Model-Theoretic Analysis of Asher And Vieu's Mereotopology

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Abstract

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In the past little work has been done to characterize the models of various mereotopological systems. This thesis focuses on Asher and Vieu's first-order mereotopology which evolved from Clarke's *Calculus of Individuals*. Its soundness and completeness proofs with respect to a topological translation of the axioms provide only sparse insights into structural properties of the mereotopological models. To overcome this problem, we characterize these models with respect to mathematical structures with well-defined properties – topological spaces, lattices, and graphs. We prove that the models of the subtheory RT^- are isomorphic to p-ortholattices (pseudocomplemented, orthocomplemented). Combining the advantages of lattices and graphs, we show how Cartesian products of finite p-ortholattices with one multiplicand being not uniquely complemented (unicomplemented) gives finite models of the full mereotopology. Our analysis enables a comparison to other mereotopologies, in particular to the *RCC*, of which lattice-theoretic characterizations exist.

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“Es ist nicht das Wissen, sondern das Lernen,
nicht das Besitzen, sondern das Erwerben,
nicht das Dasein, sondern das Hinkommen,
was den größten Genuß gewährt.”
(Carl Friedrich Gauß)

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CHAPTER 1

Introduction

In the area of knowledge representation the use of ontologies for representing and organizing knowledge becomes increasingly popular. Ontologies are shared conceptualizations of specific domains that define semantics for all concepts and relations. Such ontologies can be described by an abundance of languages in various degrees of formality. For logical inference using ontologies, so-called formal ontologies are necessary: they describe the involved concepts and relations up to a formality which allows automatic reasoning systems to determine the validity of inferences. Reasoners are used to check the truth of entailments in a theory or a specific model instance. In this sense, the term ontology in Artificial Intelligence has a meaning that differs from understanding of Ontologies in philosophy. Formal ontologies are understood to enable logical entailments by the sole use of the formal specification, but do not refer to Formal Ontologies as defined in philosophy (see [**Gua98**]). We are mainly interested in ontologies defined in the language of first-order logic (FOL), since its expressiveness allows complex descriptions while well-defined inference methods exist. In this sense, we consider ontologies to be equivalent to axiomatic systems.

Several ontologies for mereotopology have been developed during the last few decades. Most of them are more of theoretical nature, as far as we know there are few applications of them. However, this seems not to be caused by lack of interest in the area, as recent work on upper ontologies shows. On the contrary, mereotopological concepts are applied in a wide breadth of areas ranging from spatial modeling and reasoning to physical, biological, and chemical models that orders, e.g. physical or natural, parts (in manufacturing processes, anatomic models, or modeling of complex chemical structures in different levels of granularity) based on connection relations. Uncertainty about differences in mereotopological systems, in particular about their implicit assumptions, seem to be a major source of confusion that hinders forthright application of even well-developed mereotopological theories. This problem arises with the various theories in different ways: some lack any formal representations, requiring the user to reason about intended interpretations and providing no formal reasoning framework; others are already formalized, e.g. in first-order logic but lack a rigorous presentation of the allowed and forbidden models. This thesis focuses on one instance of the latter problem. We analyze the models of the mereotopology proposed by Asher and Vieu in 1995 using formalisms of well-understood mathematical disciplines in the style of a representation theorem. The purpose is straightforward: we want to understand what kind of models the axiomatic system describes and what properties these models have in common. The goal is to characterize at least the finite models of Asher and Vieu's mereotopology in terms of classes of structures defined in topology, lattice theory, and graph theory. Although the authors developed the theory from a characterization of the intended models, this characterization is a mere rephrasing of the axioms. Looking at the characterization does not reveal much insight about the theory and its models. Of course, the axiomatic system describes exactly the same models as demonstrated by their soundness and completeness proof, but the given characterization does not give any hints about applicability for certain practical purposes, the restriction of the theories, its hidden assumptions, and relations to other theories, i.e. either mereotopologies or simply mereology, topology, set theory or algebraic systems.

1. Motivation

The primary motivation of this work is to give a better insight into the axiomatic theory and to uncover problems and assumptions that users of the ontology should be aware of. A characterization of the models of the axiomatic theory allows us to reuse knowledge about the mathematical structures in the mereotopological theory. We attempt an characterization up to elementary equivalence of the models of RT. This is sufficient because two models that are elementary equivalent cannot be distinguished by a first-order sentence, hence everything we entail for one such model is true for the other model as well with respect to what we can express in first-order logic. That allows us to rely on the soundness and completeness proof of RT_0 with respect to RT_T that implies elementary equivalence between the two classes of models; hence we can use the known axioms and conditions of either theory. But only for the models of a subset of the theory, called RT^-, we can give a full-blown representation theorem. However, later we restrict ourselves to the finite models of RT_0, and by showing that every finite model of RT_0 also gives a finite model of RT_T, we characterize the finite models of RT_0 (or subtheories thereof) up to isomorphism.

Another point for motivating this work is the fact that such a characterization can link the theory to other mereotopological systems, no matter whether those use the same primitives or not, in a standard mathematical fashion and not only on a philosophical level. In previous work, Biacino and Gerla [**BG96**] characterized the models of Clarke's *Calculus of Individuals* [**Cla81, Cla85**] in terms of lattices. They showed that a subset of the axioms of Clarke, i.e. axioms A1 to A4, also called a *connection structure*, are characterizing the complete orthocomplemented lattices. Together with an axiom requiring the existence of a common point of two connected individuals, [**BG96**] prove that it is equivalent to the complete Boolean algebras. Since the system of Asher and Vieu [**AV95**] heavily relies on the work of Clarke, it is interesting to know how the changes proposed by Asher and Vieu alter the class of associated models, particularly in a lattice-theoretic description. The surprising outcome here will be that the models change fundamentally, they are much more restricted and therefore reveal additional implicit structural properties. Details on the lattice-theoretic characterization and their relation to the work of Biacino and Gerla is presented in chapter 4 on page 34. Likewise, a characterization of the *Region Connection Calculus* (RCC), originally proposed in [**RCC92**] has been conducted in [**Ste97, Ste00**]. Stell uses a similar notion of *Connection Algebras* to describe the *Region Connection Calculus*.

While going through this model analysis, the mereological and topological substructures are exhibited as well. They are related to the concept of *contact algebras* [**DW05, DWM99, DW06**] and the more restricted definition of *connection structures* [**BG96**]. This relationship will also be discussed in chapter 4 from page 34 on.

A longer term question arising from this work is an exhaustive comparison of different mereotopological approaches within a strictly defined mathematical context, such as topological spaces, graph representations, or lattices. As it turns out, lattices seems to be best suited for such a research as they provide a very intuitive way of modeling parthood relations. Therefore, this work can be seen as a first step towards a comparison of the mereotopological framework with respect to their models, rather than arguing for or against the underlying philosophical assumption. We are more interested in a rigid mathematical study to provide the community interested in mereotopological systems and relations with a model-theoretic view on mereotopology.

Ultimately, this approach can lead to a revision of mereotopology constructed purely from a mathematical structure, such as a certain class of lattices, or their combinations (e.g. Cartesian products of lattices). A so-defined mereotopology might either generalize existing axiomatizations by relaxing some restrictions or might turn out to characterize mereotopological structures in a novel way. Interesting classes of lattices that seem specifically promising are Stone [**Joh82**] and Heyting algebras [**Joh82, Vic89**] as well as pseudocomplemented or orthocomplemented lattices (see [**Grä98**] as general reference on lattices). Stell mentions using Heyting and co-Heyting algebras for defining pointless topology [**SW97, Ste97**]. A common example of a Heyting algebra is the set of open sets of a topological space (see [**Hal63**]), which links back to the mereotopology of Asher and Vieu [**AV95**] which is defined in terms of a topological space where the open sets play an important role defining mereological concepts. Similarly, Stone spaces are mentioned by [**Joh83**] for formalizing an uniform theory of pointless topology. A promising branch of research is the generalization of mereotopology in terms of a pointless topology. Linking the two together in a meaningful sense would contribute tremendously to a general framework of pointless topology that might make mereotopology as a separate theory redundant. Since Clarke's work influences many following authors, we revisit his work in the revised version of Asher and Vieu on the way towards an axiomatization of pointless topology. Asher and Vieu already made a large step forward by formalizing the theory purely in first-order logic and removing any concrete references to regions or individuals as sets of points. The only other mereotopological systems that we know of being formalized in first-order logic are the *RCC* [**RCC92**] and [**Got96**]. However, some of their definitions still imply the existence of point sets, hence another step might be necessary.

From an ontological perspective, in the presence of more and more emerging upper ontologies this analysis and further work in the same direction can guide the selection of a proper, i.e. universal, axiomatization of mereotopology. Currently, the most common upper ontologies such as SUMO, DOLCE, and BFO incorporate some mereotopological concepts [**NP01, MBG$^+$03, Gre03**]. A special application that might need a more fine-grained axiomatization is the area of geographic information systems (GIS) where mereotopologies can be see as a generalization of spatial and spatio-temporal theories (of any dimension).

2. Outline

For the characterizations of the models of Asher and Vieu three different techniques are employed. First of all, we use topological spaces as an instrument for analyzing the structures and revealing their properties. This attempt seems natural, since the intended models were already constructed using topological spaces. Nevertheless, we will see that this attempt does not lead to a characterization as exhaustive as hoped for. But instead, this failed attempt clears the way for an alternative characterization using lattice-theoretic concepts from universal algebra. Other mereotopologies have been characterized partly or completely using this manner; however, all of the analyzed models have a rather straightforward characterization in terms of Boolean or Heyting algebras. The lattice approach is more fruitful than the analysis with topological spaces and leads to a full characterization of the finite models of a subset of the mereotopology. We show that this subset is isomorphic to the class of complete atomic p-ortholattices (the intersection of double-pseudocomplemented with orthocomplemented lattices). The additional restrictions that the full mereotopological theory of Asher and Vieu have to obey, e.g. with respect to modularity, symmetry, distributivity (and weaker forms thereof), and complementedness, are singled out. These substantial restrictions to the "not-so-nice" substructures of p-ortholattices are references as *non-modular* complete atomic p-ortholattices although they go beyond non-modularity. It turns out that the most difficult part is the characterization up to isomorphism when including external connection. It is easily verifiable that the models are all in the restricted class of lattices, but the tough question is to show that any such lattice does indeed yield a model of the theory. Lattices alone fail to show the reverse homomorphism.

The third attempt is purely graph-theoretic - we show how every model can be represented as a graph based on its extension of the connection relation. We break the theory into a topological and a mereological part and show equivalences to certain graph classes. However, the models including external connection can be only classified as dually chordal graphs which are too general. A stricter vertex ordering, a maximum neighborhood inclusion ordering, that defines a subclass of dually chordal graphs but contains all the models of the theory (including external connection) is presented. But the converse cannot be proved; it is conceived that the class of graphs that yield maximum neighborhood inclusion orderings also comprises graphs that cannot be associated to models of the mereotopological theory.

To take advantage of a graph-theoretic representation and to overcome some intrinsic restrictions of lattice theory that fail to characterize the models including external connection up to isomorphism, we apply a combination of both approaches. This allows us to elegantly prove that a restricted set of p-ortholattices is in fact isomorphic to the class of models of the mereotopology without weak contact. This yields a way of constructing models from any lattice in the restricted set of p-ortholattices. The last section of the thesis focuses on the weak contact relation that further restricts the models to direct products of specific p-ortholattices. Basic properties of the lattices participating in such products are identified and we give an example of such an product which is the smallest model of the full theory of Asher and Vieu.

In conclusion,, some of the problems of the axiomatization that were identified during the analysis are addressed in a broader context and in comparison to other work on mereotopology. Generalizations of the proposed method for the analysis of other mereotopologies are discussed and an outlook for further research in the area is presented. Finally the thesis contains an overview of the lattice-theoretic implications of the analysis, in particular with respect to the intersection of orthocomplemented and pseudocomplemented lattices. This class of lattices has not received much attention previously.

Remark 1. At some points in the thesis, we discuss direct implications of intermediate results on the intended models of the mereotopology in comparison to other mereotopological frameworks. Those comments are always marked as "Remark" and separately numbered from definitions, theorems, lemmas, propositions, and so on.

CHAPTER 2

The Mereotopology RT of Asher and Vieu

Reasoning about space in general and qualitative spatial reasoning (QSR) in particular has received significant attention in the last decade. Its application ranges from simple problems of path-finding to industry automation, cognitive robotics, computer vision applications, bio-informatics, and most commonly GIS systems [**CH01**]. Qualitative spatial reasoning abstracts from quantitative concepts like distances and angles and therefore does not deal with possibly infinite or continuous units, restricting the complexity of the reasoning task. Most QSR frameworks use a combination of topological (expressing connectedness) and mereological (expressing parthood) relations. These mereotopological systems draw a lot of attention in philosophic and logic communities and recently also in reasoning and knowledge representation research. However, the differences and similarities between various mereotopologies with respect to model-theoretic and reasoning perspectives are not yet well understood. [**CV99**] was the first work classifying mereotopologies with respect to their underlying topology and mereology and the fusion of them to coherent theories. In the next section, we will give a short introduction to mereology and topology and then show how they relate to each other in mereotopological theories. The following section deals with the different proposed mereotopologies, their current status, and some intrinsic problems of these axiomatizations.

1. Introduction to Mereotopology

1.1. Topology. Mereotopology in general is a composition of topological (from Greek *topos*, "place") notions of connectedness with mereological (from Greek *méros*, "part") notions of parthood. In mathematics, topology has reached a certain level of maturity in the last century with a common understanding of its basic concepts. By topology we mean specifically the area known as general topology or *point-set topology*, whereas other divisions of topology such as algebraic and geometric topology are not considered at all. Point-set topology relies as the name implies on traditional set theory and uses all its well-known concepts such as containment (subsets), intersection, union, and Cartesian products. General topology extends these set-theoretic notions with concepts of interior, closure (and hence open and closed sets), limit points and neighborhoods but most importantly with the concept of connectedness. In this thesis, topology is illuminated from the perspective of *topological spaces* in chapter 3. This is a very natural approach, since the intended models of the axiomatization from [**AV95**] that we are interested in (called RT_T), are constructed using a topological space $\mathcal{T} = \langle S, \mathcal{O} \rangle$ on the set of points S (the *underlying set*, named X in [**AV95**]) and the topology $\mathcal{O}$ on it. Different notations for topological spaces are common. We stick to the notation of Munkres [**Mun00**]. For basic definitions and theorems for topological spaces, we refer to section 1 on page 24 in chapter 3 (topological characterization). More details on point-set topology and topological spaces can be found in [**HY88, Mun00, Men90**].

1.2. Mereology. Mereology is a much younger discipline and evolved from a philosophical logic perspective. It investigates the relations between parts, in particular parthood structures and relative complementation. Earliest references reach back to the beginning of the 20th century, e.g. Edmund Husserl ("Logical investigations" 1901 [**Hus01**]) and Alfred N. Whitehead [**Whi29**]. The term mereology itself was coined in the 1920s by Stanisław Leśniewski in his work in Polish, translated in [**Lus62**] and analyzed in [**Grz55**]. A more formal approach was conducted by Leonard and Goodman in [**LG40**] in 1940. The latter two works are now accepted as classical examples for extensional mereology. Now "Parts" by Simmons [**Sim87**] is the standard reference for mereology. Casati and Varzi approach the topic in a more rigorous way and give a classification of varying strength of mereology based on the properties of closure, extensionality, and atomism [**CV99**]. The basic relation in mereology is that of an part to a whole. This is most commonly expressed as *proper part*, $<$, where no part is a proper part of itself (irreflexive partial order). For convenience, the reflexive parthood relation $\leq$ is often used as well (reflexive partial order). Such a reflexive parthood relation is also referred to as *Ground Mereology* [**Var96**]. Rewritten using first-order logic, a ground mereology is defined in the following way.

DEFINITION 2.1. [**Var96**] Ground Mereology **M** is the theory defined by the following proper axioms for the parthood predicate P:

$$\forall x\, [P(x, x)] \tag{P1}$$

$$\forall x, y\, [P(x, y) \wedge P(y, x) \rightarrow x = y] \tag{P2}$$

$$\forall x, y, z\, [P(x, y) \wedge P(y, z) \rightarrow P(x, z)] \tag{P3}$$

In addition, most mereologies define the concepts of *overlap*, *union*, and *intersection* of two individuals. General sums and products, i.e. the union and intersection of arbitrary many individuals, also referred to as *fusion*, are also widespread. Mereological theories define a *whole*, i.e. an individual that everything else is part of - the so-called domain or universe, and relative to the whole a complement for every element. More controversial is whether any such theory may consist of *atoms*, i.e. individuals without proper parts. Atoms are indivisible not in the physical sense, but on a more abstract level so that they are the smallest parts of interest. Some theories are atomless while others explicitly force the existence of atoms [**Sim87**]. The same controversy is inherent in mereotopology as we will see shortly. Mereotopological theories can as well be defined atomless, or atomic, or making no assumption about atomism at all.

1.3. Mereotopology = Topology + Mereology. Neither topology nor mereology are by themselves powerful enough to express part-whole relations. Topology can also be seen as a theory of wholeness, but has no means of expressing parthood relations. Connection does not imply a parthood relation between two individuals, as well as disconnection does not prevent parthood. Just consider the example of countries - there exist many countries, e.g. the United States, that are not self-connected. Alaska should be considered part of the United States but is by no intuitive means connected to the other states. The same applies for Hawaii, although the kind of separation is different here: Alaska is separated by Canada from the continental US, whereas Hawaii is solely separated by the Pacific ocean. If we consider landmass only, then Alaska and the continental US are part of a self-connected individual, namely continental North America, whereas Hawaii is separated from this landmass. On the other hand mereology is not powerful enough to reason about connectedness. As the previous example shows, two individuals being part of a common individual does not imply that this sum is self-connected. Hence, parthood is not sufficient to model connectedness.

Consequently, to be able to reason about self-connected individuals, ways to combine mereology with topology are necessary. Previously, Casati and Varzi [**CV99**] classified mereotopologies by the way how the two independent theories are merged. One way to bridge this gap between them is the extension of mereology with a topological primitive [**Var94**]. Most notably the work by Tarski [**Tar56**] employs this approach as does Smith's works on mereotopology [**Smi96**]. Tarski [**Tar56**] and a recently proposed simplification by Bennett [**Ben01**] define a proper part predicate as mereological primitive (as defined in the mereology by Leśniewski, see [**Grz55**]) extended by the topological primitive of a sphere (arising from Euclidean geometry). Smith [**Smi96**] uses the reflexive parthood relation for defining mereology, extended by a mereotopological primitive of an interior part (comparable to a non-tangential part in Clarke's [**Cla81**] and Asher and Vieu's [**AV95**] mereotopologies). Clearly, mereology alone cannot define the notion of interior part, since interior requires some concept of connection or neighborhood. [**CV99**] presents a more general approach that extends each mereological concept by a corresponding interior or internal concept, e.g. for the parthood relation they define an internal part, for overlap an internal overlap relation, etc.

