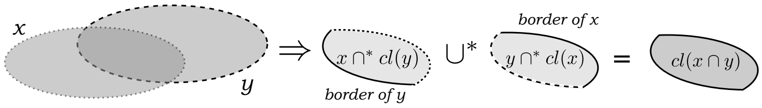


Fig. 4: The intuitions of the conditions (RP_1) and (RP_2)

6.1 (RP_1)

The first property, (RP_1), can be motivated as follows (compare also Fig. 4). Suppose a and b are regular open regions, i.e., regular and open sets, that together fill the whole space, i.e., $a \cup b = X$. The element $a \cap^* \mathrm{cl}(b)$ is the regular set $a \cap b$ together with that portion of its border that is also a border of b. Similarly, $\mathrm{cl}(a) \cap^* b$ is $a \cap b$ together with that portion of its border that is also a border of a. Because we have $a \cup b = X$ the region $a \cap b$ cannot have any point on its border that is neither in the border of a nor in the border of b. We obtain $(a \cap^* \mathrm{cl}(b)) \cup^* (\mathrm{cl}(a) \cap^* b) = \mathrm{cl}(a \cap b)$ or in other words that $a \cap^* \mathrm{cl}(b)$ is a complement of $\mathrm{cl}(a) \cap^* b$ in the interval $[a \cap b, \mathrm{cl}(a \cap b)]$. For general regular sets, i.e., not necessarily regular open sets, this generalizes to the following theorem.

Theorem 4 *Let $\langle X, \tau \rangle$ be a topological space. Then we have*

$$(a \cap^* \mathrm{cl}(b)) \cup^* (\mathrm{cl}(a) \cap^* b) = \mathrm{cl}(a) \cap^* \mathrm{cl}(b)$$

for all $a, b \in \mathrm{RT}(X)$ with $a \cup b = X$.

Proof First notice that $\subseteq$ of the equation in questions is always true. Now, to prove $\supseteq$, assume $x \in \mathrm{cl}(a) \cap^* \mathrm{cl}(b)$. From $a \cup b = X$ we obtain that $x \in a$ or $x \in b$. Suppose w.l.o.g. that $x \in a$, and, consequently, $x \in a \cap (\mathrm{cl}(a) \cap^* \mathrm{cl}(b))$. From

$$\begin{aligned} \mathrm{int}(\mathrm{cl}(a) \cap^* \mathrm{cl}(b)) &= \mathrm{int}(\mathrm{cl}(a) \cap \mathrm{cl}(b)) && \text{Lemma 5(7)} \\ &= \mathrm{int}(\mathrm{cl}(a)) \cap \mathrm{int}(\mathrm{cl}(b)) \\ &= \mathrm{int}(a) \cap \mathrm{int}(b) && a, b \text{ regular} \\ &\subseteq \mathrm{int}(a) \end{aligned}$$

we obtain $\mathrm{int}_{\mathrm{cl}(a) \cap^* \mathrm{cl}(b)}(a) = a$ using Lemma 7(7). We conclude

$$\begin{aligned} a \cap (\mathrm{cl}(a) \cap^* \mathrm{cl}(b)) &= a \cap^* \mathrm{cl}(a) \cap^* \mathrm{cl}(b) && \text{Lemma 9} \\ &= a \cap^* \mathrm{cl}(b) \\ &\subseteq (a \cap^* \mathrm{cl}(b)) \cup^* (\mathrm{cl}(a) \cap^* b), \end{aligned}$$

i.e., $x \in (a \cap^* \mathrm{cl}(b)) \cup^* (\mathrm{cl}(a) \cap^* b)$. □

The previous theorem shows that the quasiidentity

(RP_1) $x^\perp \leq y$ implies $x \cdot y^{**} + x^{**} \cdot y = x^{**} \cdot y^{**}$ for all $x, y \in L$

holds in all representable Stonian p-ortholattices. Since the theory of Stonian p-ortholattices is equational as demonstrated in Section 3.1, it is interesting to know whether an additional property/axiom such as (RP1) can also be formalized as an equation.

Lemma 10 *Let $\langle L, +, \cdot, {}^*, {}^\perp, 0, 1\rangle$ be a Stonian p-ortholattice. Then following statements are equivalent:*

(RP_1) $x^\perp \leq y$ *implies* $x \cdot y^{**} + x^{**} \cdot y = x^{**} \cdot y^{**}$ *for all* $x, y \in L$.
$(\mathrm{RP}_1^=)$ $x \cdot (x^* + y^{**}) + x^{**} \cdot (x^\perp + y) = x^{**} \cdot y^{**}$ *for all* $x, y \in L$.

Proof $(\mathrm{RP}_1) \Rightarrow (\mathrm{RP}_1^=):$ We have $x^\perp \leq x^\perp + y$ which implies

$$\begin{aligned}
& x \cdot (x^* + y^{**}) + x^{**} \cdot (x^\perp + y) & \\
&= x \cdot (x^\perp + y)^{**} + x^{**} \cdot (x^\perp + y) & \\
&= x^{**} \cdot (x^\perp + y)^{**} & (\mathrm{RP}_1) \\
&= x^{**} \cdot (x^* + y^{**}) & x^* = x^{\perp **} \\
&= x^{**} \cdot x^* + x^{**} \cdot y^{**} & S(L) \text{ distributive} \\
&= x^{**} \cdot y^{**} & x^{**} \cdot x^* = 0.
\end{aligned}$$

$(\mathrm{RP}_1^=) \Rightarrow (\mathrm{RP}_1):$ Suppose $x^\perp \leq y$. Then $y = x^\perp + y$ and we conclude

$$\begin{aligned}
x \cdot y^{**} + x^{**} \cdot y &= x \cdot (x^\perp + y)^{**} + x^{**} \cdot (x^\perp + y) & \\
&= x \cdot (x^* + y^{**}) + x^{**} \cdot (x^\perp + y) & \\
&= x^{**} \cdot y^{**}. & (\mathrm{RP}_1^=)
\end{aligned}$$

This completes the proof. □

Example 3 *Consider $x = a^{++}$ and $y = b^\perp$, first in C_{14} (cf. Fig. 2):*

$$\begin{aligned}
a^{++} \cdot (b^+)^{**} + (a^{++})^{**} \cdot b^+ &\neq (a^{++})^{**} \cdot (b^+)^{**} \\
a^{++} \cdot b^* + a^{**} \cdot b^+ &\neq a^{**} \cdot b^* \\
a^{++} \cdot b^+ + a^{++} \cdot b^+ &\neq a^{**} \cdot b^* \\
a^{++} \cdot b^+ &\neq a^{**} \cdot b^*
\end{aligned}$$

Hence C_{14} does not satisfy (RP_1). *Now consider the same elements in C_{18} (cf. Fig. 2):*

$$\begin{aligned}
a^{++} \cdot (b^+)^{**} + (a^{++})^{**} \cdot b^+ &= (a^{++})^{**} \cdot (b^+)^{**} \\
a^{++} \cdot b^* + a^{**} \cdot b^+ &= a^{**} \cdot b^* \\
a^{**} \cdot b^* &= a^{**} \cdot b^*
\end{aligned}$$

Since C_{18} is a topological Stonian p-ortholattice, (RP_1) *holds for all its elements. The differenec between C_{14} and C_{18} lies in the fact that the intersections $a^{++} \cdot b^*$ and $a^{**} \cdot b^+$ in C_{18} are strictly greater than $a^{++} \cdot b^+$. Each of them contains complementary parts of the boundary of $a^{++} \cdot b^+$ and their sum $a^{**} \cdot b^*$ contains the full boundary again. In C_{14}, the intersections $a^{++} \cdot b^*$ and $a^{**} \cdot b^+$ are identical and contain no part of the boundary, hence their sum is open again.*

6.2 (RP_2)

The next property that we want to consider is related to (RP_1) in the following sense. In the situation that motivated the first property we concluded that the element $a \cap^* \mathrm{cl}(b)$ is a complement of $\mathrm{cl}(a) \cap^* b$ in the interval $[a \cap b, \mathrm{cl}(a \cap b)]$. ($\mathrm{RP}_2$) requires that this complement must be unique.