The second approach for merging mereology and topology is the reverse of the previous: taking topology as basis and defining mereology on top of it using only topological primitives. The argument given for such a definition is the greater generality of topology. Topology is assumed to be the more powerful of the two theories, subsuming mereology in its entirety [**Var94**]. Whitehead in *Process and Reality* [**Whi29**] and De Laguna were the first to use this paradigm. Today, it is the most widespread approach: Clarke choose it for his *Calculus of Individuals* [**Cla81**], and all the work based on Clarke inherited his assumptions, e.g. Asher and Vieu [**AV95**], the *RCC* by the group around Randell, Cohn, and Bennett [**RCC92, CBGG97a, Ben01**], Gotts [**Got94**], and Pratt and Schoop's polygonal mereotopology [**PS97**]. Due to the same origin all of these works use a single primitive of connectedness (or contact) and express parthood in terms of connection which limits the expressiveness. The *RCC* is a more simplified framework compared to [**Cla81**], and [**AV95**]: it does not distinguish individuals from their interiors and closures. The authors argue that such a distinction is unnecessary for spatial reasoning aspects. But surprisingly they still distinguish tangential and non-tangential parts as well as overlap and external connection.

A third, less common suggestion how to combine topology and mereology was presented by Eschenbach and Heydrich in [**EH95**]. They employ the mereological framework of Leonard and Goodman [**LG40**] and claim it to be more general than topology and thus better extendable to mereotopology. However, they extend the classical mereology by quasi-mereological notions such as overlap (quasi-mereological because it combines mereology with some topological idea, e.g. connection) in order to be able to define wholeness. Apart from a mereological primitive (discreteness), [**EH95**] proposes to use the unary predicate *region* instead of *connection*. The authors show how they are able to define Clarke's theory by such an extended mereology.

Most mereotopologies are described in terms of first-order axioms. However, they either lack proofs of soundness and completeness [**Cla85, Smi96, BGM96, Var96**] or the proof is based on a rephrased model definition as in [**AV95**]. Only the *Region Connection Calculus* [**RCC92, CBGG97a**] and the framework of Pratt and Schoop [**PS97**], which is limited to planar polygonal mereotopology, provide formal proofs that actually give insight into the possible models. But to better understand the relation between different mereotopologies, we need to identify their actual models and compare them to each other other. To achieve that we use a strict analysis of the models from the axioms alone instead of interpreting the theories and their models on a high level.

2. Axiomatization RT_0 of Asher and Vieu

2.1. First-Order Logic. For all axiomatizations a standard first-order logic with equality is used. We usually quantify all free variables, in cases they are not fully quantified, universal quantification is assumed. For negation we use the symbol $\neg$, for conjunctions $\wedge$ and for disjunctions $\vee$. Implication is denoted by $\rightarrow$. Equality uses the standard symbol $=$ and equivalence (two-directional implication) is written as $\equiv$. All of the relations we consider are predicates in the logic, i.e. C, P, PP, O, EC, $WCont$, etc. are all binary relations, whereas OP, CL, Con are unary relations. Subset and superset relations are also considered binary predicates. Interiors (*int* or simply i), closures cl (or c), and complements $\sim$ denote functions. We use a^* as a constant describing the (unique) universal individual as used by Asher and Vieu [**AV95**]. Later we will see that this corresponds to the set S on which the topological space $\langle S, \mathcal{O} \rangle$ is defined (see chapter 3 on page 23).

2.2. The theory RT_0. The first-order theory defined by Asher and Vieu [**AV95**], called RT_0, uses a single primitive, the *connection* (also called contact) relation C. The authors use as basis the theory of Clarke's *Calculus of Individuals* as proposed in [**Cla81**] and extended in [**Cla85**]. However, some changes are applied to make the theory first-order definable: (1) the explicit fusion operator is eliminated, [**AV95**] claims it unnecessary; and (2) the concept of *weak contact*, $WCont$, is added. Moreover, the theory aims to eliminate trivial models by requiring at least an *external connection* as well as a *weak contact*. Two ontological and cognitive criticisms are also addressed. Contrary to Clarke, the proposed theory prevents individuals having a set of individuals as parts by having only individuals (also called elements or regions) as elements for the first-order theory. Finally an awkward model arising from Clarke's axiomatization that contains two (or more) externally connected individuals whose interiors amount to the entire space is fixed, see figure 4.1 in chapter 3. Oddly, this model is despite its external connection between the two largest elements considered disconnected in Clarke's system. It will be connected in Asher and Vieu. Notice that we show later that this model is a model only of a subset of axioms proposed by Asher and Vieu, but it is in the class of trivial models eliminated in the fully theory RT_0.

Notice that the theory RT_0 does not contain an explicit mereology RT_P, i.e. it does not define a set of axioms that is limited to the parthood relation. In this sense, Asher and Vieu's mereotopology follows the strategy called "Topology as Basis for Mereology" in [**CV99**] that is based on the assumption that topology is more general and fundamental than mereology. The parthood relation P is defined in terms of the primitive connection relation C. This limits the theory in the sense that within the theory of RT_0 we cannot express stronger notions of parthood than what we can express with connection. Due to the single primitive C, parthood between elements is completely defined through the connectivity of individuals. For a more elaborate discussion on the philosophical issues related to this kind of axiomatization, we refer the reader to [**CV99**] and [**Var07**].

2.3. Essential concepts. Asher and Vieu define a set of concepts that are considered essential to build models of their mereotopological ontology. By essential we mean that these definitions are necessary to read the axioms A1 to A13. In other words these definitions are not just a conservative extension of A1 to 13, but essential part of the theory. On the contrary are the enhancing concepts covered in the next subsection conservative extensions - they give us no new restrictions for the extensions of the here defined concepts (and thus for C).

Most of these concepts have been previously defined by Clarke [**Cla85**]. Others [**BGM96, Smi96, RCC92**] define instead of *tangential part* and *non-tangential part* the concepts of *tangential proper part* (TPP) respectively *non-tangential proper part* (NTPP). All of these concepts are necessary to understand the axioms. In this sense they are used in the fashion of macros. The other relations, amongst others *self-connectedness*, are not necessary for building the models. Instead they are enhancing in the way that they are desired to describe intuitive notions in the domain, but the models are not changed by them but just interpreted in a certain (intended) way.

Part P(x,y). A traditional (reflexive) partial order (transitive, symmetric) that corresponds to the axioms of Ground Mereology (see definition 2.1 on page 8) as defined by Casati and Varzi [**CV99**].

(D1) $$P(x,y) \equiv_{def} \forall z\,[C(z,x) \rightarrow C(z,y)]$$

Overlap O(x,y). Two individuals overlap if they have some part in common. In a spatial or physical theory it would mean that they occupy the same space, e.g. a room overlaps with the building it is part of in the sense that whatever space the room occupies, the building occupies as well. This example shows that parthood is a special case of overlap, whereas we can think of *partial overlap* in the sense that neither individual is part of the other, but both have some part, i.e. a proper part, in common. For example, the Rocky Mountains and Canada overlap, since obviously the Rocky Mountains do not take up the entire country of Canada, but on the other sides there are parts of the Rocky Mountains that are outside Canada, e.g. in the United States. Note that for overlap to hold, two individual actually have to have a named individual in common.

(D3) $$O(x, y) \equiv_{def} \exists z\, [P(z, x) \wedge P(z, y)]$$

External Connection EC(x,y). External connection is understood as two individuals being connected by their borders only, so that they have no common part. Hence, it is necessary that the sets of parts of the two individuals are disjoint. Moreover, we will later see that this implies that their interiors are disconnected.

(D4) $$EC(x, y) \equiv_{def} C(x, y) \wedge \neg O(x, y)$$

Non-Tangential Part NTP(x,y). A non-tangential part is intuitively a part that does not touch the border of the greater individual. Special cases must be considered for open individuals, since they naturally have no border that is included in the set of points they are defined over.

(D6) $$NTP(x, y) \equiv_{def} P(x, y) \wedge \neg\exists z\, [EC(z, x) \wedge EC(z, y)]$$

Interior i(x), Open OP(x). The interior $i(x)$ is conceived to be the greatest open part (not necessarily proper) y of x, i.e. $\langle x, y\rangle \in P$. If the individual itself is open, i.e. $\langle x\rangle \in OP$, then it should be interior of itself. This is equivalent to the definition of interiors in topological spaces, compare to fact 3.4 on page 25.

(D8) $$OP(x) \equiv_{def} x = i(x)$$

Since parthood P is reflective, i.e. $\forall x P(x, x)$ holds, it is noteworthy to observe that open individuals are then in fact non-tangential parts of themselves whereas closed individuals are tangential parts of themselves. However, *clopen* individuals, i.e. elements that are both open and closed, only non-tangential part of themselves. One might expect them to be both tangential and non-tangential part of themselves. On the contrary, every individual that is neither open nor closed is a tangential part of itself which matches our intuition that it must consist of some part of the border.

Closure c(x), Closed CL(x). Closure $c(x)$ defined by D7, and closed $CL(x)$ defined by D9 are interpreted in the standard topological sense, i.e. the complement of an open individual is closed. Intuitively, the closure of an individual includes its boundary. Exceptions apply to topological wholes, since these are elements that are open and closed but have no real boundary.

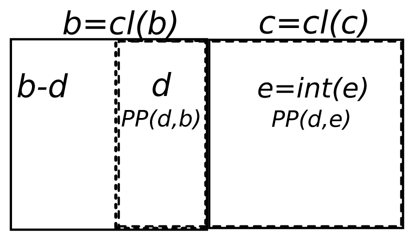


FIGURE 1. Example of weak contact: $\langle \mathbf{b}, \mathbf{c} \rangle \in WCont^{\mathcal{M}}$
Only **d** and $\mathbf{a}^*$ are open elements $\geq \mathbf{b}, \mathbf{c}$ and both are connected to **b** and **c**.

Weak Contact WCont(x,y). Weak contact $WCont$ uses the definition of the closure $c(x) =_{def} -i(-x)$ which itself uses the complement of i and the complement of x. The complement of x is defined for each $x \neq a^*$ and $c(x)$ does only exist if $x \neq a^*$. Intuitively, weak contact requires the closures of two individuals x and y to be not connected, but any greater neighborhood that the closure of one of the individuals is contained in to be connected to the other individual. The definition of weak contact might only deviate from our common-sense understanding of two elements touching each other without sharing common borders when we consider disconnected spaces.

(D11) $$WCont(x,y) \equiv_{def} \neg C(c(x), c(y)) \wedge \forall z\, [(OP(z) \wedge P(x,z)) \rightarrow C(c(z), y)]$$

2.4. Axioms. The theory RT_0 is defined by the following axioms.

Reflexivity of C

(A1) $$\forall x\, [C(x,x)]$$

Symmetry of C

(A2) $$\forall x, y\, [C(x,y) \rightarrow C(y,x)]$$

Idempotence of C

(A3) $$\forall x, y\, [\forall z\, (C(z,x) \equiv C(z,y)) \rightarrow x = y]$$

Existence of an universally connected individual u, commonly referred to as a^*.

(A4) $$\exists x \forall u\, [C(u,x)]$$

Existence of a sum for any pair of individuals.

(A5) $$\forall x, y \exists z \forall u\, [C(u,z) \equiv (C(u,x) \vee C(u,y))]$$

Existence of an intersection for any pair of overlapping individuals.

$$\text{(A6)} \qquad \forall x, y\,[O(x,y) \rightarrow \exists z \forall u\,[C(u,z) \equiv \exists v\,(P(v,x) \wedge P(v,y) \wedge C(v,u))]]$$

Part of what this axiom expresses is: The overlap of two individuals requires that these individuals have a common part that is connected to both of them. Thus we can consider a special case of this axiom (it is weaker than the axiom itself): no matter what we choose for z the most inner term must hold for $u = z$ s.t.:

$$\text{(A6')} \qquad \forall x, y\,[O(x,y) \rightarrow \exists z\,[C(z,z) \equiv \exists v\,(P(v,x) \wedge P(v,y) \wedge C(v,z))]]$$

$$\text{(A6'')} \qquad \forall x, y\,[O(x,y) \rightarrow \exists z, v\,[C(z,z) \equiv (P(v,x) \wedge P(v,y) \wedge C(v,z))]]$$

The existence of a (unique) complement for each individual.

$$\text{(A7)} \qquad \forall x\,[\exists y\,(\neg C(y,x)) \rightarrow \exists z \forall u\,[C(u,z) \equiv \exists v\,(\neg C(v,x) \wedge C(v,u))]]$$

The existence of a (unique) interior for each individual.

$$\text{(A8)} \qquad \forall x \exists y \forall u\,[C(u,y) \equiv \exists v\,(NTP(v,x) \wedge C(v,u))]$$

Defining the closure operation c as a function.

$$\text{(A9)} \qquad c(a^*) = a^*$$

The intersection of open individuals is also open.

$$\text{(A10)} \qquad \forall x, y\,[(OP(x) \wedge OP(y) \wedge O(x,y)) \rightarrow OP(x \cap y)]$$

Existence of two externally connected individuals.

$$\text{(A11)} \qquad \exists x, y\,[EC(x,y)]$$

Existence of two individuals of weak contact as defined by $WCont$. Weak contact is understood in the sense that e.g. two objects or regions touch without being connected. A common example is a glass on a table: the glass and the table do not share any boundary, they are closed (at least to this side), but there is also no other object (even air) between the glass and the table it stands on.

$$\text{(A12)} \qquad \exists x, y\,[WCont(x,y)]$$

Existence of a unique open neighborhood for every element.

$$\text{(A13)} \qquad \forall x \exists y\,[P(x,y) \wedge OP(y) \wedge \forall z\,[(P(x,z) \wedge OP(z)) \rightarrow P(y,z)]]$$

2.5. Enhancing concepts. Here we list the concepts that are defined as part of the theory RT by Asher and Vieu, but that are not necessary for the model construction. The extensions of these concepts can be directly derived from the extension of the other concepts. Since no axioms are enforcing restrictions onto them, we can wait for their evaluation until all other extensions are determined. In this sense they are enhancing - they try to capture some intuitive notion but are not directly used in the axioms.

Proper Part PP(x,y). It is the irreflexive subset of the extension of parthood, i.e. $PP(x,y) \rightarrow P(x,y)$ and $P(x,y) \wedge x \neq y \rightarrow PP(x,y)$.

(D2) $$PP(x,y) \equiv_{def} P(x,y) \wedge \neg P(y,x)$$

Tangential Part TP(x,y). It is the counterpart of the before-mentioned relation non-tangential part. In this sense, every parthood relation either describes a tangential- or non-tangential parthood relation as well show by a theorem later on.

(D5) $$TP(x,y) \equiv_{def} P(x,y) \wedge \exists z \, [EC(z,x) \wedge EC(z,y)]$$

Self-connectedness CON(x). The definition of a self-connected individual x, i.e. $\langle x \rangle \in CON^{\mathcal{M}}$ in D10 is a variation of Clarke's definition [**Cla81, Cla85**] used to classify individuals as being self-connected or disconnected. For an element to be disconnected in means it can be partitioned into at least two sets of parts where there is no connection between them. Self-connectedness is one of the motives for the introduction of mereotopologies, since it describes a notion that neither parthood nor connectedness alone can express by itself. However, for the model-theoretic analysis it is of less importance, since it poses no actual restrictions on the models. Its extension can be determined after a model has been constructed.

(D10) $$Con(x) \equiv_{def} \neg \exists y, z \, [x = y + z \wedge \neg C(cy, cz)]$$

3. Intended Models RT_T of Asher and Vieu

In the original paper [**AV95**], the intended models of the axiom system RT_0 that we just summarized are given in terms of topological spaces. Each model must be build from a non-empty topological space (X, T) with T denoting the set of open sets of the space and with all standard definitions of interior and closure operators *int* and *cl* as well as open and closed as properties in the topological interpretation and $\sim$ as relative complement with respect to X. The (intended) mereotopological models are then defined as the structures $RT_T = \langle Y, f, [\![]\!] \rangle$ where the set Y must meet the following conditions (i) to (viii).

(i) $Y \subseteq \mathcal{P}(X)$ and $X \in Y$

(ii) $\forall x \in Y \, (int(x) \in Y \,\&\, int(x) \neq \emptyset \,\&\, int(x) = int(cl(x)))$ (full interiors)

(iii) $\forall x \in Y \, (cl(x) \in Y \,\&\, cl(x) = cl(int(x)))$ (smooth boundaries)

(iv) $\forall x \in Y \, (int(\sim x) \neq \emptyset \rightarrow \sim x \in Y)$

(v) $\forall x, y \in Y \, (int(x \cap y) \neq \emptyset \rightarrow (x \cap^* y) \in Y)$

(vi) $\forall x, y \in Y\ ((x \cup^* y) \in Y)$

(vii) $\exists x, y \in Y\ ((x \cap y) \neq \emptyset \,\&\, int(x \cap y) = \emptyset)$

(viii) $\exists x, y \in Y\ ((cl(x) \cap cl(y)) = \emptyset \,\&\, \forall z \in Y\ [(open(z) \,\&\, x \subseteq z) \rightarrow y \cap cl(z) \neq \emptyset])$

where the operations $\cup^*$ and $\cap^*$ are defined in the following way;

(EQ1) $$x \cup^* y = x \cup y \cup int(cl(x \cup y))$$

(EQ2) $$x \cap^* y = x \cap y \cap cl(int(x \cap y))$$

With respect to this characterization of the models, Asher and Vieu provide a completeness and soundness proof in their paper [**AV95**]. However, the proof is not very helpful for understanding the models of the theory.. The conditions as stated above are a mere rephrasing of the axioms. We can easily show how the conditions are reflected in the axioms in a mapping one by one. Only the connections structures characterized by A1 to A3 are defined independently of these conditions.

(1) Condition (i) translates to axiom A4 as the existence of a universally connected element (the set X is an element of the theory);
(2) Condition (ii) entails the existence of a non-empty interior of all elements like A8 does;
(3) Condition (iii) entails the existence of a closure for all elements which is implicitly given by the definition D7 of the closure in terms of the uniquely identified interiors and complements, together with A9 to ensure that the universal element has a closure;
(4) Condition (iv) directly entails the existence of a unique complement for every element in Y;
(5) Condition (v) guarantees the existence of a unique intersection for any pair of elements as long as this intersection is not empty. This is a translation of axiom A6;
(6) Condition (vi) guarantees the existence of unique sums for every pair of element and is thus equivalent to A5 in RT_0
(7) Condition (vii) entails the existence of at least a pair of externally connected elements which is equivalent to A11 if we replace it with the definition of EC;
(8) Condition (viii) requires the existence of a pair of weakly connected elements in the same way A12 formulates it with the definition of weak contact from D11.

But this characterization of the models of RT by their intended models is far from being sufficient in order to evaluate these models. The conditions as given do not form any known properties from which we can derive common properties and theorems about the models. For instance, it is not clear how the conditions relate to each other and put additional, non-obvious constraints on another. For this reason, we want to get a better understanding of the models by characterizing them as some class of well-known and well-understood structures.

For the rest of the thesis we rely on the model-theoretic equivalence of RT_T and RT_0. For this reason, we refer to the system simply as RT where the reader can choose which definition is preferred or more adequate. We use both RT_T and RT_0 for the proofs depending on which allows a more elegant solution. Usually the characterization RT_T can be applied in a more direct way. In chapters 4 and 5 we only consider finite models. By showing that every finite model of RT^- (a subset of RT_0) gives rise to a finite model of a restricted RT_T, we prove equivalence between the finite models of RT^- and the restricted models of RT_T.

4. JEPD Relations

For a first step of a characterization of the models of RT_0, we try to find a set of jointly exhaustive, pairwise disjunct (JEPD) basic relations. This has been done successfully for the *RCC* [**RCC92, CBGG97b**] resulting in a hierarchy with a set of 8 basic relations (called RCC-8): PO (partial overlap), TPP (tangential proper part), NTPP (non-tangential proper part) and the inverse of both, equality (i.e. “x is part of y” and “y is part of x” entails that x and y are equal), EC (external connection) and DC (disconnected) [**RN07**]. For these eight basic relations of *RCC* it is easily provable that they satisfy the JEPD condition. Similar basic relations have been found for Allan’s interval calculus [**All83**] and for Schlieder’s oriented line segment calculus [**Sch95**]. For Egenhofer’s mereotopological framework [**Ege91**] based on boundaries, nine such basic relations have been detected.

However, the semantics of the primitive C in the *RCC* deviates from Clarke’s conception [**Cla85**]: two individuals are considered connected in *RCC* if their closures share a common point. On the contrary, the individuals itself have to share a common point in Clarke’s and Asher and Vieu’s axiomatization. This leads to a distinction between closed and open regions by the axioms we use here; this distinction is not present in *RCC*. Further implications of this richer theory are discussed in detail in [**CBGG97b**]. Since some of the oddness of Clarke’s axioms has been eliminated by Asher and Vieu, the theorems given by Clarke in [**Cla81**] and [**Cla85**] cannot been taken for granted anymore in the modified theory. The next sections develop a set of JEPD relations for the dyadic relations of the theory RT_0, i.e. C, O, P, TP, NTP, EC and $WCont$. Obviously, the monadic relations

Open OP and Closed CL will not be represented in this JEPD lattice and need special attention.

4.1. Clarke's theorems. A lattice of JEPD relations relies on the subsumption of one relation through another. We start with the relations that are easily identified from the axioms. From the definition of TP and NTP we see that both are subsumed by P. Consequently TP^{-1} and NTP^{-1} (note both TP and NTP are not symmetric, but actually antisymmetric) are subsumed by P^{-1}. The definitions of TP and NTP entail that they and their inverses are both exhaustive subsets of P respectively P^{-1}. Further, we easily see that EC is subsumed by C. Additional subsumption relations are identified by some of the theorems taken from [**Cla81**] as listed below. All the theorems are directly entailed by the axioms A1 to A10 and A13 together with necessary definitions. The proofs are only outlined in parentheses.

Properties of parthood.

T0.5 $\forall x P(x,x)$ (Reflexivity of P, proof by D1)

T0.6 $\forall x,y,z\,[(P(x,y)\wedge P(y,z))\rightarrow P(x,z)]$ (Transitivity of P, proof by D1)

T0.7 $\forall x,y\,[(P(x,y)\wedge P(y,x))\equiv x=y]$ (Antisymmetry of P, requires P and P^{-1} to be disjunct except for the case $P\cap P^{-1}=\mathfrak{I}$, proof by D1 and A3)

T0.37 $\forall x,y\,[TP(x,y)\rightarrow\neg NTP(x,y)]$ (TP and NTP are disjunct, directly follows from T0.7 and D5, D6)

T0.40 $\forall x,y\,[P(x,y)\equiv TP(x,y)\vee NTP(x,y)]$ (TP and NTP are exhaustive with respect to P, proof by contradiction from D5 and D6)

Relation between parthood and connection.

T0.9 $\forall x,y,z\,[(P(x,y)\wedge C(z,x))\rightarrow C(z,y)]$ (from D1: substitute $P(x,y)$ by $\forall w\,(C(w,x)\rightarrow C(w,y))$ and choose $w=z$)

T0.10 $\forall x,y\,[C(x,y)\equiv\exists z\,(P(z,y)\wedge C(x,z))]$(direction $\rightarrow$ follows directly from T0.9 and use D1 for the other direction and choose $z=y$)

T0.11 $\forall x,y\,[P(x,y)\rightarrow C(x,y)]$ (P subsumes C, directly follows from T0.10 with T0.9)

Properties of overlap.

T0.17 $\forall x O(x,x)$ (Reflexivity of O, prove by substituting $O(x,x)$ with its definition from D3 and choosing $z=x$)

T0.18 $\forall x,y\,[O(x,y)\rightarrow O(y,x)]$ (Symmetry of O, prove using D3 for both occurrences of O and choosing $z=x$)

T0.19 $\forall x, y\, [O(x, y) \rightarrow C(x, y)]$ (O subsumes C, prove from D3 and substitute the left side of the equivalence with T0.11, choose $z = x$ and use A1)

Relation between parthood and overlap.

T0.21 $\forall x, y\, [P(x, y) \rightarrow O(x, y)]$ (P subsumes O,prove from D3 with $z = x$)

Properties of external connection.

T0.24 $\forall x\, [\neg EC(x, x)]$ (Irreflexivity of EC, prove from D4, A1, D3, and T0.5)

T0.25 $\forall x, y\, [EC(x, y) \rightarrow EC(y, x)]$ (Symmetry of EC, proof by D4 and symmetry of C and O in A2, T0.18)

Relation between external connection and overlap.

T0.27 $\forall x, y\, [EC(x, y) \rightarrow \neg O(x, y)]$ (EC and O are disjunct, directly follows from D4)

T0.28 $\forall x, y\, [C(x, y) \equiv EC(x, y) \vee O(x, y)]$ (EC and O are exhaustive with respect to C, proof: the direction $\leftarrow$ follows directly from D3 and D4, the direction $\rightarrow$ follows from $EC(x, y) \vee O(x, y)$ by using D4 and thus obtaining $EC(x, y) \vee O(x, y) \equiv (C(x, y) \vee O(x, y)$. Hence $C(x, y) \rightarrow EC(x, y) \vee O(x, y))$

4.2. JEPD lattice. Using these theorems, in particular theorems T0.7, T0.11, T0.37, T0.40, T0.19, T0.21, T0.27, and T0.28, we are able to build the subsumption lattice. Notice that with T0.19 we have a subsumption relation between O and C. We previously noted that C is also subsumed by EC (see definition of EC in D4). Through D4 we know that O and EC are disjunct T0.27. With D4 we entailed through T0.28 that O and EC are jointly exhaustive and disjunct with respect to C.

However, we cannot determine the exhaustive disjoint subsets of the overlap relation O. We already showed that P and P^{-1} subsume O, but by no guarantee exhaustively. Thus we introduce an additional predicate *partial overlap* PO as in [**RCC92**] that contains $\langle x, y\rangle$ if it is neither in the extension of P nor of P^{-1}, but $\langle x, y\rangle \in O$ (not to be mistaken for the *proper overlap* relation as defined by [**BGM96**]).

$$PO(x, y) \equiv_{def} O(x, y) \wedge \neg P(x, y) \wedge \neg P(y, x) \tag{3}$$

Finally, we want to find a place of $WCont$ in the lattice. Its underlying intuition is quiet clear: two individuals that are in weak contact are not connected, nor are their closures, but nothing "separates" their closures. Thus we are tempted to define a disconnected relation $\neg C$ and attach $WCont$ as a child. However, the

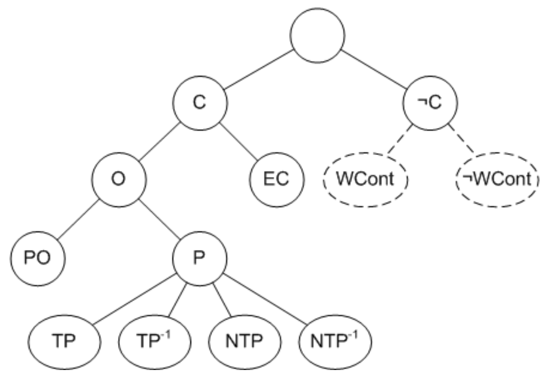


FIGURE 2. Complete subsumption lattice for the dyadic predicates of the mereotopological theory of Asher and Vieu [**AV95**]

model theoretic implication of the axiomatization with respect to the closure of any individual is far from obvious. Instead of drawing premature conclusions, we opt to postpone checking models with respect to $WCont$ at a later stage.

The JEPD subsumption lattice contains seven relations: TP , TP^{-1}, NTP , NTP^{-1}, PO, EC, $\neg C$. Deviating from the RCC relations, we do not have an equality relation as additional basic relation, although it can be easily be defined as the intersection of TP and TP^{-1} as well as NTP and NTP^{-1}. Nevertheless, equality cannot be easily distinguished in the lattice.

5. Subtheories of RT_0

In this thesis we approach the models of Asher and Vieu's axiomatization in a modular way. Because the mereotopology of Asher and Vieu follows the strategy "Topology as Basis for mereology" [**CV99**], it can be broken down into a "core" topology and axioms and definitions that extend it with mereology. In the following chapters, especially the lattice-theoretic part, we consider models of simplified theories that satisfy only a subset of the axioms A1 to A13. Only towards the end of the lattice-theoretic characterization, we work with the full theory RT_0. To distinguish the different subsets of RT_0 (i.e. subsets of the axioms A1-A13 of Asher and Vieu) we use the following notations.

RT_C the topological theory of the connection relation C consisting of the axioms A1 to A3 which corresponds to ground topology (**T**) (see [**CV99**]) that is extensional by axiom A3. Effectively, it is equivalent to a Strong Mereotopology (**SMT**) as defined in [**CV99**] and to a extensional weak contact algebras as defined in [**DW06**], satisfying the axioms C0 - C3 and C5e of [**DW06**], and hence C is a contact relation (see [**DWM99**]).

RT_P — the mereological theory consisting of the axioms A1, A2, and A4-A8. It also assumes the definitions D1 to D4. Notice that axiom A3 is not assumed to hold here. Effectively, it is partial order by P (or irreflexive partial order by PP). It is mainly referred to in the graph-theoretic characterization in chapter 5.

RT^- — $\equiv RT_0 \setminus \{A11, A12\}$. This theory excludes the axioms that require existence of an external connection and existence of a weak contact. Obviously this theory is a superset of RT_0 with equivalent structural properties, but eliminating trivial models (as claimed by [**AV95**]).

RT^-_{EC} — $\equiv RT_0 \setminus \{A12\} \equiv RT^- \cup \{A11\}$. This theory only considers models that contain at least one external connection, but does not put any restrictions on the existence of weak contact. We will later see that the step from RT^- to RT^-_{EC} is the most crucial one as it adds a complexity to the models that is reluctant to be captured nicely.

RT_0 — $\equiv RT^-_{EC} \cup \{A12\}$. It is the full mereotopological theory as defined by Asher and Vieu [**AV95**]. If we explicitly mean the models of RT, we refer to the the set of all structures RT_T.

Throughout the thesis, the unary operation $\sim$ denotes the topological complement as defined in a model of RT_0 or a subset thereof. The unary operation orthocomplement is denoted by $^\perp$ and pseudocomplements are denoted by $'$ as long as they are unambiguous. To differentiate between join- and meet-pseudocomplements, they are referred to as *jpc* and *mpc*, respectively. Variables are written in italic, whereas in models, concrete individuals (or the respective lattice elements or vertices in graphs) are written upright.

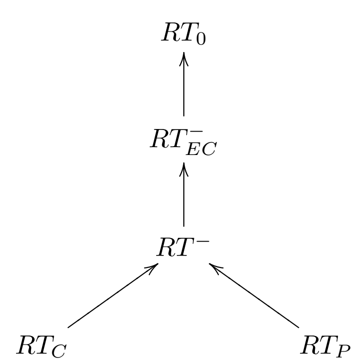


FIGURE 3. Relations between subtheories of the axioms

CHAPTER 3

Topological Characterization

This chapter is dedicated to an analysis of the topological spaces that arise from the mereotopology of Asher and Vieu. This is a very natural approach taking into account the construction of the intended models of RT from topological spaces. So the question arises whether the conditions imposed on the models can be described in terms of the rich taxonomy of separation in topological spaces. All the concepts defined in the models, e.g. complements, interiors, and closures, are directly taken from the underlying topological spaces. However, Asher and Vieu did not restrict themselves to any class of topological spaces. Nevertheless, some common characteristics of topological spaces such as separation is inherent in the conditions and the resulting axioms. Despite, we show why this approaches gives only limited insight into the models and fails to fully characterize them.

The chapter follows the idea of Düntsch and Winter [**DW05**] who characterized the models of the *Region Connection Calculus* [**RCC92, CBGG97a**] in terms of topological spaces. Düntsch and Winter conclude that earlier assumptions of the topological representations of RCC being regular spaces is in fact too strong. Their proof shows that even regularity alone (i.e. T_3) is not always satisfied. This is ascribed to various possible ways of constructing topological spaces from a model of the RCC. Asher and Vieu's mereotopology exhibits the same issue: different topological spaces can be constructed from the theory RT_0 for a single model $\mathcal{M}$. For this reason, we define an embedding topological space that is limited to the necessary points of the topological space (X, T). One could arbitrarily enlarge any point set X with the effect of every closed subset of X containing every additional point.

The first section of this chapter gives an overview over relevant topological concepts and theorems. The following section considers the spaces constructed in the completeness of [**AV95**], whereas section 3 tries to find properties of the resulting spaces of Asher and Vieu using the classical notion of separation axioms. The chapter concludes with a special consideration of the infinite models allowed in RT. Regularity is translated to the topological definitions of semi-regular; in contrast it is shown that regularity cannot be mapped to local connectedness in the embedding spaces. Finally, we compare the results to some previous topological characterizations obtained for the *RCC*.

1. Topological Spaces

This section provides basic definitions and results for topological spaces as found in the standard literature. For more details on general topology we refer e.g. to [**HY88, Mun00, Men90**]. The definitions and concepts are introduced for the purpose of the upcoming characterization of the models of *RT* from a topological perspective.

DEFINITION 3.1. [**HY88**] Let S be a non-empty set and $\mathcal{O}$ a collection of subsets of S, called *open sets*, such that

(O1) $S \in \mathcal{O}$ and $\emptyset \in \mathcal{O}$,

(O2) The union of any number of open sets is an open set,

(O3) The intersection of a finite number of open sets is an open set.

$\mathcal{O}$ is called a *topology* on S, the underlying set, and the ordered pair $\langle S, \mathcal{O} \rangle$ is called a *topological space*. By definition any subset of $\mathcal{O}$ is an *open element* (where $\emptyset$ and S are open). A set $N \in \mathcal{O}$ is called a *neighborhood* of a point $p \in S$ if N contains p, i.e. the set N is an open set containing the point p. A subset $N \subseteq S$ of the topological space $\langle S, \mathcal{O} \rangle$ is defined to be closed if the difference $S - N$ is an open set, i.e. $(S - N) \in \mathcal{O}$. Note that a set can be open and closed at the same time, it is then called *clopen*. Similarly, a subset of S can be neither open nor closed. If the empty set $\emptyset$ and the set S are the only sets that are both open and closed, the space is connected. A characterization of the set of closed sets $\mathcal{C}$ equivalent to the definition of topological spaces can be derives as follows.

THEOREM 3.2. [**HY88**] *The closed sets $\mathcal{C}$ of a topological space $\langle S, \mathcal{O} \rangle$ satisfy the following properties:*

(C1) $S \in \mathcal{C}$ and $\emptyset \in \mathcal{C}$,

(C2) The intersection of any number of closed sets is an closed set,

(C3) The union of a finite number of closed sets is an closed set.

Two important concepts for topological spaces are the *interior* and the *closure* of a subset A of S.

DEFINITION 3.3. [**HY88, Men90**] Let $A \subseteq S$ be a subset of a topological space $\langle S, \mathcal{O} \rangle$. A point p is in the closure of A if for each neighborhood N of p, $N \cap A \neq \emptyset$. The closure of A is denoted by $\bar{A} = Cl(A)$.

The point p is in the interior of A if there is a neighborhood $N \subseteq A$ so that N contains p. The union of all open sets contained in A is called the interior $Int(A)$ of A.

From this definition, the definition of topological spaces, and the closure of closed sets, we immediately can show that the following holds.

THEOREM 3.4. [**Mun00**] *The interior of A is the largest open set N contained in A, i.e. $N \subseteq A$.*

The closure of A is the smallest closed set C containing A, i.e. $C \supseteq A$.

Obviously, the subsumption relation $Int(A) \subseteq A \subseteq Cl(A)$ always hold for any $A \subseteq S$. Moreover, the following fact follows immediately.

THEOREM 3.5. [**Bou66**] *A set A is open if and only if it coincides with its interior, $A = Int(A)$.*

A set A is closed if and only if it coincides with its closure, i.e. $A = Cl(A)$.

Further concepts and theorems are introduces as needed.