Theorem 5 *Let $\langle X, \tau\rangle$ be a topological space. Then $a \cup b = X$ and $\mathrm{int}(c) \subseteq a$ and $a \cap^* c \subseteq b$ implies $c \subseteq b$ for all $a, b, c \in \mathrm{RT}(X)$.*

Proof Assume $x \in c$. Since $a \cup b = X$ we have $x \in a$ or $x \in b$. The case $x \in b$ is trivial so that we assume $x \in a$. From $\mathrm{int}(c) \subseteq a$ we obtain $\mathrm{int}(c) \subseteq \mathrm{int}(a)$, and, hence, $\mathrm{int}_c(a) = a$ from Lemma 7(7). This implies $a \cap c = a \cap^* c$ by Lemma 9 so that we conclude $x \in a \cap c = a \cap^* c \subseteq b$. □

We have just shown that the quasiidentity

(RP_2) $x^{\perp} \leq y$ and $z^{++} \leq x$ and $x \cdot z \leq y$ implies $z \leq y$ for all $x, y, z \in L$

holds in all representable Stonian p-ortholattices. As before, we are interested in an equational version of (RP_2). Notice that $a \leq b$ can always be rewritten in equational form, thus ($\mathrm{RP}_2^=$) below is indeed an equational variant of (RP_2).

Lemma 11 *Let $\langle L, +, \cdot, ^*, ^{\perp}, 0, 1\rangle$ be a Stonian p-ortholattice. Then following statements are equivalent:*

(RP_2) *$x^{\perp} \leq y$ and $z^{++} \leq x$ and $x \cdot z \leq y$ implies $z \leq y$ for all $x, y, z \in L$.*
($\mathrm{RP}_2^=$) *$z \leq x^{\perp} \cdot z^* + (x + z^{++}) \cdot z$ for all $x, z \in L$.*

Proof (RP_2) $\Rightarrow$ ($\mathrm{RP}_2^=$) : Set $x := x + z^{++}, y := x^{\perp} \cdot z^* + (x + z^{++}) \cdot z, z := z$ in (RP_2), then we have $(x+z^{++})^{\perp} = x^{\perp} \cdot z^{++\perp} = x^{\perp} \cdot z^* \leq x^{\perp} \cdot z^* + (x+z^{++}) \cdot z$ and $z^{++} \leq x + z^{++}$ and $(x + z^{++}) \cdot z \leq x^{\perp} \cdot z^* + (x + z^{++}) \cdot z$. From ($\mathrm{RP}_2$) we conclude $z \leq x^{\perp} \cdot z^* + (x + z^{++}) \cdot z$.
($\mathrm{RP}_2^=$) $\Rightarrow$ (RP_2) : Suppose $x^{\perp} \leq y$ and $z^{++} \leq x$ and $x \cdot z \leq y$. Then (a) $x = x + z^{++}$ and (b) $x^{\perp} + x \cdot z \leq y$, and, hence by substituting (a) in (b), $x^{\perp} \cdot z^* + (x + z^{++}) \cdot z = (x + z^{++})^{\perp} + (x + z^{++}) \cdot z = x^{\perp} + x \cdot z \leq y$. We conclude

$$\begin{aligned} z &\leq x^{\perp} \cdot z^* + (x + z^{++}) \cdot z && (\mathrm{RP}_2^=) \\ &\leq y. \end{aligned}$$

This completes the proof. □

Example 4 *Consider $x = b^+$, $y = a^{++}$, and $z = a^{**} \cdot b^*$; first in C_{14} (cf. Fig. 2). We have all three preconditions of* (RP_2) *satisfied:*

$$\begin{aligned} &(i) && b^{+\perp} = b^{**} \leq a^{++} \\ &(ii) && (a^{**} \cdot b^*)^{++} = a^{++} \cdot b^+ \leq b^+ \\ &(iii) && b^+ \cdot (a^{**} \cdot b^*) = a^{++} \cdot b^+ \leq a^{++} \end{aligned}$$

But $a^{**} \cdot b^{*} \nleq a^{++}$ *violates the consequent of* (RP$_2$).

In C_{18}*, the last precondition is not satisfied:*

$$(iii) \qquad b^{+} \cdot (a^{**} \cdot b^{*}) = a^{**} \cdot b^{+} \nleq a^{++}$$

Therefore (RP$_2$) *trivially holds. Again, since* C_{18} *is a topological Stonian p-ortholattice,* (RP$_2$) *must hold for all its elements.*

6.3 Meet of interiors $\mathrm{int}_b(a)$

The final three properties of this section deal with the interior operation $\mathrm{int}_b(a)$ for regular sets in a topological space. First we show that the meet of the interiors of a with respect to b and c are unaffected by additional applications of interiors with respect to b and c. Afterwards we prove a symmetry between the interior of a with respect to b and the interior of b with respect to a. Finally, we prove a localized version of distributivity. This results in the sentences (M), (S), and (D) which must hold for all elements in any representable Stonian p-ortholattice since all three conditions can be expressed as quasiidentities.

Theorem 6 *Let* $\langle X, \tau \rangle$ *be a topological space. Then we have*

$$\mathrm{int}_b(\mathrm{int}_b(a) \cap^{*} \mathrm{int}_c(a)) = \mathrm{int}_b(a) \cap^{*} \mathrm{int}_c(a)$$

for all $a, b, c \in \mathrm{RT}(X)$.

Proof By Lemma 7(9) it is sufficient to show

$$\mathrm{int}_b(a) \cap^{*} \mathrm{int}_c(a) = \mathrm{int}_b(\mathrm{int}_c(a)).$$

To this end we compute

$$\begin{aligned} \mathrm{cl}(\mathrm{int}(\mathrm{int}_c(a) \cap b)) &= \mathrm{cl}(\mathrm{int}(\mathrm{int}_c(a)) \cap \mathrm{int}(b)) && \text{Lemma 5(3)} \\ &= \mathrm{cl}(\mathrm{int}(a) \cap \mathrm{int}(b)) && \text{Lemma 7(6)} \\ &= \mathrm{cl}(\mathrm{int}(a \cap b)). \end{aligned}$$

Now, suppose that d is regular. Then we have

$$\begin{aligned} & d \subseteq \mathrm{int}_b(a) \cap^{*} \mathrm{int}_c(a) \\ \iff\ & d \subseteq \mathrm{int}_b(a) \text{ and } d \subseteq \mathrm{int}_c(a) \\ \iff\ & d \subseteq a \text{ and } d \cap b \subseteq \mathrm{cl}(\mathrm{int}(a \cap b)) \text{ and } d \subseteq \mathrm{int}_c(a) && \text{Lemma 8} \\ \iff\ & d \cap b \subseteq \mathrm{cl}(\mathrm{int}(a \cap b)) \text{ and } d \subseteq \mathrm{int}_c(a) && \text{since } \mathrm{int}_c(a) \subseteq a \\ \iff\ & d \cap b \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{int}_c(a) \cap b)) \text{ and } d \subseteq \mathrm{int}_c(a) && \text{see above} \\ \iff\ & d \subseteq \mathrm{int}_b(\mathrm{int}_c(a)), && \text{Lemma 8} \end{aligned}$$

i.e., $\mathrm{int}_b(a) \cap^{*} \mathrm{int}_c(a) = \mathrm{int}_b(\mathrm{int}_c(a))$. □

We have just shown that the equation

(M) $\mathrm{int}_y(\mathrm{int}_y(x) \cdot \mathrm{int}_z(x)) = \mathrm{int}_y(x) \cdot \mathrm{int}_z(x)$

holds in all representable Stonian p-ortholattices.