2. Embedding Topological Space

In the upcoming considerations, we focus on the embedding of the structure $RT_T = \langle Y, f, [\![]\!] \rangle$ in a topological space as stated in the completeness proof of Asher and Vieu [**AV95**]. In particular, we consider the set of points $X = a^*$ where a^* denotes the unique universal individual in a model of RT_T that is identified through axiom $A4$. Using the notation of the original paper, we choose $X = \Sigma_U =_{def} \bigcup \{\Omega_{[c_n]} | c_n \in \Sigma_C\}$, the set X over which the space ranges being the union of all points occurring in any of the relevant individuals (an individual is represented by the equivalence class $[c_k]$ over constants Σ_C occurring in the sentences Σ of a RT_0-consistent saturated set) of a specific model of RT_0. Intuitively, we understand the set of points X of the topological space to contain exactly those points that are contained in some set contained in Y. The set of points $\Omega_{[c_n]}$ for each equivalence class $[c_n]$ representing an individual in the theory is defined in terms of ultrafilters in [**AV95**]. Each set of points is chosen maximal with respect to the set of points that could be chosen. Notice the difference between finiteness in the models of RT_0 and finiteness of the point set X which underlies the intended model as captured by RT_T. For finite models of RT_0, meaning that the extension of $C^{\mathcal{M}}$ is finite and equivalently that the number of individuals is finite, one can still construct an underlying topological space over an infinite point-set. However, as we will see, the resulting space is not T_0. Vice versa, topological spaces over infinite point sets do not necessarily yield models with infinite number of individuals.

Analogously, we can define the topology that is created when embedding a structure RT_T in a topological space as:

$$\text{(EQ1)} \quad T = \Sigma_U^T = \{\emptyset\} \cup \left\{\Omega_{[c_n]} | c_n \in \Sigma_C \wedge \Sigma \vdash OP(c_n)\right\} \cup \left\{\bigcup Z | Z \subseteq \left\{\Omega_{[c_n]} | c_n \in \Sigma_C \wedge \Sigma \vdash OP(c_n)\right\}\right\}$$

This is simply the empty set (explicitly not needed in structures of the mereotopology but required in any topological space by definition, see axiom O3 of definition 3.1), together with the set of open sets contained in Y of the structure RT_T (the sets that are open in a model of RT_T by being in the extension of OP) and arbitrary unions of sets of open sets of Y (last part of the above definition). Notice that by the last part it is implied that the union of any possible combination of open sets is included. Then this topology T is uniquely identified for each given set X and point sets $\Omega_{[c_n]}$. Since the points included in the equivalence classes can be chosen freely, the topology T is not uniquely identified through X and the classes $[c_n]$ alone. The identification of the points included in $\Omega_{[c_n]}$ for each equivalence class $[c_n]$ is crucial.

We summarize that every model of RT_0 is embeddable in some topological space. This is an expected result, since the completeness proof of [**AV95**] shows elementary equivalence between the models of RT_0 and RT_T, i.e. every sentence Φ consistent with RT_0 can represent a model of RT.

3. Separation Axioms

DEFINITION 3.6. [**HY88**] Separation axioms T_x:
(Axiom T_0) Given two points of a topological space S, at least one of them is contained in an open set not containing the other.

(Axiom T_1) Given two points of S, each of them lies in an open set not containing the other.

(Axiom T_2, Hausdorff axiom) Given two points of S, there are disjoint open sets, each containing just one of the two points.

(Axiom T_3) If C is a closed set in the space S, and if p is a point not in C, then there are disjoint open sets in S, one containing C and the other containing p.

Notice that a space satisfying axiom T_3 does not necessarily satisfy T_1. A space satisfying both T_1 and T_3 is called a T_3-space or *regular space* [**HY88**].

We call the topological space $(X, T) = (\Sigma_U, \Sigma_U^T)$ the *embedding space* that RT_T yields for a given T. It is minimal with respect to the set X, but not minimal in the number of points it contains. If any of the atomic elements is assigned a set of points, then these points will be topologically indistinguishable unless each point is a closed set by itself and thus each subset of points in X is an open set. Otherwise the indistinguishable points in an atom prevent the topology even from satisfying T_0.

If we have an infinite number of atoms in a model of RT, then we need an infinite set of points in X as well. On the other extreme, assign each atomic individual exactly a single point and assign each individual containing proper parts the sum of the points contained in their proper parts (with any additional point if it has only a single proper part in order to distinguish it from its single proper part; the same applies if two sets have the same proper parts, then each needs an extra point to distinguish it from another). Then, if the set of atoms is finite in the original model, the set of points will be finite as well. So for finite models, the embeddable space is always a discrete topology over all the points. In general, if the topology T contains all possible unions of subsets of open sets in a model of RT_T, the topology must be discrete if it is T_0. Notice that the universal element is denoted by a constant, e.g. a^*, in any set Σ_C. Since by definition this set contains exactly the points of X, the last part of the definition of T can take unions of arbitrary open sets of points. In particular, if we take each point $p \in X$ as open set, we always obtain the discrete topology that satisfies T_0, T_1, and T_2.

On the contrary, if any set A representing an atomic individual in a model of RT_0 contains more than a single point, and the topology would not include single point sets, then it is obviously not T_0. We can find two points in A, e.g. a_1 and a_2 that are not in distinct subsets of A, and thus are topologically indistinguishable. Note that no other set B with $B \not\supseteq A$ can contain these points: (1) if $int(A \cap B) \neq \emptyset$, then for the intersection it holds $(A \cap^* B) \in Y$, (2) otherwise $int(A \cap B) = \emptyset$ and since A and B share a point, $A \cap B \neq \emptyset$. In case (1) A was not atomic, in (2) A was not in OP, i.e. A is not an open set in the model of RT_0 (A is externally connected to B). In both cases our assumption is contradicted.

So for any finite set of individuals $[c_n]$ or, equivalently, for all the sets of points $\Sigma_{[c_n]}$ of every class $[c_n]$ being finite, the resulting topology must be discrete. Furthermore, it holds in general:

PROPOSITION 3.7. *The complete topological space as defined in the completeness proof of* [**AV95**] *is a discrete topology and thus Hausdorff (T_0, T_1, and T_2) if the model of RT_0 is finite or all atoms are represented by finite point sets.*

The discrete space is also totally disconnected and compact because it is finite.

Considering proper subsets of the union of open sets in the definition of a topology (X, T) on a set X, we can distinguish the following cases. For an infinite set of points $\Sigma_{[c_n]}$ of a single equivalence class $[c_n]$ without the number of classes $[c_n]$ being infinite, the resulting topological space contains indistinguishable points and thus is not T_0.

PROPOSITION 3.8. *If the number of equivalence classes c_n of individuals is finite, but any of its atomic regions contains an infinite number of points $\Sigma_{[c_n]}$, the incomplete topology formed from it does not satisfy T_0.*

This leaves us with the case of a model of RT_0 containing an infinite number of atoms. We analyze this case in more details in the next section.

4. Models With Infinite Sets of Atoms

As we have seen, the only interesting case - in terms of the arising topological spaces - are models on infinite number of individuals, i.e. models that contain an infinite number of atomic individuals. These models are characterized by an infinite number of sets in Y. This case subsumes the case of an atomless model - the most extreme variation of it (with uncountable number of individuals). A special, "atomic" case is a model with infinite number of atoms.

In general we cannot verify whether a topology T as constructed by the embedding is T_0.

The embedding topological space (X, T) is the minimal space with respect to the number of open sets for a given set X and given point sets $\Omega_{[c_n]}$If we assume the topology to be T_0, we can characterize the embedding topological space (X, T) more specifically in terms of semi-regular spaces.

4.1. Semi-regularity.

Semi-regularity is a weaker property than regularity which has been proved for the topological spaces of the *RCC*. Regularity turned out to be too strong for the topological spaces built from the models of *RCC*. This result can be carried over to the models of *RT*, since then all sets must be regular open for which we have no justification.

DEFINITION 3.9. [**SS78, Bou66**] A set A is called *regular open* if $A = Int(Cl(A))$. A set A is called regular closed if $A = Cl(Int(A))$.

Obviously, the open sets in Y of a model of RT_T are regular open as well as the closed sets in Y are regular closed (follows from fact 3.5, the definition of regular open sets (definition 3.9), and the *full interior* condition (ii) of the model definition of RT_T for all open sets A):

$$A = int(A) \supset A = int(cl(A)) \tag{EQ2}$$

$$A = cl(A) \supset A = cl(int(A)) \tag{EQ3}$$

The definition of the models gives a much stronger assumption, the so-called assumption of *smooth boundaries*: every part of the objects in a given n-dimensional space must be n-dimensional as well [**AV95**]. Together with the assumption of *full interiors* [**AV95**] this is equivalent to *regular* in the sense of Cohn et al. [**CBGG97a**]: $Cl(Int(x)) = Cl(x)$ and $Cl(Int(x)) = Cl(x)$. However, in RCC this is an assumption, since no explicit relation between individuals and their respective interior and closure are made. Therefore, it comes for free and is not reflected in the axioms of the theory of RCC [**RCC92**]. The same notion of this interpretation of *regular* is not captured adequately by regular open and regular closed sets as expressed above. Instead of the above equations, Asher and Vieu assume as well the unrestricted version favored by Cohn et al.

Nevertheless, we can prove semi-regularity for the embedding topological spaces from the following simple lemma and its corollary. Semi-regularity is a weaker property of topological spaces than regularity. The topological spaces arising from RCC were in [**DW05**] shown to be semi-regular, but not regular, i.e. not T_3. We can prove semi-regularity only for the infinite models of Asher and Vieu's mereotopology.

DEFINITION 3.10. [**Bou66**] A topological space X is called semi-regular (as well as its topology T) if the regular open subsets form a base of its topology T.

LEMMA 3.11. *The topology T of the topological space (X, T) embedding a model of RT_0 with infinite number of individuals contains only open sets and their unions (also open sets) in Y in the respective structure RT_T.*

COROLLARY 3.12. *For a given model of RT_0 with infinite number of individuals, in the resulting structure RT_T (with infinite Y) the set of open sets in Y defines a base (also called basis) of the topological space (X, T).*

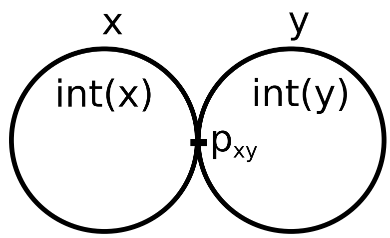


FIGURE 4. Smallest model of RT_{EC}^{-} with the boundary point p_{xy}

THEOREM 1. *For a given model of RT_0 with an infinite number of individuals, the topological space (X, T) is semi-regular.*

PROOF. Recall that theorem 3.5 requires the set of open sets contained in Y to form a base of the topological space X. Since all open sets of Y are regular open, the regular open sets, T, form trivially a base of (X, T). □

EXAMPLE 1. *Consider two closed, externally connected elements $\mathbf{x}$ and $\mathbf{y}$: they need to share a boundary point p_{xy} that lies on the boundary of $\mathbf{x}$ and $\mathbf{y}$, i.e. not in the interiors $\mathbf{int}(\mathbf{x})$, $\mathbf{int}(\mathbf{y})$ but in the closures $cl(\mathbf{x}) = \mathbf{x}$, $cl(\mathbf{y}) = \mathbf{y}$, compare figure 4.1. In short: $p_{xy} \in \mathbf{x}, \mathbf{y}$ and $p_{xy} \notin \mathbf{int}(\mathbf{x}), \mathbf{int}(\mathbf{y})$. With $\mathbf{x}$ and $\mathbf{y}$ being closed, their sum (union) $\mathbf{x} \cup^* \mathbf{y}$ must be closed as well (see theorem 3.2). If we now assume $p_{xy} \notin (\mathbf{x} \cup^* \mathbf{y})$, then there must exist a relative complement of $(\mathbf{x} \cup^* \mathbf{y})$ with respect to X, i.e. an open element containing p_{xy}. Then T_1 is satisfied again for the choice of any pair of points that includes p_{xy} and another point from either $\mathbf{x}$ or $\mathbf{y}$. Contrary, if we assume $p_{xy} \in (\mathbf{x} \cup^* \mathbf{y})$ and $\mathbf{x} \cup^* \mathbf{y}$ being a* clopen *set of points, the complementary open set (since $(\mathbf{x} \cup^* \mathbf{y})$ is a closed set) does not include the point p_{xy}. In other words, the model does not require the existence of an open set that includes p_{xy} but is does not include $\mathbf{int}(\mathbf{x})$ and $\mathbf{int}(\mathbf{y})$. However, due to the closure of a topology under arbitrary unions, the topology must include the set $\mathbf{x} \cup \mathbf{y}$ as an open set.*

This model satisfies A11 ($\mathbf{x}$ and $\mathbf{y}$ are externally connected) and with no additional individuals except for the universal $a^* = (x \cup^* y)$, it is equivalent to the model characterized as $\mathcal{L}_6$ in the lattice-theoretic part, chapter 4 with $\mathbf{x} = \sim \mathbf{int}(\mathbf{y})$ and $\mathbf{y} = \sim \mathbf{int}(\mathbf{x})$. A further extension with a disjoint region containing a weak contact will make it a model of RT without changing the point-wise interpretation of the submodel containing $\mathbf{int}(\mathbf{x}), \mathbf{int}(\mathbf{y}), \mathbf{x}, \mathbf{y}, \mathbf{a}^*$. Notice this model is in fact not T_0 if

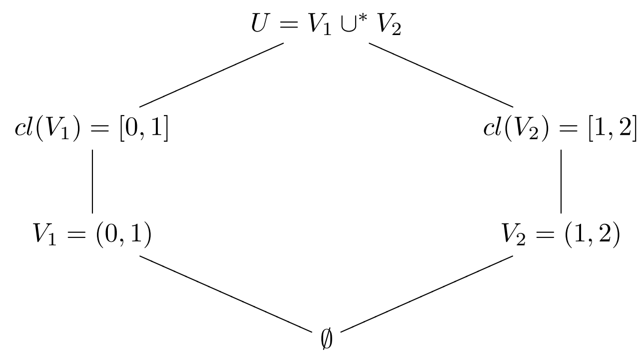


FIGURE 5. Model of RT_{EC}^{-} whose embedding topological space is not locally connected

the set of points for x and y each contain more than a single point. If they both contain exactly a single point, the space is T_0 but not T_1.

However, if we conceive the sets $\mathbf{x}$, $\mathbf{y}$, $\mathbf{x} \cup^* \mathbf{y}$ *as subsets of Y of a model with infinite number of individuals, the unions still form a base for the topology and all open sets in the base are regular open. Hence, the space is semi-regular. This follows immediately from the construction of the topology T'. Moreover, we can easily verify that then the space is T_1 and T_2.*

4.2. Local connectedness. In the previous subsection we showed how the condition of "smooth boundaries" prevents objects of different dimensions. This condition rules explicitly the existence of classical geometric objects of increasing dimensions out, such as points of zero dimension, lines of one dimension, surfaces of two dimensions, etc. Moreover, the condition of "full interiors" was expressed for the models of RT. The notion of local connectedness seems to correspond to this requirement that in an n-dimensional space no individual of fewer dimensions can exist. However, we give a small counterexample that proves this assumption wrong.

DEFINITION 3.13. [**Mun00, HY88**] A space S is *locally connected at a point x* if for every open set U containing x there is a connected open set V with $V \subseteq U$ containing x. The space S is *locally connected* if it is locally connected at each point.

PROPOSITION 3.14. *The minimal topological space (X, T) that embeds a model of RT_0 is not necessarily locally connected even if Y is infinite and each atom in Y is connected.*

EXAMPLE 2. *Consider the following substructure, see figure 4.2: two open sets* V_1 *and* V_2*, e.g.* $V_1 = (0,1)$ *and* $V_2 = (1,2)$ *with explicit closure* $cl(V_1) = [0,1]$ *and* $cl(V_2) = [1,2]$*. Now their sum is* $U = V_1 \cup^* V_2 = (0,2)$ *which is the universal element and therefore is* clopen*. Although this model does not satisfy the existential requirements for external connection and weak contact, it is still valid with respect to all other conditions of* RT_0*. In particular are the conditions (ii) and (iii) satisfied because* U *is* clopen*. Moreover is the resulting topology semi-regular with all the sets* U*,* V_1*,* V_2 *being regular open. Notice however that the two sets* V_1*,* V_2 *alone do not form a basis of the topology. The given example corresponds to the six element lattice described in chapter 4.*

So even if the atoms (in the proof V_1 and V_2) are locally connected, for the entire embedding topological local connectedness is not ensured.

4.3. Relation to RCC. Comparing this to the results of Düntsch and Winter [**DW05**], we are not surprised to see that the models of Asher and Vieu are only T_1 and semiregular (as the models of RCC are) if the number of atoms in the theory is infinite. It is well-known that the RCC is an atomless mereotopology (with most models being even uncountable), therefore all models have an infinite number of individuals [**LYL05, XL06, DWM99, Ste00**]. Contrary, the finite models, i.e. atomic models, of Asher and Vieu cannot be captured properly by an analysis using topological spaces. The forced discreteness of the resulting topologies makes the analysis in the finite case practically worthless.

The notion of weakly regular spaces arises in [**DW05**]. It is not clear yet, whether the minimal topological spaces considered here are weakly regular. However, weak regularity implies semi-regularity [**DW05**], but weak regularity is less stringent than regularity in topological spaces.

DEFINITION 3.15. [**DW05**] A weakly regular space is a topological space (X, T) that is semiregular and for each non-empty set S_1 in T there exists a non-empty set S_2 in T so that $Cl(S_2) \subseteq S_1$.

Düntsch and Winter, however, remark that a weakly regular space can be considered as the pointless version of semiregular spaces. They compare their definition to the notion of inexhaustibility as used in [**LY03**]. Since inexhaustibility is justified by RCC requiring atomless models, it can most likely not be applied to the spaces resulting from RT_0.

5. Conclusion

The topological characterization does not give much insights into the models of the mereotopology. In particular, only atomless theories can be analyzed in a meaningful way using topological spaces. Moreover, the notions of regularity applied by Asher and Vieu as well as the Manchester group are different from regularity in a topological sense.

For the finite models of Asher and Vieu - the ones that are particularly interesting from a knowledge representation and reasoning perspective - the characterization of their corresponding topological spaces always lead to discrete topologies, which are valid spaces but does not uncover many interesting properties of the models of RT. The unsatisfactory satisfactory characterization as topological spaces lets us turn our focus towards other characterization approaches, namely using lattices and graphs in the next chapters. Those structures have their strengths in representing finite models.

Notice that the topological approach is a feasible way to tackle infinite models of the theory RT. Neither lattices nor graphs are very well suited to captures those models. Although - as we see in the next chapter - the lattice properties generalize to infinite cases, the upcoming characterization focuses on finite models. In this light, the results for the special cases of infinite models which are equivalent to atomless instances of the theory are still valuable. From the methodological perspective we showed limitations of the topological characterization of mereotopological theories. It might be useful for those theories that force all models to be atomless (e.g. the RCC), but is only of limited help for finite models. Moreover, the characterizations are very different from those topological spaces usually considered in theory: the mereotopological models of Asher and Vieu are not Hausdorff, whereas mathematicians usually consider the Hausdorff property as very general and differentiate much more restricted spaces.