Example 5 *Consider $x = (a^{++} \cdot b^{+})^{\perp}$ in the inner structures of C_{14}, C_{16}, and C_{18}. Choose $y = a$ and $z = b^*$. Then in all of C_{14}, C_{16} we have*

$$\mathrm{int}_a(\mathrm{int}_a(x) \cdot \mathrm{int}_{b^*}(x)) = \mathrm{int}_a(x) \cdot \mathrm{int}_{b^*}(x)$$
$$\mathrm{int}_a([(x \cdot a + a^{\perp}) \cdot x)] \cdot [(x \cdot b^* + b^{*\perp}) \cdot x]) = [(x \cdot a + a^{\perp}) \cdot x] \cdot [(x \cdot b^* + b^{*\perp}) \cdot x]$$
$$\mathrm{int}_a([(x \cdot a + a^{\perp}) \cdot x)] \cdot [(x \cdot b^* + b^{++}) \cdot x]) = [(x \cdot a + a^{\perp}) \cdot x] \cdot [(x \cdot b^* + b^{++}) \cdot x]$$
$$\mathrm{int}_a([(b + a^{\perp}) \cdot x] \cdot [(a^* + b^{++}) \cdot x]) = [(b + a^{\perp}) \cdot x] \cdot [(a^* + b^{++}) \cdot x]$$

We continue at the last line , first with C_{14}. In C_{14} we have:

$$\mathrm{int}_a([(b + a^{\perp}) \cdot x] \cdot [(a^* + b^{++}) \cdot x]) = [(b + a^{\perp}) \cdot x] \cdot [(a^* + b^{++}) \cdot x]$$
$$\mathrm{int}_a([x \cdot x] \cdot [x \cdot x]) = [x \cdot x] \cdot [x \cdot x]$$
$$\mathrm{int}_a(x) = x$$
$$(x \cdot a + a^{\perp}) \cdot x = x$$
$$(b + a^{\perp}) \cdot x = x$$
$$x = x$$

Hence in C_{14} we have $\mathrm{int}_a(\mathrm{int}_a(x) \cdot \mathrm{int}_{b^*}(x)) = \mathrm{int}_a(x) \cdot \mathrm{int}_{b^*}(x)$. *In C_{16}, this fails:*

$$\mathrm{int}_a([(b + a^{\perp}) \cdot x] \cdot [(a^* + b^{++}) \cdot x]) \neq [(b + a^{\perp}) \cdot x] \cdot [(a^* + b^{++}) \cdot x]$$
$$\mathrm{int}_a([(a^{++} \cdot b^*)^{\perp} \cdot x] \cdot [(a^{++} \cdot b^*)^{\perp} \cdot x]) \neq [(a^{++} \cdot b^*)^{\perp} \cdot x] \cdot [(a^{++} \cdot b^*)^{\perp} \cdot x]$$
$$\mathrm{int}_a((a^{++} \cdot b^*)^{\perp} \cdot x) \neq (a^{++} \cdot b^*)^{\perp} \cdot x$$
$$\mathrm{int}_a((a^{++} \cdot b^*)^{\perp}) \neq (a^{++} \cdot b^*)^{\perp}$$
$$((a^{++} \cdot b^*)^{\perp} \cdot a + a^{\perp}) \cdot (a^{++} \cdot b^*)^{\perp} \neq (a^{++} \cdot b^*)^{\perp}$$
$$(b^{++} + a^{\perp}) \cdot (a^{++} \cdot b^*)^{\perp} \neq (a^{++} \cdot b^*)^{\perp}$$
$$(a^{**} \cdot b^*)^{\perp} \cdot (a^{++} \cdot b^*)^{\perp} \neq (a^{++} \cdot b^*)^{\perp}$$
$$(a^{**} \cdot b^*)^{\perp} \neq (a^{++} \cdot b^*)^{\perp}$$

In C_{18} the equality $\mathrm{int}_a(\mathrm{int}_a(x) \cdot \mathrm{int}_{b^*}(x)) = \mathrm{int}_a(x) \cdot \mathrm{int}_{b^*}(x)$ *holds again:*

$$\mathrm{int}_a([(b + a^{\perp}) \cdot x] \cdot [(a^* + b^{++}) \cdot x]) = [(b + a^{\perp}) \cdot x] \cdot [(a^* + b^{++}) \cdot x]$$
$$\mathrm{int}_a([(a^{**} \cdot b^{+})^{\perp} \cdot x] \cdot [(a^{++} \cdot b^*)^{\perp} \cdot x]) = [(a^{**} \cdot b^{+})^{\perp} \cdot x] \cdot [(a^{++} \cdot b^*)^{\perp} \cdot x]$$
$$\mathrm{int}_a((a^{**} \cdot b^{+})^{\perp} \cdot (a^{++} \cdot b^*)^{\perp}) = (a^{**} \cdot b^{+})^{\perp} \cdot (a^{++} \cdot b^*)^{\perp}$$
$$\mathrm{int}_a((a^{**} \cdot b^*)^{\perp}) = (a^{**} \cdot b^*)^{\perp}$$
$$((a^{**} \cdot b^*)^{\perp} \cdot a + a^{\perp}) \cdot (a^{**} \cdot b^*)^{\perp} = (a^{**} \cdot b^*)^{\perp}$$
$$((b^{++} + a^{\perp}) \cdot (a^{**} \cdot b^*)^{\perp} = (a^{**} \cdot b^*)^{\perp}$$
$$(a^{**} \cdot b^*)^{\perp} \cdot (a^{**} \cdot b^*)^{\perp} = (a^{**} \cdot b^*)^{\perp}$$
$$(a^{**} \cdot b^*)^{\perp} = (a^{**} \cdot b^*)^{\perp}$$

Again, (M) *is not only true for these particular elements, but generally valid for all elements in the topological Stonian p-ortholattice C_{18}.*

6.4 Symmetry of $\mathrm{int}_b(a) = a$

We now show that $\mathrm{int}_b(a) = a$ and $\mathrm{int}_a(b) = b$ can be used interchangeably in topological Stonian p-ortholattices and thus also in representable ones.

Theorem 7 *Let $\langle X, \tau\rangle$ be a topological space. Then $\mathrm{int}_b(a) = a$ iff $\mathrm{int}_a(b) = b$ for all $a, b \in \mathrm{RT}(X)$.*

Proof Consider the following computation

$$\begin{aligned} a \subseteq \mathrm{int}_b(a) &\iff a \cap b \subseteq \mathrm{cl}(\mathrm{int}(a \cap b)) && \text{Lemma 8} \\ &\iff b \cap a \subseteq \mathrm{cl}(\mathrm{int}(b \cap a)) \\ &\iff b \subseteq \mathrm{int}_a(b). && \text{Lemma 8} \end{aligned}$$

This implies the assertion since $\mathrm{int}_b(a) \subseteq a$ and $\mathrm{int}_a(b) \subseteq b$ are trivial. □

Notice that the property of the previous theorem is a conjunction of two quasi-identities so that we have just shown that

(S) $\mathrm{int}_y(x) = x$ iff $\mathrm{int}_x(y) = y$ for all $x, y \in L$

holds in all representable Stonian p-ortholattices. As before, we are interested in an equational version of (S).

Lemma 12 *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a Stonian p-ortholattice. Then following statements are equivalent:*

(S) $\mathrm{int}_y(x) = x$ *iff* $\mathrm{int}_x(y) = y$ *for all* $x, y \in L$.
($\mathrm{S}^=$) $\mathrm{int}_{\mathrm{int}_x(y)}(x) = x$ *for all* $x, y \in L$.