CHAPTER 4

Lattice-Theoretic Characterization

In this chapter we represent the finite models[1] of Asher and Vieu's mereotopology [**AV95**] as lattices over the set of individuals of a model of RT_0 supplemented by the empty set $\emptyset$ as infimum of the lattice. Recall that the goal in mereotopology is to model part-whole relations. Relative complementation is the most important concept to achieve this: one can express that an individual and its complement together make up the whole. Complementation is a characteristic property of lattices as well. Complementation and modularity are the most important means to classify lattices. Utilizing complementation in lattices allows us to characterize the finite models of RT^- up to isomorphism. For finite models showing isomorphism amounts to proving elementary equivalence to the associated lattices. In general, elementary equivalence is a weaker notion than isomorphism and only expresses that two structures satisfy the same set of sentences of a given language, here first-order logic.

Definition 4.1. [**Mar02**] We say that two $\mathfrak{L}$-languages $\mathfrak{M}$ and $\mathfrak{N}$ are elementarily equivalent if $\mathfrak{M} \models \Phi \iff \mathfrak{N} \models \Phi$.

In particular, we analyze the lattices with respect to the complementation properties collected in Stern [**Ste99**]. More general lattice properties, such as completeness and atomicity give us a homomorphic characterization of the models RT^-_{EC} as complete atomic (doubly) pseudocomplemented ortholattices, or in short complete atomic p-ortholattices On the contrary, unique complementation is explicitly ruled out as property for the models of Asher and Vieu.

In lattice theory, apart from complementation two commonly used properties are distributivity and a weaker form thereof, modularity. Modularity is a necessary but not sufficient condition for distributivity and further weakenings exist as well, such as semimodularity, weak modularity, and orthomodularity [**Ste99**]. A less well-known concept is semidistributivity (see [**Ste99**] for definitions). However, we show that the lattices of the full theory RT_0 are never modular and never distributive.

[1] By finite we mean the models with finite number of individuals, not models that yield finite topological spaces. In particular, as shown in the previous chapter, we can always find an embedding in an infinite space, but with the consequence that the space will not even be T_0.

Some of the weaker properties such as semi-modularity and orthomodularity are not satisfied in any of the lattices whereas weak modularity and semi-distributivity (a weakening of distributivity in a different direction) apply to some of the lattices. Notice that this can be solely attributed to axiom A11 requiring the existence of an external connection. Removing this axiom as done in RT^- gives models that are p-ortholattices, including the models that satisfy any of the before-mentioned specialized modularity properties. For the finite models of RT^-, we prove in section 6 that they are isomorphic to the finite (complete atomic) p-ortholattices.

Contrary to the characterization of Clarke's system obtained by Biacino and Gerla [**BG96**] and the results on RCC obtained in [**DW05**], the models of Asher and Vieu's mereotopology do not simply correspond to Boolean algebras. The difference to Clarke's *Calculus of Individuals* this is not surprising, since [**AV95**] already noted problems with his definition of external connection that allows external connection to be mapped to overlap. Without any externally connected elements, the lattices of Asher and Vieu's mereotopology would not imply strictly non-modular lattices as well. The definitions of external connection of Asher and Vieu and RCC are identical, but since in RCC there is no requirement to always model interiors and exteriors as specified elements of the domain, any element can have external connections, but separate non-open regions (as interiors) are not forced by the axioms. Only elements that are non-tangential parts of themselves cannot be externally connected. This thesis reveals that the definition of external connection influences the models significantly and underlines the necessity to put more emphasis on the evaluation of external connection relations in their respective mereotopologies.

1. Prerequisites

This section introduces the basic concepts used throughout the remainder of the chapter. We show how each model of RT_0 (and thus of RT) can be associated with a unique algebraic structure that is a lattice. Some standard properties of lattices are then carried forward to the models of RT_0.

Definition 4.2. [**Grä98**] A partially ordered set (*poset*) is a binary relation over a set that is reflexive, antisymmetric, and transitive.

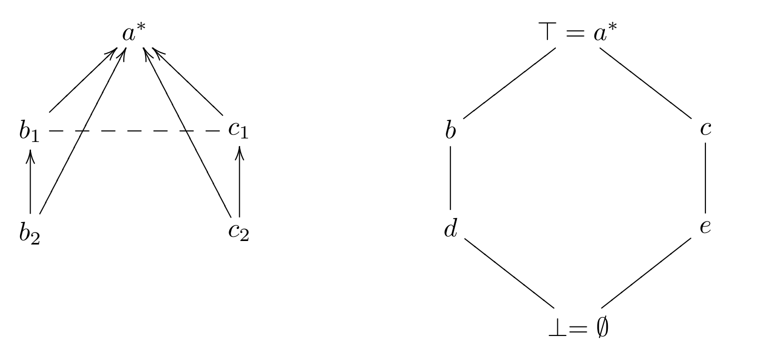
A lattice is a poset in which each pair of elements has a unique supremum (the least upper bound of their join) and a unique infimum (the greatest lower bound of their meet).

Notice that the definition refers to strict partially ordered sets, also called reflexive partially ordered sets as opposed to irreflexive partial orders. In the subsequent work we assume reflexivity when using the term *partial order* or *poset* unless otherwise stated. By definition lattices satisfy associative, commutative, absorption, and idempotent laws [**Grä98**].

We consider the lattices formed from the structures $RT_T = \langle Y, f, [\![]\!] \rangle$ with Y being the set of individuals (or regions, or point sets). There exists then some corresponding model $\mathcal{M}$ of RT_0 with the partial order given by its parthood extension, i.e. $x \leq y \iff \langle \mathbf{x}, \mathbf{y} \rangle \in \mathbf{P}^{\mathcal{M}})$. For any such model of RT_0 we define an algebraic structure $\mathcal{L}$ over the poset $(Y \cup \emptyset, \leq)$ that is a lattice with the join defined as the union of two individuals (as defined by axiom A5) in RT_0, and the intersection being defined as the intersection of two individuals in RT_0 (by A6). Thus, the algebraic structures can be always regarded as lattices

(EQ4) $$\mathcal{L} = (Y \cup \emptyset, \wedge, \vee) \equiv (Y \cup \emptyset, +, \cdot) \equiv (Y \cup \emptyset, \cup^*, \cap^*)$$

The first part expresses uses the classic lattice definition over the poset $Y \cup \emptyset$ (the partial order is implicitly assumed) with $\vee$ being the join (union) operator and $\wedge$ being the meet (intersection) operator. The second notation refers to the same lattice in terms of models of RT_0 (the axioms from section 2) , where $+$ is the union of two individuals (equivalent to the lattice join) and $\cdot$ is their intersection (equivalent to the lattice meet). The third notation refers to the intended models of the structure (defined by the conditions of RT_T, see section 3) in terms of their topological operations “regular sum” and “regular intersection” of two individuals. Elementary equivalence between the theories RT_0 and RT_T has been established by Asher and Vieu through their completeness proof, therefore we can refer to RT_0 and RT_T synonymously as long as we consider the full class. However, for the finite



(a) parthood (directed edges) and external connection (dashed edges) in a a model $\mathcal{M}$ of RT^-_{EC}

(b) lattice L_6 resulting from the model $\mathcal{M}$

FIGURE 6. Example of a lattice construction from a model $\mathcal{M}$ of RT^-_{EC}

models of RT^- and RT^-_{EC} we show there always exist equivalent finite models of subsets of the conditions of RT_T, so we can use the definitions of RT_T instead of RT_0 if we restrict ourselves to finite models of RT_0. Each lattice $\mathcal{L} = (Y \cup \emptyset, \cup^*, \cap^*)$ resulting from a model $\mathcal{M}$ of RT_T uses the union and intersection $\cup^*$ and $\cap^*$ defined by Asher and Vieu as join and meet operators, respectively:

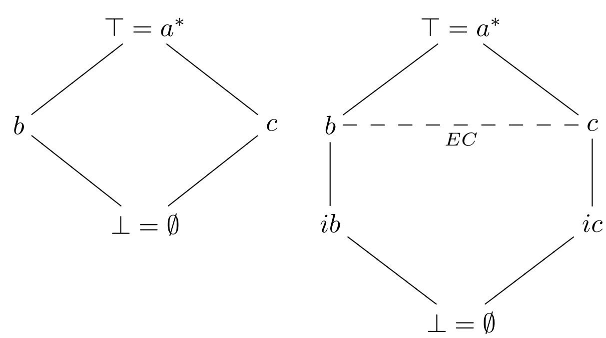
(EQ5) $$x \cup^* y = x \cup y \cup int(cl(x \cup y))$$

(EQ6) $$x \cap^* y = x \cap y \cap cl(int(x \cap y))$$

LEMMA 4.3. *For any model $\mathcal{M}$ of RT, the algebraic structure $\mathcal{L}^{\mathcal{M}} = (Y \cup \emptyset, \cup^*, \cap^*)$ is a lattice.*

PROOF. Clearly, the defined structure is a lattice, since the parthood relation $\mathbf{P}^{\mathcal{M}}$ for any model $\mathcal{M}$ of RT_0 gives a structure of RT_T and hence a poset $\langle Y \cup \emptyset, \mathbf{P}^{\mathcal{M}} \rangle$: We proved previously that the binary relation P is always reflexive, antisymmetric, and transitive (see T0.5, T0.6, and T.0.7 in section 4.1) and thus is the extension $\mathbf{P}^{\mathcal{M}}$ for any model $\mathcal{M}$. That each pair in the poset $Y \cup \emptyset$ has a unique supremum and infimum with the join operation $\cup^*$ and the meet operation $\cap^*$ is enforced by the conditions (v) and (vi) of RT_T. □

Here we see the inherent similarity between lattices and mereotopological systems in general: both are based on a partial order and define unique sums (though not all mereotopological theories do) and unique intersections. A lattice to a single model $\mathcal{M}$ of RT_0 is denoted by $\mathcal{L}^{\mathcal{M}}$ in the following whereas the class of lattices that can be associated to some model of RT_0 will be referred to as $\mathfrak{L}_{RT}$. I.e. $\mathcal{L}^{\mathcal{M}} \in \mathfrak{L}_{RT}$ if and only if $\mathcal{M}$ is a model of RT_0. Since we restrict our analysis to the finite



(a) finite model of RT^- (b) finite model of RT^-_{EC}

FIGURE 7. Finite models of RT^- and RT^-_{EC} as lattices

models of RT_0, we assume every lattice $\mathcal{L}^{\mathcal{M}}$ to be that of a finite model $\mathcal{M}$ unless otherwise stated.

REMARK 2. The question what kind of mereotopological models other latticed structures might yield naturally arises, especially considering meet-semi-lattices. This could give rise to a mereotopology that considers relevant individuals on a coarse scale and makes assumptions about the individuals that exist as their respective intersections, but that does not require the existence of arbitrary sums of pairs of individuals, as criticized frequently in mereotopology. Common examples argue that the sum of e.g. my right thumb with the peak of the CN tower usually do not make sense at all. A meet-semi-lattice could fix this issue by only requiring intersections to exist, but not necessarily sums of every pair of individuals.

1.1. Finite models of RT^- and RT^-_{EC}. First, we claim that there actually exist finite models of the theory RT_0. Notice that the class of lattices $\mathfrak{L}_{RT}$ is non-empty. At the moment we are unable to prove that. But we are able to prove the weaker fact about the classes $\mathfrak{L}_{RT^-}$ and $\mathfrak{L}_{RT^-_{EC}}$.

LEMMA 4.4. *There exist finite, non-trivial models satisfying RT^- and RT^-_{EC}.*

PROOF. The model defined by $\langle a^*, b\rangle, \langle a^*, c\rangle \in C^{\mathcal{M}}$ (with all reflexive and symmetric tuples also contained in$C^{\mathcal{M}}$) satisfies all axioms of RT^- and is of finite domain $\{a^*, b, c\}$ and hence is a finite model of RT^-.

The model defined by $\langle a^*, b\rangle, \langle a^*, ib\rangle, \langle a^*, c\rangle, \langle a^*, ic\rangle, \langle b, ib\rangle, \langle c, ic\rangle, \langle b, c\rangle \in C^{\mathcal{M}}$ (again with all reflexive and symmetric tuples also contained in $C^{\mathcal{M}}$) with $\langle b, c\rangle, \langle c, b\rangle \in EC^{\mathcal{M}}$ satisfies all axioms of RT^-_{EC} and has a finite domain $\{a^*, b, c, ib, ic\}$ and thus is a finite model of RT^-_{EC}. □

For the lattices of these two smallest models of RT^- and RT^-_{EC}, see figure 1.

Now in a second step we show that every model of RT^- and every model of RT^-_{EC} is isomorphic to a finite structure satisfying the conditions (i) to (vi) of RT_T, and (i) to (vii) RT_T, respectively. I.e. any finite model of RT^- is isomorphic to a model of $RT_T = \langle Y, f, [\![]\!]\rangle$ with the conditions (i) to (vi), where Y is a finite set of individuals. The soundness and completeness proofs of [**AV95**] only say that any model of RT_0 is elementary equivalent to a model of RT_T. It does not guarantee that a finite model of RT_0 necessarily yields a finite model of RT_T. That this is true at least for the finite models of RT^- (yielding finite models of (i) to (vi) of RT_T) is shown in the next proof.

THEOREM 2. *Every finite model of RT^{-} is isomorphic to a structure $RT_T = \langle Y, f, [\![]\!] \rangle$ over a topological space (X, T) satisfying conditions (i) to (vi) where $Y \subseteq \mathcal{P}(X)$ and Y is finite.*

PROOF. We follow along the lines of the completeness proof of [**AV95**]. We know that a finite model of RT_0 can be captured by a finite set of (finite)[2] sentences Σ which can only contain a finite number of equivalence classes $[c_n]$. We then can construct the topological space (Σ_U, Σ_U^T) according to the embedding in a topological space, see 2. Reconsider the topology Σ_U^T we can construct

$$\Sigma_U^T = \{\emptyset\} \cup \{\Omega_{[c_n]} | c_n \in \Sigma_C \wedge \Sigma \vdash OP(c_n)\} \\ \cup \left\{\bigcup Z | Z \subseteq \{\Omega_{[c_n]} | c_n \in \Sigma_C \wedge \Sigma \vdash OP(c_n)\}\right\}$$

Since the number of equivalent classes of open elements, i.e. the cardinality of $\{[c_n] \,| c_n \in \Sigma_C \wedge \Sigma \vdash OP(c_n)\}$, is finite, the second set of the union is finite. Moreover are arbitrary finite union of such sets finite as well, and hence Σ_U^T is finite. Then (Σ_U, Σ_U^T) is a structure that satisfies the conditions of RT_T which has been proved for the general case in the completeness proof of [**AV95**].

We claim that there always exists a finite set Y if Σ_U^T is a finite set for the following reason: it is clear that the number of open sets in Y is finite, and so is the number of closed sets in Y (each closed set must be the complement of an open set in Σ_U^T, but only a finite number of these exist). It is less clear that there always exists a model with a finite number of individuals that are neither open nor closed. But recall that $\Sigma_U = \bigcup \{\Omega_{[c_n]} | c_n \in \Sigma_C\}$. If we use the maximal point sets $\Omega_{[c_n]}$, $\Omega_{[-c_n]}$ for the equivalence classes $[c_n]$ and $[-c_n]$, then $X = \Omega_{[c_n]} \cup \Omega_{[-c_n]}$ must always hold. Otherwise either $\Omega_{[c_n]}$ or $\Omega_{[-c_n]}$ are not maximal. But then all sets $\Omega_{[c_n]} \in \Sigma_U^T$ are not only open but also closed, because $\Omega_{[-c_n]} \in \Sigma_U^T$. So every set in the topology Σ_U^T is *clopen*. If we choose the minimal embedding space (Σ_U, Σ_U^T), it will be T_0, and immediately the topology Σ_U^T is discrete (compare to the results from chapter 3). We claim that such topology Σ_U^T with only *clopen* sets always satisfies the conditions (i) to (vi) if we choose $\Sigma_U^T = Y$. By definition of Σ_U and Σ_U^T, condition (i) of RT_T holds. Conditions (ii) and (iii) hold trivially because $int(x) = x$ and $cl(x) = x$ hold for every $x \in Y$. Condition (iv) is satisfied because all sets in Σ_U^T are *clopen*; conditions (v) and (vi) are satisfied because we can find a minimal embedding space that gives a discrete topology Σ_U^T which contains all

[2]Axiom schemata are not allowed in standard FOL

possible subsets of $X = \Sigma_U$ and therefore all unions and all intersections of sets in Σ_U. With finite Σ_U, we thus have constructed a model of RT_T with finite Y. $\square$

Theorem 2 now allows us to use the specifications of RT_T and RT_0 interchangeable even if we restrict ourselves to finite models of RT^-. The same applies for RT^-_{EC}, where the proof is analogue to the above proof, but for all pairs of individuals x, y that participate in the extension of EC, i.e. $\langle x, y\rangle \in EC^{\mathcal{M}}$, the non-empty point sets $\Omega_{xy} = \Omega_{[x]} \cap \Omega_{[y]}$ (which are not closed because their complement is not in Σ_U) representing the intersections $x \cap^* y$ are removed from the finite topology Σ_U to form Y. Notice that they have to be in Σ_U to ensure closure under finite intersections. The new set $Y = \Sigma_U \setminus \left(\Omega_{[x]} \cap \Omega_{[y]} | \langle x, y\rangle \in EC^{\mathcal{M}}\right)$ still satisfies all conditions (i) to (vi), but now additionally condition (vii) because if the extension of $EC^{\mathcal{M}}$ is not empty, i.e. there exists some pair x, y so that $\langle x, y\rangle \in EC^{\mathcal{M}}$, then $\Omega_{xy} = \Omega_{[x]} \cap \Omega_{[y]}$ is a non-empty set, but it is not in Y and so is no subset thereof. Hence $int(\Omega_{[x]} \cap \Omega_{[y]}) = \emptyset$.

1.2. Existence of non-open, non-intersecting individuals. In the characterization of the structures RT in terms of lattices the following proposition is based on a simple observation but is momentous in its effect on the models of RT. Recall that by definition two individuals x and y being externally connected means their set-theoretic intersection is non-empty (i.e. they have at least one point in common) but the sets do not overlap, i.e. there exists no common subset $z \in Y$ of x and y. Intuitively, it is understood as two individuals being connected but not overlapping in any part. As an example, consider a spatial interpretation: two countries share a boundary by which they are externally connected, but there is no place (in particular no county, city, house, street, or garden) that belongs to both countries at the same time.

Proposition 4.5. *In each model of RT^-_{EC} and RT_0 two non-open, non-intersecting (but connected) individuals must exist.*

Proof. Condition (vii) of RT_T requires the existence of $x, y \in Y$ s.t. $x \cap y \neq \emptyset \wedge int(x \cap y) = \emptyset$ (definition of external connection). It requires two regions x and y to share at least a point, but to not share an interior point (note that $int(x \cap y) = int(x) \cap int(y)$), i.e. they share only boundary points. This is equivalent to the verbal explanation given in Clarke's work **[Cla81]**. Since for open regions $o = int(o)$ holds, if either one of x and y would be open, it could not contain any boundary points that it can potentially share. Thus for the existence of two externally connected regions x and y, these regions must be non-open (but not necessarily closed). The non-intersection of x and y with respect to a common parts follows trivially, since this common part would have a non-empty interior (by condition (ii) of RT_T) and thus violate the condition of external connection EC in the equivalent model of RT_0 or RT^-_{EC}. □

Later, we will see that this proposition is a key argument for proving non-unique complementation, non-modularity, and non-orthomodularity. Asher and Vieu argue that the axiom A11 (or similarly the condition (vii)) prevents trivial models. However, only the prevented trivial models can be modular, orthomodular, and uniquely complemented. The proofs show that all non-trivial models are non-modular, non-orthomodular, and not uniquely complemented. In section 6 we actually show that $RT^- = RT_0 \setminus \{A11, A12\}$ would allow modular, orthomodular, and uniquely complemented lattices.

Before turning our attention to complementation and modularity, we prove that the lattices are complete and atomic for all finite models. This will also help to simplify other proofs.

2. Completeness

As we have just seen, the sum $\cup^*$($+$ in RT_0) and the intersection $\cap^*$ ($\cdot$ in RT_0) translate to join and meet in the associated lattices. But we know more about the sum and intersections: for any pair of individuals in a model of RT_0 a unique sum and a unique intersection (if not empty) must exist. In lattices the closely related concepts of completeness exists. Completeness in a lattice essentially tells the same: for any pair of lattice elements, there must exist a unique join and meet.

THEOREM 4.6. [**Grä98**] *Let P be a poset in which $\bigwedge H$ exists, for all $H \subseteq P$. Then P is a complete lattice.*

FACT 4.7. [**Grä98**] *The following are equivalent by duality of completeness*

(1) P is a complete lattice

(2) $\bigwedge H$ exists for all $H \subseteq P$

(3) $\bigvee H$ exists for all $H \subseteq P$

THEOREM 3. *Every finite lattice in $\mathfrak{L}_{RT}$ is a complete lattice.*

PROOF. Condition (3) is satisfied by the models of the mereotopology due to condition (vi), $\forall x, y \in Y\, [x \cup^* y \in Y]$, in [**AV95**]. Thus the lattices associated with the finite models of the mereotopology as defined in EQ4 are complete lattices. □

The proof shows that completeness of the lattices is nothing but the complete existence of sums, the so-called arbitrary mereotopological sum of two individuals, and the existence of arbitrary intersections of any two individuals in the models of RT. Having both operations (join and meet) in the algebraic system makes completeness a dual property: we can either take arbitrary (finitely) many joins of atoms of the lattice or arbitrary (finitely) many intersections of dual-atoms (see definition 4.8) in the lattice. Hence the existence of both join and meet distinguishes the structures from the more generic semi-lattices. Notice that in general finite lattices are complete. Since we only consider the finite models of RT, completeness can always be assumed.

REMARK 3. Completeness does not necessarily apply to infinite models, since even though the join and meet of every pair of individuals must exist, this is not clear for arbitrary unions or intersections. However, if we cannot prove completeness for all models, including the infinite ones, of RT, then the models arising from RT are not in the subclass of models of Clarke's connection structures, which are complete ortholattices. Completeness for Clarke's system follows from the unrestricted fusion operator. But Asher and Vieu eliminated it from their theory, while claiming that it is unnecessary. However, if that allows incomplete lattices, the models are actually changed by this step. This needs further investigation.

3. Atomicity

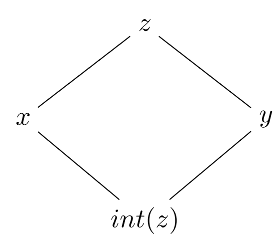
A lot of theorems and definitions about special kinds of lattices rely on lattices being atomic or atomistic, where the latter one is a strengthening of atomic. Especially when considering symmetries in lattices, these concepts are important, but also for certain kinds of complementation, e.g. relative complementation. From a mereotopological perspective we are interested to know whether the lattices are atomic or even atomistic to understand how individuals are built from smaller parts.

DEFINITION 4.8. **[MM70]** In a lattice we say that b *covers* a and write $a \prec b$ when $a < b$ and moreover $a < c < b$ is not satisfied by any c. An element p of a lattice L with 0 is called an atom if $0 \prec p$.

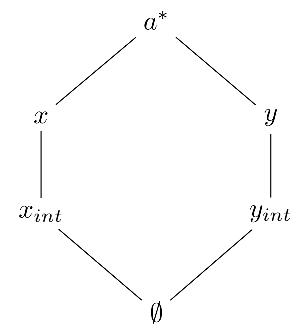
A lattice L with 0 is called atomic when every non-zero element $a \in L$ contains an atom.

L is called atomistic when every non-zero element $a \in L$ is the join of atoms contained in a.

An element d of a lattice with top element 1 is called a *dual-atom* (or a *hyperplane*) if $d \prec 1$.



(a) non-atomistic sublattice of a lattice $\mathcal{L}^{\mathcal{M}}$



(b) atomic, but non-atomistic lattice $\mathcal{L}_6$

FIGURE 8. Examples for non-atomistic lattices

Notice that every finite lattice is atomic, hence we immediately derive from considering finite models only that all lattices in $\mathfrak{L}_{RT}$ are atomic.

THEOREM 4. *Every finite lattice in* $\mathfrak{L}_{RT}$ *is atomic.*

The infinite models of the theory of Asher and Vieu are more tricky to handle. If the infinity is restricted to the number of atoms, then we can still prove that such a model yields an atomic lattice. However, if we deal with atomless models of RT then we cannot prove that the resulting lattices are atomic. This underlines the equivalence between non-atomic lattices and atomless mereotopological theories as expressed in the following corollary.

COROLLARY 4.9. *A model* $\mathcal{M}$ *satisfying the axioms of RT is atomic if and only if the lattice* $\mathcal{L}^{\mathcal{M}}$ *is atomic.*

Now, consider atomicity. Unfortunately atomicity does not hold for models of RT and even not for finite models. The lattices of the finite models are not atomistic because we can must have individuals that only differ by the external connections from their interior. Because external connections are not reflected in the lattices, an individual can cover another individual without having an extra part. Then this "greater" individual is not simply the join of atoms, since it contains something (the difference to its "smaller" individual containing the same set of atoms) that is not reflected in some atom. Consider two elements $\mathbf{x}$ and $\mathbf{y}$ in a model $\mathcal{M}$ of RT_0 that overlap, i.e. have a common part $\mathbf{z}$. Nothing prevents $\mathbf{x}$ and $\mathbf{y}$ to have $\mathbf{z}$ as their only part. Even if $\mathbf{z}$ is atomistic, the elements $\mathbf{x}$ and $\mathbf{y}$ are not required to be, since they are "more" than joins of atoms. Otherwise they would be identical. But they can be externally connected to different elements, and by axiom $A3$ be then not identical. The same occurs if an element has an interior, but this interior is its only part. Then the element itself must be distinguished from its interior by some external connection, but is not simply the join of atoms (i.e. the interior or the atoms the interior consists of), otherwise it would again be identical with its interior.

The proof for non-atomicity of all lattices of RT can be extremely simplified by a relation between atomistic pseudocomplemented, section-semicomplemented lattices and Boolean lattices. Then it follows directly that all models (not just the finite models) of RT are strictly not associated to atomistic lattices. See subsection 4.4 for the proof.

4. Complementation

Relative complements are a standard notion in topological spaces. Moreover, the models of RT define complements for each individuals with respect to the "whole". The purpose of this chapter is to find a meaningful description of these complements in terms of lattices. Fortunately, complementation is a very common property for lattices as well and there exist several specialized kinds of complementation. We analyze which of these complementation properties hold for the lattices in $\mathfrak{L}_{RT}$ in order to characterize the models of RT in terms of classes of complemented lattices.

Definition 4.10. [**Grä98**] A complemented lattice is a bounded lattice (with the infimum 0 and supremum 1) in which each element x has a complement y s.t. $x \wedge y = 0$ and $x \vee y = 1$.

Notice that this holds even for models that are atomless. If a model has an infinite set of individuals, then the universally connected individual must be still defined and can serve as supremum. Moreover, because we add the empty set to the individuals in the lattice and define it to be covered by all and only atoms. Therefore the lattice is still bounded although it might contain an infinite set of atoms.

Notation 4.11. We use the complementation operations $\sim x$ and $-x$ when referring to the models of RT_T and RT_0, respectively. In lattices, x' denotes a (not necessarily unique) complement.

THEOREM 5. *Let $\mathcal{M}$ be a model of RT_0. Then $\mathcal{L}^{\mathcal{M}}$ is a complemented lattice with $(a^*)' = \emptyset$ and $(\emptyset)' = a^*$. In $\mathcal{L}^{\mathcal{M}}$, $a' = b$ holds if $-a = b$ holds in $\mathcal{M}$.*

PROOF. Assume there exists an individual a' being greater (in the partial order P) than the universal individual a^*, i.e. $\langle a^*, a' \rangle \in PP$: by definition D1 every individual that a^* is connected, a' must be connected to as well. Since by A4 a^* is connected to all other individuals in the theory, a' is hence also connected to all other individuals in the theory. However, then by A3 $a^* = a'$ follows immediately and a^* cannot be part of a'. This also shows that the supremum of the lattice is unique. On the other side we define $\emptyset$ to be the infimum, i.e. that every other element in the lattice is greater than $\emptyset$. Hence the lattices $\mathcal{L}^{\mathcal{M}} = (Y \cup \emptyset, \cap^*, \cup^*, \emptyset, a^*)$ in $\mathfrak{L}_{RT}$ are upper- and lower-bounded, i.e. are bounded lattices.

The last step required shows that there exists a complement for each element satisfying the condition mentioned in definition 4.10. From condition (iv) of RT_T we know that such a complement exist for a set A if the interior of the complement of A is not empty. To cover the case that the interior of the complement is the empty set, we have defined the lattice over the set $Y \cup \emptyset$. □

This theorem links together the topological complement $-\mathbf{x}$ for individuals in models of RT with the lattice complement $\mathbf{x}'$ in the associated lattices in $\mathfrak{L}_{RT}$. We can further derive following corollary to connect the models of RT with the lattices they yield.

COROLLARY 4.12. *Each model $\mathcal{M}$ of RT gives rise to a complemented lattice $\mathcal{L}^{\mathcal{M}}$ with the zero element (empty set) removed.*

Throughout the rest of the thesis, we will reference the complemented lattices not by their long definition $\mathcal{L}^{\mathcal{M}} = (Y \cup \emptyset, \cap^*, \cup^*, \emptyset, a^*)$ that explicitly specifies the infimum ("zero element") $\bot = \emptyset$ and the supremum ("top element") $\top = a^*$, but abbreviate it by $\mathcal{L}^{\mathcal{M}}$.

The next subsections take a closer look at different kinds of complementation. We use a classification of complemented lattices from [**Ste99**] (see also [**BJ72**]), figure 1.14, with an error corrected. Notice that this is specific to bounded lattices.

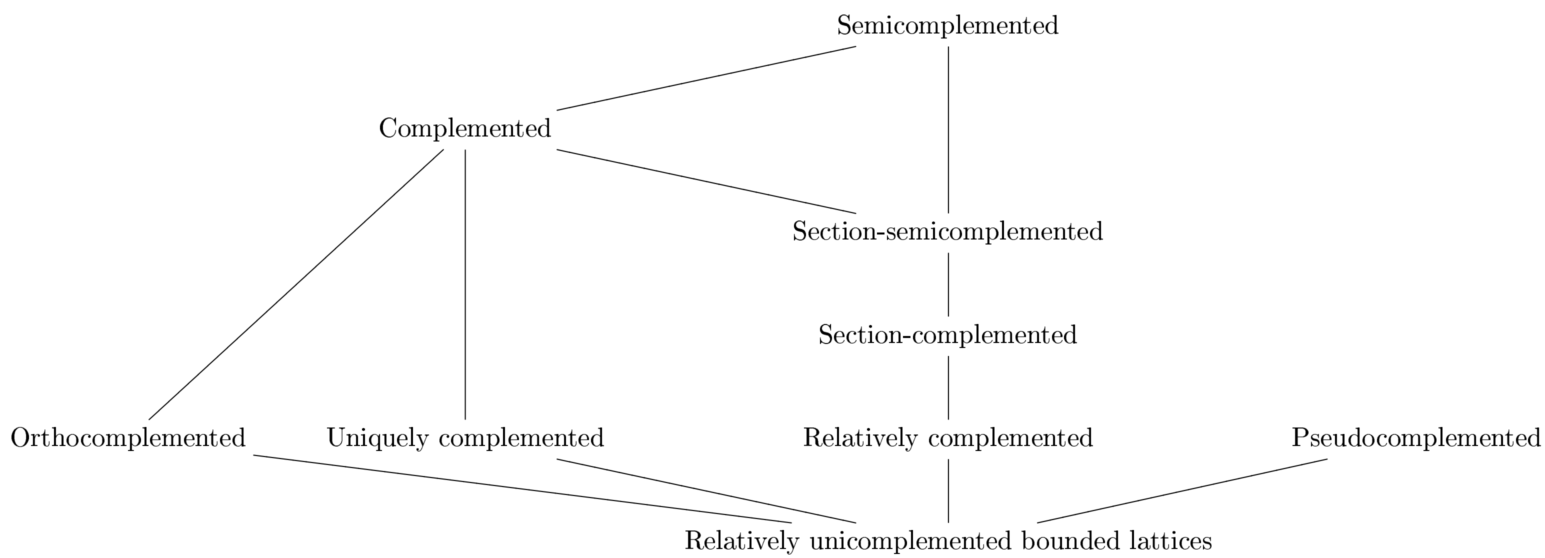


FIGURE 9. Relationships between different kinds and degrees of complementation in bounded lattices, corrected version from [**Ste99**]

4.1. Unique complementation.

DEFINITION 4.13. A uniquely complemented lattice is a complemented lattice in which for each element x only one element $x' = y$ exists such that $x \wedge y = 0$ and $x \vee y = 1$.

The theorem 6 demarcates the models of RT_0 from the uniquely complemented lattices. It is a stronger version of theorem 5 proved above.

THEOREM 6. *Every lattice in $\mathcal{L}_{RT}$ is complemented, but no lattice in $\mathfrak{L}_{RT}$ is uniquely complemented.*

PROOF. Assuming 5, only the part that the lattices cannot be uniquely complemented is still open. We now show that the proposition 4.5 yields at least two individuals that do not have unique complements. From proposition 4.5 we know at least two non-open regions, call them x and y, exist. From the fact that for open regions $x = int(x)$ holds and from the condition (ii) of RT_T (existence of an interior for every individual) we know the interiors $x_{int} = int(x)$ and $y_{int} = int(y)$ are distinct from x and y, respectively. Let us now denote the (topological) complement of x as $\sim x$, it follows (a) $x \cup^* \sim x = a^*$ and (b) $x \cap^* \sim x = \emptyset$ (from the criteria of a complement). We claim that then (a') $x_{int} \cup^* \sim x = a^*$ and (b') $x_{int} \cap^* \sim x = \emptyset$ must also hold. Since $x_{int} \leq x \leq cl(x)$, it (b') follows immediately from (b). Now let us assume that $\exists c\, [x_{int} \cup^* \sim x = a^* - c]$, i.e. that the join of x_{int} with $\sim x$ is a proper part of a^* and there exists a set of points c that distinguished the join of x_{int} with $\sim x$ from a^*. Then $c \in Y$ must be a part of x, but not of x_{int} in terms of point set inclusion, i.e. $c \subset x$ and $c \not\subset x_{int}$. Again by the existence of interiors

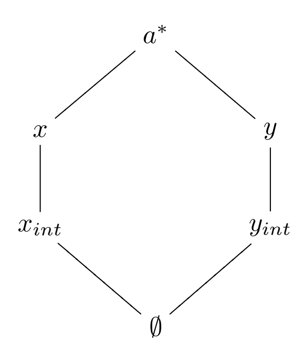


FIGURE 10. Failure of unique complementation in models of RT
x_{int} and y_{int} must exist, but both y and y_{int} are complements of x and, vice versa, x and x_{int} are both complements of y.

(condition (ii) of RT_T) $int(c) \in Y$ and $int(c) \neq \emptyset$. Then $int(c) \subset x$ and both x_{int} and $int(c)$ are open sets. Thus $x_{int} \cup int(c)$ is an open set and $x \supseteq x_{int} \cup int(c)$. However, this violates the definition of the interior as the largest open set, since $x_{int} \subseteq x_{int} \cup int(c)$. So either x_{int} is not the interior of x or $c = \emptyset$. The first case contradicts our assumption, and in the latter case $x_{int} \cup^* \sim x = a^* - c$ then yields (a') $x_{int} \cup^* \sim x = a^*$ which is also a contradiction to our assumption. With (a') and (b') proved, we know that $\sim x$ is the complement of x and x_{int}. Vice versa, we now know that both x and x_{int} are complements of $\sim x$. The same argumentation applies for y. Thus condition (vii) of RT_T requiring the existence of an external connection forces the associated lattices to be strictly not uniquely complemented. □

Notice that the proof uses the structures defined by RT_T and not by the axioms of RT_0. However, because we previously showed the equivalence between them, the proof also applies for the models satisfying the axioms RT_0.

REMARK 4. Notice that the notions of a complement $-x$ in the topological sense and in the lattice-theoretic x' sense are quite different. The elements of the models of RT have unique topological complements (the complement is defined set-theoretic) whereas the lattices containing the same elements are not uniquely complemented since the complement is only defined in terms of the join and meet operations.