Proof (S) $\Rightarrow$ ($\mathrm{S}^=$) : From Lemma 7(4) we obtain $\mathrm{int}_x(\mathrm{int}_x(y)) = \mathrm{int}_x(y)$ so that the implication '$\Rightarrow$' (S) for $\mathrm{int}_x(y)$ and x implies $\mathrm{int}_{\mathrm{int}_x(y)}(x) = x$.
($\mathrm{S}^=$) $\Rightarrow$ (S) : Assume $\mathrm{int}_x(y) = y$. Then we have $\mathrm{int}_y(x) = \mathrm{int}_{\mathrm{int}_x(y)}(x) = x$ by ($\mathrm{S}^=$). The converse implication follows analogously. □

Example 6 *Consider again $x = (a^{++} \cdot b^+)^\perp$ and $y = a$. In Example 5 we already showed that in C_{14} we have $\mathrm{int}_a(x) = x$. Now consider $\mathrm{int}_x(a)$:*

$$\begin{aligned} \mathrm{int}_x(a) &= (a \cdot x + x^\perp) \cdot a \\ &= (b + x^\perp) \cdot a \\ &= (b + a^{++} \cdot b^+) \cdot a \\ &= (a^{++} \cdot a \\ &= a^{++} \\ &\neq a \end{aligned}$$

Hence (S) *is violated in* C_{14}. *But in* C_{18} (S) *always holds. E.g. we have* $int_x(a) = a^{++} \neq a$ *as in* C_{14} *but we also have:*

$$\begin{aligned}\mathrm{int}_a(x) &= (x \cdot a + a^{\perp}) \cdot x \\ &= (b + a^{\perp}) \cdot x \\ &= (a^{**} \cdot b+)^{\perp} \cdot x \\ &= (a^{**} \cdot b+)^{\perp} \\ &\neq a\end{aligned}$$

On the other hand we have $\mathrm{int}_a((a^{**} \cdot b^*)^{\perp}) = (a^{**} \cdot b^*)^{\perp}$ *in* C_{18} *as demonstrated in Example 5 and we also have:*

$$\begin{aligned}\mathrm{int}_{(a^{**} \cdot b^*)^{\perp}}(a) &= (a \cdot (a^{**} \cdot b^*)^{\perp} + (a^{**} \cdot b^*)^{\perp\perp}) \cdot a \\ &= (b^{++} + (a^{**} \cdot b^*)) \cdot a \\ &= a \cdot a \\ &= a\end{aligned}$$

Hence both sides of (S) *are satisfied in this particular instance. Again,* (S) *must hold for arbitrary elements in* (C_{18}) *because it is a topological Stonian p-ortholattice.*

6.5 Localized distributivity

Our last property is based on the observation that topological Stonian p-ortholattices exhibit distributivity for some triples of elements even though they are generally not distributive. Again, we use the interior and closure with respect to another element to state when distributivity holds.

Theorem 8 *Let* $\langle X, \tau \rangle$ *be a topological space. Then* $\mathrm{int}_b(a) = a$ *and* $\mathrm{int}_c(a) = a$ *and* $\mathrm{cl}_c(b) = b$ *implies* $a \cap^* (b \cup^* c) = (a \cap^* b) \cup^* (a \cap^* c)$ *for all* $a, b, c \in \mathrm{RT}(X)$.

Proof We have

$$\begin{aligned}a \cap^* (b \cup^* c) &\subseteq a \cap (b \cup^* c) \\ &= a \cap (b \cup c) && \text{Corollary 2(2)} \\ &= (a \cap b) \cup (a \cap c) \\ &= (a \cap^* b) \cup (a \cap^* c) && \text{Lemma 9} \\ &\subseteq (a \cap^* b) \cup^* (a \cap^* c).\end{aligned}$$

The converse inclusion holds in every lattice. □

The previous theorem shows that the quasiidentity

(D) $\mathrm{int}_y(x) = x$ and $\mathrm{int}_z(x) = x$ and $\mathrm{cl}_z(y) = y$ implies $x \cdot (y+z) = x \cdot y + x \cdot z$ for all $x, y, z \in L$.

holds in all representable Stonian p-ortholattices. As before, we are interested in an equational version of (D). Unfortunately, we are only able to provide such a version if we assume (M).

Lemma 13 *Let* $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ *be a Stonian p-ortholattice satisfying* (M). *Then following statements are equivalent:*

(D) $\mathrm{int}_y(x) = x$ *and* $\mathrm{int}_z(x) = x$ *and* $\mathrm{cl}_z(y) = y$ *implies* $x \cdot (y + z) = x \cdot y + x \cdot z$ *for all* $x, y, z \in L$.

($\mathrm{D}^=$) $\mathrm{int}_{\mathrm{cl}_z(y)}(x) \cdot \mathrm{int}_z(x) \cdot (\mathrm{cl}_z(y) + z) = \mathrm{int}_z(x) \cdot \mathrm{cl}_z(y) + \mathrm{int}_{\mathrm{cl}_z(y)}(x) \cdot z$ *for all* $x, y, z \in L$.

Proof (D) $\Rightarrow$ ($\mathrm{D}^=$) : Property (M) implies

$$\mathrm{int}_{\mathrm{cl}_z(y)}(\mathrm{int}_{\mathrm{cl}_z(y)}(x) \cdot \mathrm{int}_z(x)) = \mathrm{int}_{\mathrm{cl}_z(y)}(x) \cdot \mathrm{int}_z(x)$$

as well as

$$\mathrm{int}_z(\mathrm{int}_{\mathrm{cl}_z(y)}(x) \cdot \mathrm{int}_z(x)) = \mathrm{int}_{\mathrm{cl}_z(y)}(x) \cdot \mathrm{int}_z(x).$$

Using Corollary 1(4) and (D) we obtain

$$\begin{aligned}
&\mathrm{int}_{\mathrm{cl}_z(y)}(x) \cdot \mathrm{int}_z(x) \cdot (\mathrm{cl}_z(y) + z) \\
&= \mathrm{int}_{\mathrm{cl}_z(y)}(x) \cdot \mathrm{int}_z(x) \cdot \mathrm{cl}_z(y) + \mathrm{int}_{\mathrm{cl}_z(y)}(x) \cdot \mathrm{int}_z(x) \cdot z && \text{(D)} \\
&= x \cdot \mathrm{int}_z(x) \cdot \mathrm{cl}_z(y) + \mathrm{int}_{\mathrm{cl}_z(y)}(x) \cdot x \cdot z && \text{Lemma 7(3)} \\
&= \mathrm{int}_z(x) \cdot \mathrm{cl}_z(y) + \mathrm{int}_{\mathrm{cl}_z(y)}(x) \cdot z. && \text{Lemma 7(1)}
\end{aligned}$$

($\mathrm{D}^=$) $\Rightarrow$ (D) : Assume $\mathrm{int}_y(x) = x$ and $\mathrm{int}_z(x) = x$ and $\mathrm{cl}_z(y) = y$. Then we compute

$$\begin{aligned}
x \cdot (y + z) &= \mathrm{int}_{\mathrm{cl}_z(y)}(x) \cdot \mathrm{int}_z(x) \cdot (\mathrm{cl}_z(y) + z) \\
&= \mathrm{int}_z(x) \cdot \mathrm{cl}_z(y) + \mathrm{int}_{\mathrm{cl}_z(y)}(x) \cdot z && (\mathrm{D}^=) \\
&= x \cdot y + x \cdot z.
\end{aligned}$$

This completes the proof. □

Lemma 14 *Let* $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ *be a Stonian p-ortholattice. Then* (D) *is equivalent to*

($\mathrm{D}^{\mathrm{dual}}$) $\mathrm{cl}_y(x) = x$ *and* $\mathrm{cl}_z(x) = x$ *and* $\mathrm{int}_z(y) = y$ *implies* $x + y \cdot z = (x + y) \cdot (x + z)$ *for all* $x, y, z \in L$.