From the lattices being never uniquely complemented, we can derive that they are not Boolean lattices, since all Boolean lattices are uniquely complemented, see figure 4. A more general result was obtained in [**McL56**]:

THEOREM 4.14. [**McL56**] *Every complemented, atomic lattice with unique comparable complements is modular.*

Later we provide alternative proofs that shown that the models of RT are not associated to modular nor distributive lattices.

4.2. Pseudocomplementation. Pseudocomplementation is a common property for lattices that relaxes the conditions of unique complementation. Every uniquely complemented lattices is also join- and meet-pseudocomplemented, but the converse is not always true. Intuitively, pseudocomplementation captures that every element in a lattice should have a unique greatest (for meet-pseudocomplementation) and a unique smallest (for join-pseudocomplementation) pseudocomplement in the sets $\{x \in L | x \wedge a = \emptyset\}$ and $\{x \in L | x \vee a = \emptyset\}$, respectively. In fact, we can show that these elements are related to the closure and interior of individuals in the

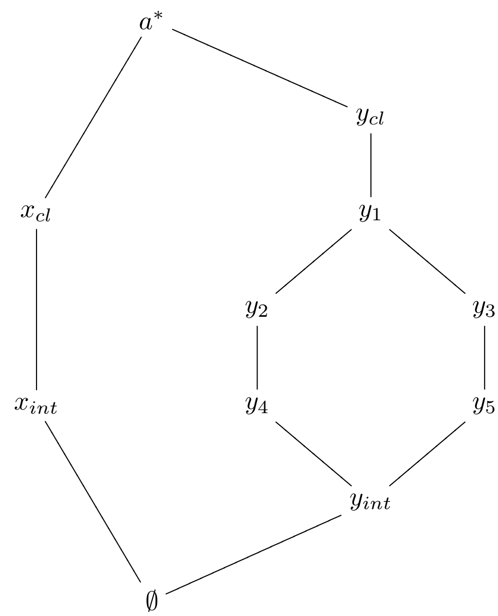


FIGURE 11. Example of a double pseudocomplemented lattice
In this example y_{int} is the join-pseudocomplement of x_{cl} and x_{int} whereas y_{cl} is the meet-pseudocomplement of x_{cl} and x_{int}. The lattice is neither uniquely complemented nor orthocomplemented.

topological understanding. For an example of meet- and join-pseudocomplemented lattice see figure 11.

DEFINITION 4.15. [**Grä98**] Let L be a lattice with infimum 0, an element a' is a meet-pseudocomplement (or simply pseudocomplement) of $a \in L$ if and only if $a \wedge a' = 0$ and $\forall x\, (a \wedge x = 0 \Rightarrow x \leq a')$.

[**Ste99**] Let L be a lattice with supremum 1, an element a' is a join-pseudocomplement of $a \in L$ if and only if $a \vee a' = 1$ and $\forall x\, (a \vee x = 1 \Rightarrow x \geq a')$.

[**Ste99**] A lattice that is meet-pseudocomplemented and join-pseudocomplemented is often called a *double p-lattice*.

LEMMA 4.16. *In any lattice in $\mathfrak{L}_{RT}$ for any complement a' of a, $cl(a')$ and $int(a')$ are complements of a.*

PROOF. With $int(a') \leq a' \leq cl(a')$ it is clear that $int(a') \wedge a = 0$ and $cl(a') \vee a = 1$. Thus we only proof (i) $cl(a') \wedge a = 0$ and (ii) $int(a') \vee a = 1$ follows by duality. Assume contrary to (i) that $cl(a') \wedge a > 0$ then there exists a common part - call it b - of both a and $cl(a')$, i.e. $b \leq a$ and $b \leq cl(a')$, but $b \not\leq a'$. But then b needs to have an interior $int(b)$ defined. Notice that moreover a' has some interior $int(a')$

that is distinct from $int(b')$. So reconsider a', $int(a') \neq int(cl(a'))$ holds because $int(a') \cup^* int(b) \subseteq cl(a')$. Hence we have a violation of condition (ii) of RT_T and our assumption that $cl(a') \wedge a > 0$ was wrong.
From duality of meet and join in the lattice every $cl(a)$ having an orthocomplement that is an interior, i.e. $int(a')$ if $cl(a') \wedge a = 0$ holds then also $int(a') \vee a = 1$ holds. □

LEMMA 4.17. *Any lattice in $\mathfrak{L}_{RT}$ is meet- and join-pseudocomplemented.*

PROOF. We know every lattice in $\mathfrak{L}_{RT}$ is complemented, hence there exists for every a some a' so that $a \wedge a' = 0$ and $a \vee a' = 1$. From the lemma 4.16 we know that for every a', $int(a')$ and $cl(a')$ are also complements of a. Now we have two claims: (i) first every element b with $b > cl(a')$ has a non-zero meet with a and thus cannot be meet-pseudocomplement, and every element c with $c < int(a')$ has a join with a that is not 1. (ii) second we claim that every element b with $a \wedge b = 0$ or $a \vee b = 1$ satisfies the condition $b \leq cl(a')$ or $b \geq int(a')$, respectively.
(i) Assume b with $b > cl(a')$ and $b \wedge a = 0$ exists. Then the extension of C in which b participates must subsume the extension of C in which $cl(a')$ participates. If the extensions of O where b or $cl(a')$ participate are the same then either $cl(a')$ is not closed (b has an additional another external connection) or b and $cl(a')$ have the same extensions of C and are by A3 identical. If the extension of O in which b participates is strictly greater than the one of $cl(a')$, then b must overlap with some part of a and $b \wedge a = 0$ does not longer hold. In both cases we derive a contradiction.
(ii) From (i) we know there exists no such b with $b > cl(a')$ so that $b \wedge a = 0$. Now we have to prove that no other element b exists with $b \wedge a = 0$ that is incomparable to $cl(a')$. Notice that every element b is either comparable to a or $-a$, see proposition 6.3 in chapter 6. Assume a' to be the orthocomplement of a (in the next subsection we will show that such an element must always exist). If b is comparable to a then obviously $a \wedge b = 0$ does not hold. Therefore b must be comparable to $-a$. The trivial case is $cl(a') = -a$. Otherwise the sum $b \cup^* cl(a')$ overlaps in some part(s) with a ($cl(a')$ is already maximally connected to a without overlap, see the argument for (i)), which in turn requires one part (either of b or $cl(a')$, or of a third element) to overlap with a. That would mean either b or $cl(a')$ overlaps with a and $a \wedge b = 0$ or $a \wedge cl(a') = 0$ does not hold. Hence no such b can exist. From (i) and (ii) together with lemma 4.16, $cl(a')$ must be the meet-pseudocomplement of a.
The proof for the join-pseudocomplements is analogous, we omit it here. □

From the proof it follows directly that $cl(a')$ is the meet-pseudocomplement of any a and equivalently $int(a')$ to be the join-pseudocomplement of any a. We summarize this in the following theorem.

THEOREM 7. *Every lattice in $\mathfrak{L}_{RT}$ is a double p-lattice where for any complement* $\mathbf{x}'$ *of* $\mathbf{x}$ *the individuals denoted by* $cl(\mathbf{x}')$ *and* $int(\mathbf{x}')$ *are the meet- and join-pseudocomplements of* $\mathbf{x}$.

Notice that not in general not every pseudocomplement must be a complement, i.e. a meet-pseudocomplement x' of x needs not to satisfy $x \vee x' = 1$ and a join-pseudocomplement x' of x needs not to satisfy $x \wedge x' = 0$. However, in the lattices in $\mathfrak{L}_{RT}$ every pseudocomplement is also a complement, which is a direct consequence of theorem 7 and lemma 4.16.

COROLLARY 4.18. *In any lattice in $\mathfrak{L}_{RT}$ all meet- and join-pseudocomplement are complements.*

In terms of the mereotopology the theorem shows that all possible (lattice) complements of an individual have the same closure (and interior), which is also always the greatest (smallest) of all possible complements in the lattice.

Meet- and join-pseudocomplementedness does not imply complementedness, due to the simple fact the the meet-complement can be different from the join-complement. In [**Ste99**], p.26 such an example is given. However, Chameni-Nembua and Monjardet [**CNM93**] proved the following fundamental relation between meet- and join-pseudocomplementedness in complemented lattices.

THEOREM 4.19. [**CNM93**] *For a complemented lattice L, the following conditions are equivalent:*

(1) L is meet-pseudocomplemented

(2) L is join-pseudocomplemented

(3) L is a double p-lattice (dual pseudocomplemented)

NOTE 4.20. Chameni-Nembua and Monjardet use for condition (3) of 4.19 the term *pseudocomplemented* when referring to both meet- and join-pseudocomplemented. We occasionally use it if it is unambiguous. Since the lattices we are concerned with are all complemented, meet-pseudocomplemented and double pseudocomplemented imply one another anyways. Nevertheless, note that in the literature it is more common to use the terms pseudocomplement and meet-pseudocomplement synonymously and pseudocomplementedness does then not generally imply join-pseudocomplementedness.

4.3. Orthocomplementation. Another type of complementation in lattices is orthocomplementation. It is usually covered in the context of orthomodularity which we will consider later, since so far we are only interested in complementation in the lattices. We know that the lattices corresponding to the models of RT are pseudocomplemented, but strictly not uniquely complemented. However, as we have seen in the previous section, pseudocomplementation is a rather weak kind of complementation. For instance is the lattice is figure 11 pseudocomplemented, but does not resemble a model of RT. Using orthocomplementation, we will bridge this gap. Orthocomplemented lattices have been used previously in [**BG96**] for the characterization of the connection structures of Clarke's mereotopological system. Moreover, Iturrioz [**Itu83, Itu86**] demonstrated that there exists nice topological representations of orthomodular lattices, which are by definition orthocomplemented. We should notice that orthocomplementation and orthomodularity do not receive much attention in general lattice theory, but are mainly applied in the description of quantum-mechanical systems in the field of quantum logic. Orthomodular and orthocomplemented lattices emanated from lattice-theoretical work by Birkhoff and von Neumann in the 1930s on the logical structure of physical theories, in particular that of quantum mechanic systems equivalent to the characterizations of Hilbert space [**BvN36**].

Definition 4.21. [**Bly05**] An involution lattice is a bounded lattice L together with an antitone mapping $^{\perp} : L \to L$ such that $x = x^{\perp\perp}$ for every $x \in L$.

An ortholattice (i.e. orthocomplemented lattice) is an involution lattice $(L, ^{\perp})$ in which the involution is an orthocomplementation in the sense that $x \wedge x^{\perp} = 0$ for every $x \in L$.

Theorem 4.22. [**Kal83**] *A bounded lattice is an ortholattice if and only if there exists a unary operation* $^{\perp} : L \to L$ *so that (1) to (3) hold:*

(1) $\forall x \left[x = x^{\perp\perp}\right]$ *(involution law)*

(2) $\forall x \left[x \wedge x^{\perp} = \perp\right]$ *and* $\forall x \left[x \vee x^{\perp} = \top\right]$ *(complement laws)*

(3) $\forall x, y \left[x \leq y \equiv x^{\perp} \geq y^{\perp}\right]$ *(order-reversing law)*

Note since the lattices are bounded, $\forall x \left[x \vee x^{\perp} = \top\right]$ follows from $\forall x \left[x \wedge x^{\perp} = \perp\right]$ and vice versa by duality. One could substitute property (3) with an equivalent property $\forall x, y \left[(x \cap y)^{\perp} = x^{\perp} \cup y^{\perp}\right]$ [**Nob06**].

To prove orthocomplementation for the lattices in $\mathfrak{L}_{RT}$, we first have to find an suitable orthocomplementation function $^{\perp}$ that satisfies the properties (1) to (3). Obviously, both the definition of the (not-unique) complement as in section 4 and of the pseudocomplement as in section 4.2 do not satisfy the involution condition.

However, the topological complementation $\sim$ seems to satisfy the properties. Just its lattice-theoretic formalization is nontrivial. We need to find a way to distinguish the topological complement from the other potential complements for each lattice element. For the moment it is sufficient to show that the topological complement is always an orthocomplement in a lattice $\mathcal{L}^{\mathcal{M}}$, since we know that the topological complement must exist for any individual except a^* in a structure RT_T. The unary operation orthocomplement $^{\perp}$in the lattices will be the (topological) complement relation $\sim$ in RT_T with $\sim(a^*) = \emptyset$ and $\sim(\emptyset) = a^*$.

THEOREM 8. *Every lattice in $\mathfrak{L}_{RT}$ is orthocomplemented when choosing the topological complement $\sim$ as orthocomplementation (involution) operation $^{\perp}$.*

PROOF. In order to determine whether $\sim$ satisfies the conditions of an orthocomplementation operation, we check the properties from theorem 4.22 for the topological complement:

(1) Involution law: $\forall x\, [x = \sim\sim x]$

(2) Complement law: $\forall x\, [x \cap^* \sim x = \emptyset]$ (or the dual of it)

(3) Order-reversing law: $\forall x, y\, [x \leq y \equiv \sim x \geq \sim y]$

Choosing $\sim a^* = \emptyset$ and $\sim \emptyset = a^*$ makes $\sim$ a complete function on the set $Y \cup \emptyset$. For the topological complement property (1) applies naturally (from its set-theoretic definition). For (2), we know that for the topological complements $x \cap^* \sim x = \emptyset$ holds (by its set-theoretic definition). Thus, if the lattices satisfy property (3), then the topological complement $\sim$ makes the lattices that are associated with the models of RT ortholattices. Again, we apply the set-theoretic definition of the topological complement defined over an topological space. Consider x and y as sets of points: $x \leq y$ (in the lattice) if and only if $x \subseteq y$. If $x = y$ then $\sim x = \sim y$ and (3) holds trivially. Hence assume $x \subset y$, then all the points in $y \setminus x$ (nonempty) must be part of the complement of x, i.e. $y \setminus x \subseteq \sim x$. Since all points that are in both x and y are in neither complement and all points in neither sets are in both complements, $\sim y$ must be a proper subset of $\sim x$, i.e. $\sim x = a^* \setminus (x \cap y)$ and $\sim y = a^* \setminus (x \cap y) \setminus (y \setminus x)$. And it follows: $a^* \setminus (x \cap y) \setminus (y \setminus x) \subseteq a^* \setminus (x \cap y)$ and with $y \setminus x$ being distinct from $x \cap y$ and by the assumption being non-empty: $a^* \setminus (x \cap y) \setminus (y \setminus x) \subset a^* \setminus (x \cap y)$ and thus $\sim y < \sim x$, and the order-reversing law (3) is satisfied. $\square$

LEMMA 4.23. **[Kal83]** *(p. 26) Let L be an ortholattice. The following statements are equivalent:*

(1) L is a Boolean algebra,

(2) L is uniquely complemented.

Since each lattice associated with the mereotopology is not uniquely complemented, it is not Boolean. Later we use the properties of modularity to show in a different way that the lattices are non-distributivity. This result gives us a major and maybe surprising distinction from the characterization of the RCC and Clarke's system. Both were proved to yield distributive pseudocomplemented lattices, whereas now we showed that the lattices resulting from models of RT are pseudocomplemented and orthocomplemented, but not uniquely complemented and hence not distributive and not Boolean (like the models of Clarke's system).

In the literature complemented, pseudocomplemented lattices are commonly abbreviated double p-lattices (or algebras), see theorem 4.19. Orthocomplemented lattices are shortly called ortholattices. Conform to this convention, we refer to the (double) pseudocomplemented, orthocomplemented lattices in short as *p-ortholattices*. That they are complemented follows immediately from orthocomplemented (recall that complementedness does not follow from double pseudocomplementedness). Hence the p-ortholattices are always doubly pseudocomplemented by theorem 4.19.

4.4. Section-semicomplementation. In a relatively complemented lattice every element e has a relative complement in any interval $[a, b]$ containing the element **[Grä98]** ($a \leq e \leq b$). For the lattices in $\mathcal{L}^{\mathcal{M}}$ being relatively complemented that would mean that every individual is relatively complemented (in the topological sense) to any element that it is a part of. One can easily see that this does not hold in general for the lattices in $\mathcal{L}^{\mathcal{M}}$ since an individual and its interior (or closure) can have the exactly same parts. For an example see figure 10: the elements of the interval $[x_{int}, x]$ are clearly not relatively complemented. The weaker notions section-semicomplementation and section-complementation are listed only for the purpose of a complete classification in terms of the complementation properties from figure 4 and by showing not section-semicomplementedness we can easily rule out the stronger properties..

Janowitz related pseudocomplemented and section-semicomplemented lattices to the Boolean algebras.

THEOREM 4.24. **[Jan68]** *Every pseudocomplemented section-semicomplemented lattice is a Boolean algebra.*

All the lattices in $\mathfrak{L}_{RT}$ are not Boolean, therefore we can use this result to derive that the lattices in $\mathfrak{L}_{RT}$ are not section-semicomplemented because we just proved pseudocomplementation. Additionally, section-complemented lattices must be section-semicomplemented and relatively complemented lattices must be section-complemented, see figure 4

THEOREM 9. *No lattice in $\mathcal{L}_{RT}$ is section-semicomplemented, section-complemented, or relatively complemented.*

PROOF. In section 3 we proved that the lattices associated to models of RT and subsets thereof are atomic, but not atomistic. (see theorem 4). Hence, they cannot be section-semicomplemented. It immediately follows from subsumption of the lattice properties (see figure 4) that they are not section-complemented and thus not relatively complemented. □

THEOREM 4.25. [**MM70**] *An atomic lattice L is atomistic if and only if L is section-semicomplemented.*

We can use this result to immediately show that the lattices associated to models of RT are not atomistic, since we proved in section 3 that the lattices associated to finite models are atomic. If an infinite model does not correspond to an atomic lattice, it cannot correspond to an atomistic lattice either.