Proof Assume $\mathrm{cl}_y(x) = x$ and $\mathrm{cl}_z(x) = x$ and $\mathrm{int}_z(y) = y$. From the definition of cl we obtain $\mathrm{int}_{y^\perp}(x^\perp) = x^\perp$ and $\mathrm{int}_{z^\perp}(x^\perp) = x^\perp$ and $\mathrm{cl}_{z^\perp}(y^\perp) = y^\perp$. We conclude

$$\begin{aligned}
x + y \cdot z &= (x^\perp \cdot (y^\perp + z^\perp))^\perp \\
&= (x^\perp \cdot y^\perp + x^\perp \cdot z^\perp)^\perp && \text{by (D)} \\
&= (x + y) \cdot (x + z).
\end{aligned}$$

The converse implication is shown analogously. □

Example 7 *Consider the elements $x = a^*$, $y = b$, and $z = a^\perp$. First we verify that in C_{14} the preconditions of (D) are met:*

$$
\begin{array}{lrl}
(i) & int_b(a^*) &= (a^* \cdot b + b^\perp) \cdot a^* \\
& &= b^\perp \cdot a^* \\
& &= a^* \\
(ii) & int_{a^\perp}(a^*) &= (a^* \cdot a^\perp + a^{\perp\perp}) \cdot a^* \\
& &= (a^\perp + a) \cdot a^* \\
& &= a^* \\
(iii) & cl_{a^\perp}(b) &= (b + a^\perp) \cdot a^{\perp\perp} + b \\
& &= (a^{++} \cdot b^+)^\perp \cdot a + b \\
& &= b + b \\
& &= b
\end{array}
$$

But the following computation shows we do not have $a^* \cdot (b + a^\perp) = a^* \cdot b + a^* \cdot a^\perp$:

$$
\begin{aligned}
a^* \cdot (b + a^\perp) &\neq a^* \cdot b + a^* \cdot a^\perp \\
a^* \cdot (a^{++} \cdot b^+)^\perp &\neq 0 + a^\perp \\
a^* &\neq a^\perp
\end{aligned}
$$

This example shows (D) is not satisfied in C_{14}.

In C_{18}, the precondition of (D) are equally satisfied; the computations for (i) and (ii) are identical to those for C_{14}, while the following verifies (iii) in C_{18}:

$$
\begin{array}{lrl}
(iii) & cl_{a^\perp}(b) &= (b + a^\perp) \cdot a^{\perp\perp} + b \\
& &= (a^{**} \cdot b^+)^\perp \cdot a + b \\
& &= b + b \\
& &= b
\end{array}
$$

But now $a^* \cdot (b + a^\perp) = a^* \cdot b + a^* \cdot a^\perp$ holds as expected because C_{18} is a topological Stonian p-ortholattice:

$$
\begin{aligned}
a^* \cdot (b + a^\perp) &= a^* \cdot b + a^* \cdot a^\perp \\
a^* \cdot (a^{**} \cdot b^+)^\perp &= 0 + a^\perp \\
a^\perp &= a^\perp
\end{aligned}
$$

6.6 (RP_1) and (RP_2) hold in Stonian p-ortholattices satisfying (S) and (D)

To conclude this section, we will show that in a Stonian p-ortholattice (RP_1) and (RP_2) become provable if (D) is satisfied.

Lemma 15 *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a Stonian p-ortholattice satisfying* (S) *and* (D). *Then* (RP_1) *and* (RP_2) *are valid.*

Proof We want to prove (RP_2) first. Therefore, suppose $x^\perp \leq y$, $z^{++} \leq x$, and $x \cdot z \leq y$. Then we have $z^{++} = z^{++} \cdot z \leq x \cdot z \leq y$ so that $\mathrm{int}_z(x) = x$, $\mathrm{int}_z(y) = y$, and $\mathrm{cl}_y(x) = x$ follow from Lemma 7(7) and Corollary 1(8). Using (S) we obtain $\mathrm{int}_x(z) = z$ and $\mathrm{int}_y(z) = z$. We conclude

$$\begin{aligned} z &= z \cdot (x + x^\perp) \\ &\leq z \cdot (x + y) \\ &= z \cdot x + z \cdot y && \text{by (D)} \\ &\leq y + z \cdot y \\ &= y. \end{aligned}$$

In order to prove (RP_1) suppose $x^\perp \leq y$. First we want to show that $x^{**} \cdot y^{**} \leq x \cdot y^{**} + x^{**} \cdot y + y$. This follows by using x, $x \cdot y^{**} + x^{**} \cdot y + y$ and $x^{**} \cdot y^{**}$ as x, y, and z in (RP_2) by:

$$\begin{aligned} &x^\perp \leq y \leq x \cdot y^{**} + x^{**} \cdot y + y, \\ &(x^{**} \cdot y^{**})^{++} = x^{****++} \cdot y^{****++} = x^{++} \cdot y^{++} \leq x, \\ &x \cdot (x^{**} \cdot y^{**}) = x \cdot y^{**} \leq x \cdot y^{**} + x^{**} \cdot y + y. \end{aligned}$$

We have $(x \cdot y^{**} + x^{**} \cdot y)^{**} = x^{**} \cdot y^{**} \leq y^{**}$ so that $\mathrm{cl}_y(x \cdot y^{**} + x^{**} \cdot y) = x \cdot y^{**} + x^{**} \cdot y$ follows from Corollary 1(7). Furthermore, we have $x^{**} \cdot y^{**})^{++} = x^{****++} \cdot y^{****++} = x^{++} \cdot y^{++} = x^{++} \cdot y^{****++} + x^{****++} \cdot y^{++} = (x \cdot y^{**} + x^{**} \cdot y)^{++}$ and $(x^{**} \cdot y^{**})^{++} = x^{****++} \cdot y^{****++} = x^{++} \cdot y^{++} \leq y^{++}$. From Lemma 7(7) we get $\mathrm{int}_{x^{**} \cdot y^{**}}(x \cdot y^{**} + x^{**} \cdot y) = x \cdot y^{**} + x^{**} \cdot y$ and $\mathrm{int}_{x^{**} \cdot y^{**}}(y) = y$. Using (S) we obtain $\mathrm{int}_{x \cdot y^{**} + x^{**} \cdot y}(x^{**} \cdot y^{**}) = x^{**} \cdot y^{**}$ and $\mathrm{int}_y(x^{**} \cdot y^{**}) = x^{**} \cdot y^{**}$. We conclude

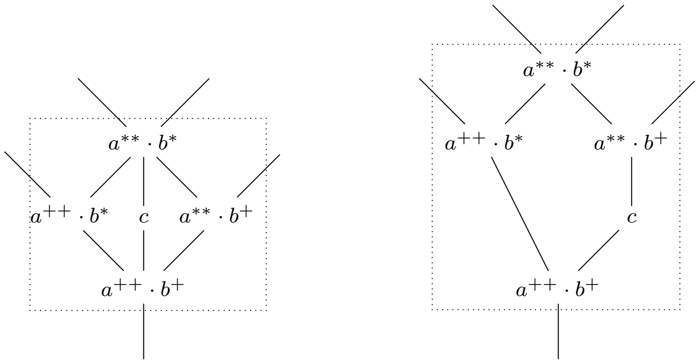
$$\begin{aligned} x^{**} \cdot y^{**} &= x^{**} \cdot y^{**} \cdot (x \cdot y^{**} + x^{**} \cdot y + y) && \text{see above} \\ &= x^{**} \cdot y^{**} \cdot (x \cdot y^{**} + x^{**} \cdot y) + x^{**} \cdot y^{**} \cdot y && \text{(D)} \\ &= x \cdot y^{**} + x^{**} \cdot y. \end{aligned}$$

This completes the proof. □

The Theorems 4 to 8 in this section show that the properties (RP_1), (RP_2), (M), (S), and (D) are necessary for a Stonian p-ortholattice to be representable. Furthermore, we showed that (RP_1) and (RP_2) are entailed by (D) in Stonian p-ortholattices. It remains open whether these properties guarantee representability of Stonian p-ortholattices.

7 Some representable and non-representable Stonian p-ortholattices

In the following we provide examples of Stonian p-ortholattice satisfying some (or all) of the properties (RP_1), (RP_2), (M), (S), and (D). In addition to the lattices C_{14}, C_{16}, C_{18} and C_{20}, which we previously introduced, we will also use C_{20}^d and C_{20}^m, whose outer structure is also given by Fig. 2(a), while their inner structure is given in Fig. 5. Moreover, we use the lattice C_{12} from Fig. 6. This lattice does not



(a) The inner structure of C_{20}^d. (b) The inner structure of C_{20}^m.

Fig. 5: The inner structure of C_{20}^d and C_{20}^m

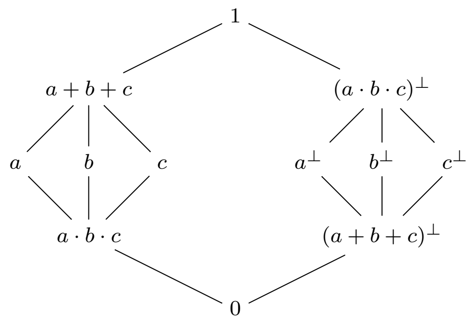


Fig. 6: The lattice C_{12}

satisfy (A12), but by Lemma 4 the lattice $C_{12} \times C_{18}$ does. Since (M) is an equation and valid in both C_{12} and C_{18}, all of the properties (RP_1), (RP_2), (M), (S), and (D) can be written as equations in $C_{12} \times C_{18}$ as well. Since all five properties are true in C_{18}, any of those five properties is true in $C_{12} \times C_{18}$ if and only if it is true in C_{12}.

In Table 1 we have summarized the properties satisfied by the different lattices. If a certain property is not valid, we list a counterexample. We checked the results in Table 1 and verified that all lattices in the table are indeed Stonian p-ortholattices by the program written in Haskell as mentioned earlier. Recall further that all examples in Table 1 satisfy ($A11$), ($A12$) and ($A13$), i.e. are also models of RT_0.

We already noted that C_{18} is the smallest representable lattice amongst the Stonian p-ortholattices that satisfy ($A12$). The next biggest example of a representable Stonian p-ortholattice satisfying ($A12$) is C_{24}, for which we also include a concrete representation in Fig. 7. As Table 1 shows, C_{18} and C_{24} are at the same time the only lattices amongst our examples in which all five properties hold; they are the only representable Stonian p-ortholattices of up to 24 elements that satisfy ($A12$).

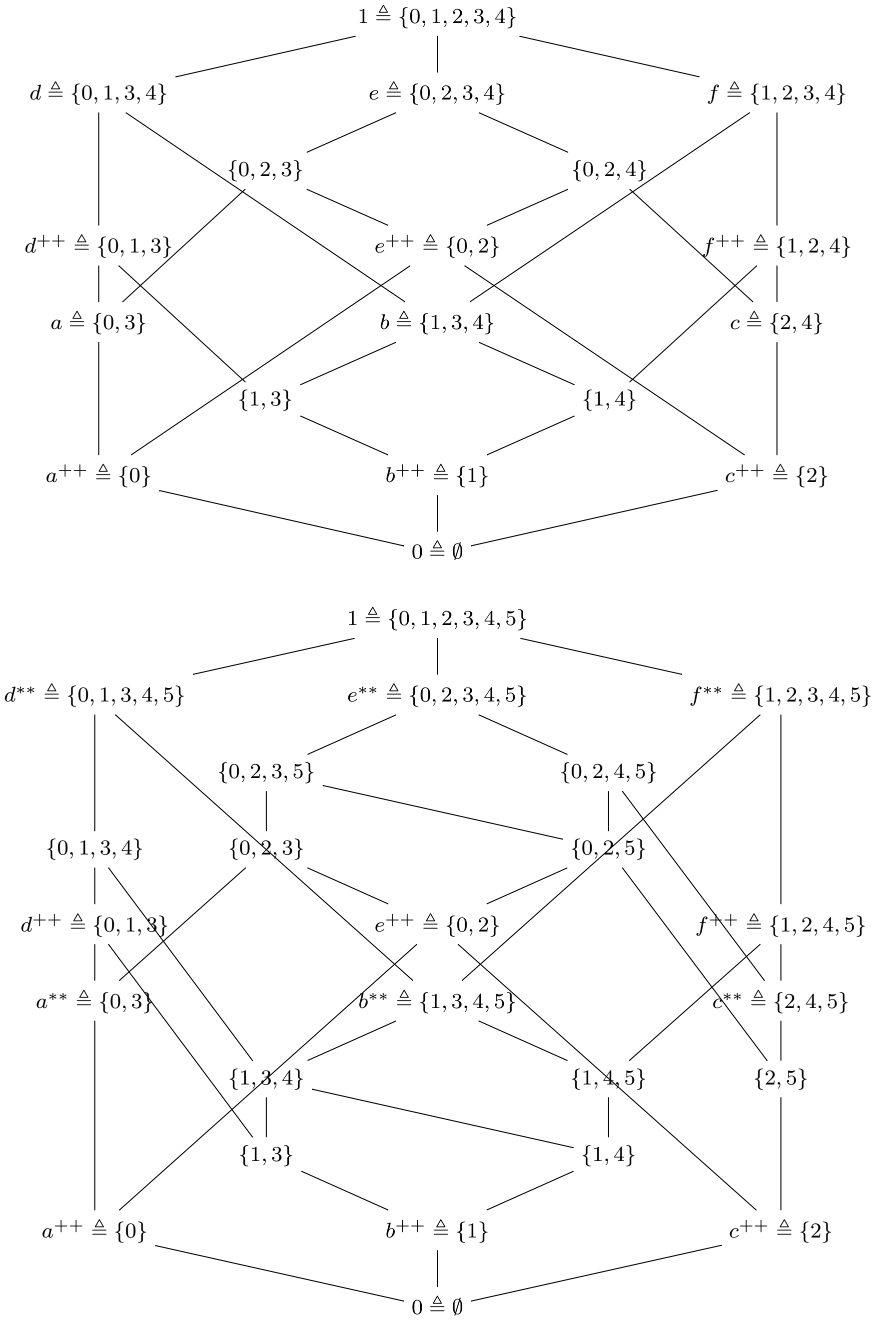


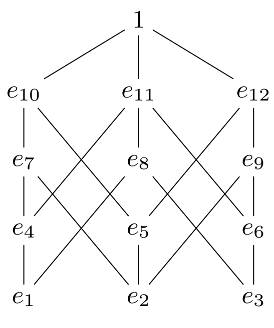
Fig. 7: Point-set representations of the Stonian p-ortholattices C_{18} and C_{24}

Table 1: Examples of Stonian p-ortholattices and counterexamples (if they exist) to the properties (RP_1), (RP_2), (M), (S), and (D) (d is an abbreviation for $a^{++} \cdot b^* + a^{**} + b^+$)

Lattice	(RP_1)	(RP_2)	(M)	(S)	(D)
C_{14}	$x = a^{++}$ $y = b^+$	$x = a^{++}$ $y = b^+$ $z = a^{**} \cdot b^*$	true	$x = a^{++}$ $y = b^*$	$x = b$ $y = a^*$ $z = b^{++}$
C_{16}	$x = a^{++}$ $y = b^+$	$x = b^+$ $y = a^{++}$ $z = a^{**} \cdot b^*$	$x = (a^{++} \cdot b^+)^\perp$ $y = a$ $z = b^*$	$x = a$ $y = b^\perp$	$x = a^*$ $y = b$ $z = a^\perp$
C_{18}	true	true	true	true	true
C_{20}	$x = a^{++}$ $y = b^+$	true	$x = (a^{++} \cdot b^+)^\perp$ $y = a$ $z = b^*$	$x = a$ $y = d^\perp$	$x = a$ $y = a^\perp$ $z = d$
C_{20}^d	true	$x = a^{++}$ $y = b^+$ $z = c$	$x = a^{**} \cdot b^*$ $y = a^{++}$ $z = c^\perp$	$x = a^{++}$ $y = c$	$x = b$ $y = a^*$ $z = c$
C_{20}^m	true	true	$x = (a^{++} \cdot b^+)^\perp$ $y = b^\perp$ $z = c$	$x = a$ $y = c$	$x = a$ $y = a^\perp$ $z = c$
C_{24}	true	true	true	true	true
$C_{12} \times C_{18}$	true	true	true	true	$x = (a, 1)$ $y = (b, 1)$ $z = (c, 1)$

7.1 A non-representable model of RT_0: the example of C_{14}

With the help of one of our examples of non-representable Stonian p-ortholattices we can now show that non-representable models of the theory RT_0 exist. For that matter, let us consider C_{14} in more detail. C_{14} is a Stonian p-ortholattice that satisfies ($A11$), ($A12$) and ($A13$), hence C_{14} with the lowest element removed is a model of RT_0 by Theorem 2. Because C_{14} violates some of the conditions of topological Stonian p-ortholattices, it is not representable in the sense of Def. 2. That also implies that C_{14} cannot be generated from the regular sets of a topological space, thereby refuting the original completeness theorem for RT_0 given in [1]. In order to make the last statement more precise, consider the following figure of $C_{14} \setminus \{0\}$ and the following formulas where e_i for $i \in \{1, \ldots, 12\}$ are constant symbols and 1 is the symbol for the universe as introduced in Section 2:



$$(\Omega)\ \forall x(x = e_1 \vee x = e_2 \vee x = e_3 \vee x = e_4 \vee x = e_5 \vee x = e_6 \vee x = e_7$$
$$\vee\, x = e_8 \vee x = e_9 \vee x = e_{10} \vee x = e_{11} \vee x = e_{12} \vee x = 1)$$
$$(\Omega_i)\ e_i \neq 1$$
$$(\Omega_{i,j})\ e_i \neq e_j$$

The set of formulas $\{\Omega\} \cup \{\Omega_i \mid i \in \{1, \ldots, 12\}\} \cup \{\Omega_{i,j} \mid i, j \in \{1, \ldots, 12\} \wedge i < j\}$ requires that any model of this set has exactly 13 elements. Now, consider the following formulas:

$(\Gamma_0)\ \forall x\ C(x, 1)$
$(\Gamma_1)\ C(e_1, e_4) \wedge C(e_1, e_7) \wedge C(e_1, e_8) \wedge C(e_1, e_{10}) \wedge C(e_1, e_{11})$
$(\Gamma_2)\ C(e_2, e_5) \wedge C(e_2, e_7) \wedge C(e_2, e_9) \wedge C(e_2, e_{10}) \wedge C(e_2, e_{12})$
$(\Gamma_3)\ C(e_3, e_6) \wedge C(e_3, e_8) \wedge C(e_3, e_9) \wedge C(e_3, e_{11}) \wedge C(e_3, e_{12})$
$(\Gamma_4)\ C(e_4, e_5) \wedge C(e_4, e_7) \wedge C(e_4, e_8) \wedge C(e_4, e_{10}) \wedge C(e_4, e_{11}) \wedge C(e_4, e_{12})$
$(\Gamma_5)\ C(e_5, e_6) \wedge C(e_5, e_7) \wedge C(e_5, e_9) \wedge C(e_5, e_{10}) \wedge C(e_5, e_{11}) \wedge C(e_5, e_{12})$
$(\Gamma_6)\ C(e_6, e_8) \wedge C(e_6, e_9) \wedge C(e_6, e_{10}) \wedge C(e_6, e_{11}) \wedge C(e_6, e_{12})$
$(\Gamma_7)\ C(e_7, e_8) \wedge C(e_7, e_9) \wedge C(e_7, e_{10}) \wedge C(e_7, e_{11}) \wedge C(e_7, e_{12})$
$(\Gamma_8)\ C(e_8, e_9) \wedge C(e_8, e_{10}) \wedge C(e_8, e_{11}) \wedge C(e_8, e_{12})$
$(\Gamma_9)\ C(e_9, e_{10}) \wedge C(e_9, e_{11}) \wedge C(e_9, e_{12})$
$(\Gamma_{10})\ C(e_{10}, e_{11}) \wedge C(e_{10}, e_{12})$
$(\Gamma_{11})\ C(e_{11}, e_{12})$

The above formulas Γ_i for $i \in \{0, \ldots, 11\}$ together with the axioms (A1) and (A2) from RT_0 (cf. Section 2) specify which elements should be in contact, and, similarly,

$(\Upsilon_1)\ \neg C(e_1, e_2) \wedge \neg C(e_1, e_3) \wedge \neg C(e_1, e_5) \wedge \neg C(e_1, e_6) \wedge \neg C(e_1, e_9) \wedge \neg C(e_1, e_{12})$
$(\Upsilon_2)\ \neg C(e_2, e_3) \wedge \neg C(e_2, e_4) \wedge \neg C(e_2, e_6) \wedge \neg C(e_2, e_8) \wedge \neg C(e_2, e_{11})$
$(\Upsilon_3)\ \neg C(e_3, e_4) \wedge \neg C(e_3, e_5) \wedge \neg C(e_3, e_7) \wedge \neg C(e_3, e_{10})$
$(\Upsilon_4)\ \neg C(e_4, e_6) \wedge \neg C(e_4, e_9)$
$(\Upsilon_5)\ \neg C(e_5, e_8)$
$(\Upsilon_6)\ \neg C(e_6, e_7)$

the formulas Υ_i for $i \in \{1, \ldots, 6\}$ together with (A2) specify which elements should not be in contact. Altogether these formulas specify precisely the contact structure of any 13-element model of RT_0. Consequently, the set

$$\Lambda = RT_0 \cup \{\Omega\} \cup \{\Omega_i \mid i \in \{1, \ldots, 12\}\} \cup \{\Omega_{i,j} \mid i, j \in \{1, \ldots, 12\} \wedge i < j\}$$
$$\cup\, \{\Gamma_i \mid i \in \{0, \ldots, 11\}\} \cup \{\Upsilon_i \mid i \in \{1, \ldots, 6\}\}$$

has at most one model (up to isomorphism). As mentioned earlier $C_{14} \setminus \{0\}$ is a model of RT_0, and, by construction of the formulas above, a model of Λ[3]. However,

[3] The model finder Paradox, an incremental SAT-based model, can automatically generate this model. See `www.cs.toronto.edu/~torsten/RegularSets/` for our axiomatization of Λ in the TPTP format and the model found by Paradox. The result can be easily reproduced using our axiomatization as input for Paradox 4.0, which is accessible from the online theorem proving environment `TPTP.org`

this model is not representable, i.e., it is not an intended model in the sense of [1], providing a counterexample to the completeness proof in [1].

We also want to demonstrate where the original completeness proof of [1] fails. The completeness proof follows the usual Henkin method. First, the authors extend the given consistent theory Σ to a maximal consistent saturated set of formulas. The remaining step is to construct a topological model based on certain subsets of equivalence classes of constant symbols Σ_C from Σ. In order to do so they define two sorts of ultrafilter constructions, one for interior points (IP) and one for boundary points (BP), as follows:

$$
\begin{aligned}
\mathrm{IP}(\alpha) \equiv_{\mathrm{def}} & \alpha \subseteq \Sigma_C \wedge \alpha \neq \emptyset \wedge \\
& \forall x, y[(x \in \alpha \wedge y \in \alpha) \rightarrow (O(x,y) \wedge x \cdot y \in \alpha)] \wedge \\
& \forall x, y[(x \in \alpha \wedge P(x,y)) \rightarrow y \in \alpha] \wedge \\
& \alpha \text{ maximal with respect to } \subseteq \text{ and the properties above} \\
\mathrm{BP}(\alpha) \equiv_{\mathrm{def}} & \alpha \subseteq \Sigma_C \wedge \\
& \exists x, y[x \in \alpha \wedge y \in \alpha \wedge EC(x,y)] \\
& \forall x, y[(x \in \alpha \wedge y \in \alpha) \rightarrow ((O(x,y) \wedge x \cdot y \in \alpha) \\
& \qquad \vee (\exists z, t(z \in \alpha \wedge t \in \alpha \wedge P(z,x) \wedge P(t,y) \wedge EC(z,t))))] \wedge \\
& \forall x, y[(x \in \alpha \wedge P(x,y)) \rightarrow y \in \alpha] \wedge \\
& \alpha \text{ maximal with respect to } \subseteq \text{ and the properties above}
\end{aligned}
$$

Based on the above they define an interpretation for every equivalence class of constants $[c] = \{c' \mid \Sigma \vdash c = c'\}$, i.e., the set of constants c' that can be proven in Σ to be equal to c, by $\Omega_{[c]} = \{\alpha \mid (\mathrm{IP}(\alpha) \vee \mathrm{BP}(\alpha)) \wedge [c] \subseteq \alpha\}$. This interpretation yields an interpretation for arbitrary constants by $[\![c]\!] = \Omega_{[c]}$. The topology is defined as the topology with the set $\{[\![c]\!] \mid \Sigma \vdash OP(c)\}$ as a basis of the open sets, i.e., the interpretation of open elements of the theory generate the topology. Notice the similarity of the definitions above to those in [7]. Now, on Page 850 of [1] the authors claim that the following property (among others)

$$(*) \qquad [\![x \cdot y]\!] = [\![x]\!] \cap^* [\![y]\!]$$

follows immediately from the axioms and definitions. We want to demonstrate that this is not the case. Since any model of Λ has exactly 13 elements, we will identify the 13 equivalence classes of the elements $e_1, \ldots, e_{12}, 1$ after saturating Λ with the corresponding element. If $M\uparrow = \{y \mid \exists x(x \in M \wedge P(x,y))\}$ denotes the upwards closure of M, then we obtain the following interior and boundary points:

$$\mathrm{IP} = \{\{e_1\}\uparrow, \{e_2\}\uparrow, \{e_3\}\uparrow\}, \qquad \mathrm{BP} = \{\{e_4, e_5\}\uparrow, \{e_5, e_6\}\uparrow\}.$$

Consequently, we get

$$
\begin{aligned}
[\![e_7]\!] &= \{\{e_1\}\uparrow, \{e_2\}\uparrow, \{e_4, e_5\}\uparrow\}, \\
[\![e_{12}]\!] &= \{\{e_2\}\uparrow, \{e_3\}\uparrow, \{e_4, e_5\}\uparrow, \{e_5, e_6\}\uparrow\}, \\
[\![e_7]\!] \cap [\![e_{12}]\!] &= \{\{e_2\}\uparrow, \{e_4, e_5\}\uparrow\}, \\
[\![e_7 \cdot e_{12}]\!] = [\![e_2]\!] &= \{\{e_2\}\uparrow\}.
\end{aligned}
$$

Furthermore, the model is finite so that $\{\emptyset, [\![e_1]\!], [\![e_2]\!], [\![e_3]\!], [\![e_7]\!], [\![e_8]\!], [\![e_9]\!], [\![e_{13}]\!]\}$ is the set of open sets, and $\{\emptyset, [\![e_4]\!], [\![e_5]\!], [\![e_6]\!], [\![e_{10}]\!], [\![e_{11}]\!], [\![e_{12}]\!], [\![e_{13}]\!]\}$ is the set of closed sets of the topology. We obtain

$$\begin{aligned}
\mathrm{cl}(\mathrm{int}([\![e_7]\!] \cap [\![e_{12}]\!])) &= \mathrm{cl}(\mathrm{int}(\{\{e_2\}\uparrow, \{e_4, e_5\}\uparrow\})) \\
&= \mathrm{cl}([\![e_2]\!]) \\
&= \mathrm{cl}(\{\{e_2\}\uparrow\}) \\
&= [\![e_5]\!] \\
&= \{\{e_2\}\uparrow, \{e_4, e_5\}\uparrow, \{e_5, e_6\}\uparrow\},
\end{aligned}$$

so that

$$\begin{aligned}
[\![e_7]\!] \cap^* [\![e_{12}]\!] &= [\![e_7]\!] \cap [\![e_{12}]\!] \cap \mathrm{cl}(\mathrm{int}([\![e_7]\!] \cap [\![e_{12}]\!])) \\
&= \{\{e_2\}\uparrow, \{e_4, e_5\}\uparrow\} \cap \{\{e_2\}\uparrow, \{e_4, e_5\}\uparrow, \{e_5, e_6\}\uparrow\} \\
&= \{\{e_2\}\uparrow, \{e_4, e_5\}\uparrow\}
\end{aligned}$$

follows, a contradiction to $(*)$. Notice that the elements e_7 and e_{12} are also integral part of the counterexample to (RP_1). Furthermore, a similar argument does not work for C_{18} since the meet $e_7 \cdot e_{12}$ will lead to a different element.

8 Conclusion and Future Work

We showed that not all Stonian p-ortholattices are topologically representable and explored five topologically motivated quasiidentities that must hold in topological Stonian p-ortholattices and thus also in representable Stonian p-ortholattices. We also showed that all five proposed properties can be expressed as equations, thus ensuring that the variety of Stonian p-ortholattices extended by all five properties is again a variety. Since (RP_1) and (RP_2) follow from (D) in Stonian p-ortholattices, the resulting equational theory can be obtained by extending the equational theory of Stonian p-ortholattices by ($\mathrm{S}^=$), ($\mathrm{M}^=$), and ($\mathrm{D}^=$). Most importantly, we conclude that an algebraic representation of the regular sets of a topological space is a Stonian p-ortholattice that satisfies (S), (M), and (D). With regard to the mereotopology RT_0, we were able to demonstrate that the mereotopology has unintended models, i.e. models which are not representable by regular sets of a topological space as claimed in [1].

Future work is twofold. First, we only showed that the properties (S), (M), and (D) are necessary for a Stonian p-ortholattice to be representable. In order to give a full (isomorphic) algebraic representation of the regular sets of a topological space, it remains to be proved whether (S), (M), and (D) are sufficient. If not, additional properties satisfied by all representable Stonian p-ortholattices need to be identified.

Separately of this issue, it remains to be investigated whether the conditions (S), (M), and (D) are independent of each other in Stonian p-ortholattices or whether e.g. (S) or (M) are provable in the presence of (D). From our examples we only know that (M) is insufficient to prove (S) or (D), and that (M) and (S) are insufficient to prove (D). Moreover, our experiments showed that all Stonian p-ortholattices containing no more than 60 elements and satisfying (D) also satisfy (S).

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