REMARK 5. The lattices are not atomistic due to the existence of overlaps $O(x, y) = z$ between elements x and y that have no other parts than the part z they overlap in. To allow the existence of such elements, they need to be externally connected in a different fashion. Intuitively this is a neglect of their other parts that actually distinguish x and y. Since if x is externally connected to another element w and y is not, then there is something in x that it connects to w but that is not in y. E.g. this can be some part of the border, therefore x and y still have the same interior - the element z that they overlap in. One should therefore consider a change in the mereotopology by adding following axiom. Intuitively, this enforces the existence of distinct parts when two elements partially overlap (i.e. neither is part of the other). It can be conceived as a weak form of relative complementation.

$$\forall x, y, z\,[[O(x, y) \wedge \neg P(x, y) \wedge \neg P(y, x)] \rightarrow \exists v\,(P(v, x) \wedge \neg C(v, y))] \tag{EQ7}$$

THEOREM 10. *No lattice in $\mathfrak{L}_{RT}$ is atomistic.*

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5. Non-modularity and Non-distributivity

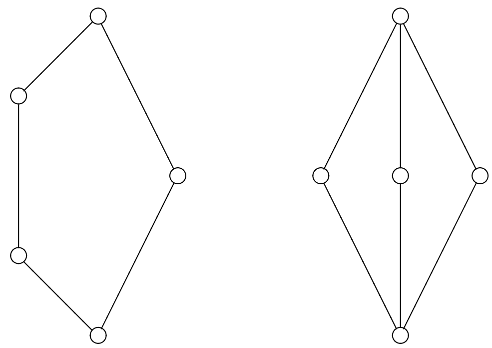
In this section we deal with lattice properties that are commonly found in mathematical literature besides complementation. These are amongst others modularity and weaker forms thereof. We first take a closer look at general modularity, since this is a strong lattice property that gives well-understood lattices.

THEOREM 4.26. [**Grä98**] *A Boolean lattice is a complemented distributive lattice.*

We already showed that the mereotopological structures of Asher and Vieu together with the empty set form complemented lattices. However, since the lattices are not uniquely complemented, they cannot be Boolean as we argued in section 4.1. Alternative proofs for the lattices in $\mathfrak{L}_{RT}$ being non-Boolean were provided by the properties of section-semicomplementation in pseudocomplemented lattices and unique complementation in orthocomplemented lattices. However, none of those directly disproved distributivity. That is one goal of this section. A large set of lattices, such as Boolean, Heyting, Brouwerian, and Stone lattices are subsets of distributive lattices. In [**Sto36**], Stone gave a representation theorem for distributive lattices and proved the more general Stone duality as correspondence between posets and certain topological spaces that could had linked back the lattice structures to topological spaces. On the opposite, non-distributivity has far-reaching implications, i.e. none of the lattices can be in any subset of distributive lattices. Failing to prove modularity, we are interested in weaker properties.

Subsequently we will turn our attention to orthomodular lattices. Orthomodular lattices are a refined set of ortholattices that have nice topological representations as demonstrated by [**Itu83, Itu86**]. Previously we showed orthocomplementedness of all lattices in $\mathfrak{L}_{RT}$, so looking at a well-known specialized set thereof is rather natural. Orthomodular lattices are in general not necessarily modular, [**Ber85**] gives an example of a non-modular orthomodular lattice. But in the reverse, proving non-orthomodularity implies non-modularity. Nevertheless, the separate proof showing why the lattices cannot be modular gives valuable insight into the forbidden sublattices of the theory RT.

Finally, we consider weaker types of modularity apart from orthomodularity. Amongst others we consider semimodular lattices, which include geometric lattices (also



(a) Pentagon lattice N_5 (b) Diamond lattice M_3

FIGURE 12. Forbidden sublattices for distributive lattices

known as matroid lattices) that received a significant amount of attention. Semimodularity is important in particular for geometric spaces. However, we see that due to the distinction of elements just by their borders, the models of RT are not even semimodular. Moreover, we show by examples that weak modularity and semidistributivity is not satisfied for all lattices in $\mathfrak{L}_{RT}$.

5.1. Modularity. Modular lattices are most commonly characterized by the absence of the pentagon (figure 5.1(a)) as sublattice (theorem 4.28). Moreover, [**Ded00**] showed that every modular non-distributive lattice has the diamond lattice M_3 (figure 5.1(b)) as sublattice. A direct consequence is the characterization of distributive lattices by the absence of pentagon and diamond as sublattices.

DEFINITION 4.27. [**Grä98**] A lattice satisfying either

(i) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ or

(ii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

is a distributive lattice.

[**DP90**] A lattice is modular if it satisfies

(i') $x \geq z \Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

In fact, we will show that the models of RT are not modular, and thus not distributive. Note that the condition (i') of modular lattices is the unrestricted from of condition (i) for distributive lattices. An alternative characterization goes back to the work of Dedekind [**Ded00**], who showed that every non-modular lattice contains a pentagon as sublattice.

THEOREM 4.28. [**Grä98**] *A lattice L is modular if and only if it does not contain a pentagon N_5.*

in order to demonstrate that there exist lattices in in $\mathfrak{L}_{RT}$ that are non-modular and hence not distributive we first give a counterexample that contains a sublattice isomorphic to the pentagon N_5. Afterward, we show that non-modularity is a strict property for the lattices in $\mathfrak{L}_{RT}$, i.e. that none of the lattices in this class can be modular.

EXAMPLE 3. *We give a model $\mathcal{M}$ of RT^-_{EC} whose lattice contains a pentagon sublattice and thus is non-modular. This model is actually the smallest model of RT^-_{EC}. We use the set $Y^{\mathcal{M}} = \{\mathbf{a}^*, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}\}$ and the extension of C as following:*

$$\{\{\langle \mathbf{x}, \mathbf{y}\rangle : \mathbf{x}, \mathbf{y} \in Y\} \setminus \{\langle \mathbf{b}, \mathbf{e}\rangle, \langle \mathbf{c}, \mathbf{d}\rangle, \langle \mathbf{d}, \mathbf{e}\rangle\}\} \in C^{\mathcal{M}} \tag{EQ8}$$

i.e. all individuals are connected to another except for the disconnected pairs $\langle \mathbf{b}, \mathbf{e}\rangle, \langle \mathbf{c}, \mathbf{d}\rangle, \langle \mathbf{d}, \mathbf{e}\rangle$. The tedious work showing that all axioms (except A12) are satisfied by $\mathcal{M}$ is omitted here, but can be easily done by the reader. The respective lattice $\mathcal{L}$ is formed over the set $Y = Y^{\mathcal{M}} \cup \{\emptyset\} = \{\mathbf{a}^*, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \emptyset\}$, figure 5.1 shows its Hasse diagram. In the model $\langle \mathbf{b}, \mathbf{c}\rangle \in EC^{\mathcal{M}}$ must hold to simultaneously satisfy A11 and A3.

By removing e.g. $\mathbf{c}$ *from this lattice we obtain a sublattice $\mathcal{L}_S$ isomorphic to the pentagon N_5. Hence by theorem 4.28 the lattice is non-modular. Further notice that we can easily extend this model $\mathcal{M}$ to satisfy the existential requirement of A12 (existence of weak contact) by adding an independent lattice over some set Z with no element in Z being connected to any of the elements $\mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$. This structure can contain a weak contact following the definition D11 of* [**AV95**]*. Since none of the elements in Z are connected we can define joins $\{\langle y, z\rangle : y \in Y, z \in Z\}$ on top of the two lattices over Y and Z. Furthermore, complements to these joins can be defined without interference with the sublattices over Y and Z. Then the described sublattice $\mathcal{L}_S$ over the set $Y' = \{\mathbf{a}^*, \mathbf{b}, \mathbf{d}, \mathbf{e}, \emptyset\}$ is still a valid sublattice. Therefore $\mathcal{M}$ is a model of the full theory RT_0 whose lattice $\mathcal{L}^{\mathcal{M}}$ is not modular.*

The next part shows that in fact all lattices in $\mathfrak{L}_{RT}$ are non-modular and hence non-distributive. We first prove that the lattice $\mathcal{L}^{\mathcal{M}}$ of any model $\mathcal{M}$ of RT (and RT^-_{EC}) contains a special 6-element sublattice $\mathcal{L}_6$ (see figure 5.1(a)) and that this sublattice always contains a sublattice isomorphic to the pentagon N_5 (figure 5.1(b)).

LEMMA 4.29. *Every model $\mathcal{M}$ of the axioms of RT_0 entails the existence of a 6-element sublattice $\mathcal{L}_6$ of $\mathcal{L}^{\mathcal{M}} = (Y \cup \emptyset, \cap^*, \cup^*, \emptyset, a^*)$ with following properties:*

(1) $\mathcal{L}_6$ is a sublattice of $\mathcal{L}^{\mathcal{M}}$ over some set $Y' = \{\mathbf{a}, \mathbf{b_1}, \mathbf{b_2}, \mathbf{c_1}, \mathbf{c_2}, \emptyset\} \subseteq Y^{\mathcal{M}}$

(2) $\mathbf{a} = \mathbf{b_n} \cup^* \mathbf{c_m}$, for $n, m \in \{1, 2\}$, is the supremum of $\mathcal{L}_6$

(3) $\emptyset = \mathbf{b_n} \cap^* \mathbf{c_m}$, for $n, m \in \{1, 2\}$, is the infimum of $\mathcal{L}_6$

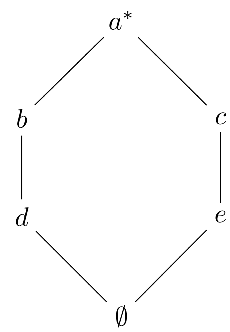


FIGURE 13. Benzene $\mathcal{L}_6$ that is a sublattice of any lattice over a model of RT

(4) $\mathbf{b_1} \cap^* \mathbf{b_2} = \mathbf{b_2}$ *and* $\mathbf{c}_1 \cap^* \mathbf{c_2} = \mathbf{c_2}$

(5) $\mathbf{b_1} \cup^* \mathbf{b_2} = \mathbf{b_1}$ *and* $\mathbf{c_1} \cup^* \mathbf{c_2} = \mathbf{c_1}$

(6) $\mathbf{a} \cup^* x = \mathbf{a}$ *and* $\mathbf{a} \cap^* x = x$ *for all* $x \in \mathcal{L}_6$

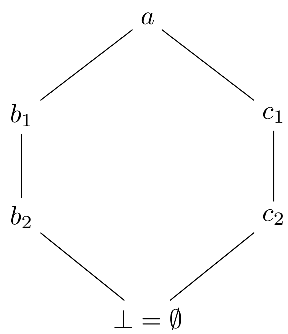
(7) $\emptyset \cup^* x = x$ *and* $\emptyset \cap^* x = \emptyset$ *for all* $x \in \mathcal{L}_6$

Thus $\mathcal{L}_6$ *is actually a lattice closed under* $\cap^*$ *and* $\cup^*$*.*

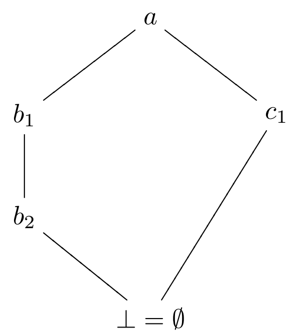
PROOF. Since the axioms force the existence of a pair of externally connected individuals which are non-open by proposition 4.5. Let us call these $\mathbf{b_1}$ and $\mathbf{c_1}$. Because of their non-openness, two open regions $\mathbf{b_2} = int(\mathbf{b_1})$ and $\mathbf{c_2} = int(\mathbf{c_1})$ must exist as interiors according to (ii) of RT_T. These regions $\mathbf{b_2}$ and $\mathbf{c}_2$ are part of and connected to the element they are interior of, $\mathbf{b_1}$ and $\mathbf{c_1}$, respectively. $\mathbf{b_2}$ and $\mathbf{c_2}$ are not connected to each other in order to satisfy the condition of external connection for $\mathbf{b_1}$ and $\mathbf{c_1}$ (see D4 or (vii) of RT_T). This set of regions Y' with $\mathbf{a} = \mathbf{b_1} \cup^* \mathbf{c_1}$ (for $\mathbf{a} = a^*$ it is actually the smallest model allowed by RT^-_{EC}) together with the empty set forms a sublattice with $\mathbf{a}$ as supremum, two branches consisting of $\mathbf{b_1}$ and $\mathbf{b_2} = int(\mathbf{b_1})$ respectively $\mathbf{c_1}$ and $\mathbf{c_2} = int(\mathbf{c_1})$, and the zero element $\emptyset$. Any model of RT_T contains at least these elements. If the lattice contains additional elements, $\mathcal{L}_6$ always forms a sublattice of it, since the elements $\mathbf{a}, \mathbf{b_1}, \mathbf{b_2}, \mathbf{c_1}, \mathbf{c_2}, \emptyset$ are closed under $\cup^*$ and $\cap^*$. Hence the axioms force any model of RT_0 or RT^-_{EC} to have $\mathcal{L}_6$ as sublattice. □

COROLLARY 4.30. *RT_0 entails the existence of a pentagon sublattice $\mathcal{L}_5 \cong N_5$ of the lattice $\mathcal{L}^{\mathcal{M}}$ for every model $\mathcal{M}$.*

PROOF. By removing an arbitrary element of the set $\{\mathbf{b_1}, \mathbf{c_1}, \mathbf{b_2}, \mathbf{c_2}\}$ from the sublattice $\mathcal{L}_6$ entailed by lemma 4.29 for any lattice in $\mathfrak{L}_{RT}$, we obtain a sublattice $\mathcal{L}_5$ that is still closed under join and meet and is a pentagon N_5. □



(a) lattice $\mathcal{L}_6$



(b) pentagon lattice $\mathcal{L}_5 \cong N_5$ as sublattice of $\mathcal{L}_6$

FIGURE 14. Six element sublattice contained in every lattice $\mathcal{L}^{\mathcal{M}}$ and one possible pentagon sublattice

It follows immediately that all the lattices that can be associated to models of RT are non-modular, since they contain a pentagon N_5 as sublattice.

THEOREM 11. *No lattice in $\mathcal{L}_{RT}$ is modular.*

5.2. Orthomodularity. Orthomodularity extends the concept of ortholattices. Each orthomodular lattice is orthocomplemented, but the converse does not hold in general. Orthomodular lattices are used primarily to describe the structure of physical elements in quantum-mechanic systems.

Since we proved orthocomplementedness for all lattices in $\mathfrak{L}_{RT}$, checking for orthomodularity seems like an obvious continuation of this characterization.

DEFINITION 4.31. [**Itu86**] An orthomodular lattice L is an orthocomplemented lattice which satisfies the orthomodular law, i.e. for all $x, y \in L$, if $x \leq y$ and $x^{\perp} \wedge y = 0$ then $x = y$.

[**Bly05**] An orthomodular lattice is an ortholattice in which the orthomodular identity $x \leq y \rightarrow y = x \vee (y \wedge x^{\perp})$ holds.

LEMMA 4.32. [**Kal83**] *Let L be an ortholattice. The following statements are equivalent:*

(1) L is orthomodular,

(2) For all $x, y \in L$, if $x \leq y$ and $y \wedge x^{\perp} = \emptyset$ then $x = y$

(3) O_6 is not a subalgebra of L. (O_6 is the benzene depicted in figure 5.1).

DEFINITION 4.33. **[Kal83]** A subalgebra of an ortholattice L is a subset M which is closed under the operations $^{\perp}$,$\wedge$, $\vee$, and which contains 0 and 1.

Beware that not every sublattice of an ortholattice is closed under orthocomplementation again, but a subalgebra of an ortholattice needs to be closed under orthocomplementation.

It is not difficult to see that condition (2) is violated for the lattices in $\mathfrak{L}_{RT}$. We already showed that every lattice in $\mathfrak{L}_{RT}$ is not uniquely complemented. If we let x_1 and x_2 be both (comparable) complements of an open individual x (think of x_2 being the interior of the externally connected individual x_1), where w.l.g. $-x \cong x^{\perp} \cong x_1$ (x_1 is the topological complement, hence closed), then $x_2 < x_1$ and $x = x_1^{\perp}$. By complementation $x_2 \wedge x = \emptyset$ holds, thus $x_2 \wedge x_1^{\perp} = \emptyset$ and it would follow $x_1 = x_2$. However, we know that there must exist such pair of distinct individuals x_1 and x_2 by proposition 4.5.

THEOREM 12. *No lattice in $\mathfrak{L}_{RT}$ is orthomodular.*

PROOF. Every lattice $\mathcal{L}^{\mathcal{M}}$ is not uniquely complemented, and then it cannot be orthomodular because it must contain a pair of elements $\mathbf{b_1}, \mathbf{b_2}$ that are comparable but have a common complement. Reconsider the model from lemma 4.29 of the sublattice that must exist in every structure RT_T: this sublattice requires $\mathbf{b_2} \prec \mathbf{b_1}$ with $\sim \mathbf{b_2} \cap^* \mathbf{b_1} = \emptyset$. In the contrary, (2) of lemma 4.32 requires $\mathbf{b_1} = \mathbf{b_2}$, an obvious contradiction to $\mathbf{b_2} \prec \mathbf{b_1}$. Proposition 4.5 forces the existence of two non-open externally connected individuals $\mathbf{b_1}$ and $\mathbf{c_1}$with interiors $int(\mathbf{b_1})$ and $int(\mathbf{b_2})$ that are proper parts of $\mathbf{b_1}$, $\mathbf{c_1}$ and no relative complements with respect to $\mathbf{b_1}$, $\mathbf{c_1}$ exist, i.e. in the corresponding model $\mathcal{M}$ of RT_0 $[\langle b_3, b_1\rangle \in PP \rightarrow \langle b_3, b_2\rangle \in P \vee \langle b_2, b_3\rangle \in P)]$ (see proof of theorem 8). Further $(\mathbf{b_1} \cap^* \mathbf{c_1}) \notin Y$ must hold for $\mathbf{b_1}$, $\mathbf{c_1}$ to be externally connected in $\mathcal{M}$. But then $int(\mathbf{b_1}) \cap^* \mathbf{c_1} = \emptyset$ and $int(\mathbf{c_1}) \cap^* \mathbf{b_1} = \emptyset$ directly follow. Thus the elements $\mathbf{b_1}$ and $int(\mathbf{b_1})$ will satisfy the condition on the left side of the orthomodular law, but not the conclusion. Thus every model must be non-orthomodular. □

Now we show that there are orthocomplemented, non-orthomodular lattices that do not correspond to any model of RT. In particular, the lattice L_6 obviously has O_6 as subalgebra and thus is non-orthomodular, but it does not correspond to a model of RT.

We previously proved (lemma 4.29) that each model $\mathcal{M}$ of RT_0 yields a sublattice $\mathcal{L}_6^{\mathcal{M}}$. However, this sublattice does for not necessarily yield O_6 as subalgebra of $\mathcal{L}^{\mathcal{M}}$: first $\mathcal{L}_6^{\mathcal{M}}$ is not closed under orthocomplementation when more elements are added, i.e. $a_1^{\perp} = b_2$ and $a_2^{\perp} = b_1$ do not always hold; and second $\mathcal{L}_6^{\mathcal{M}}$ does not necessarily contain the empty set $\emptyset$ and the universal element a^*. However, we can say that the lattice $\mathcal{L}_6^{\mathcal{M}} = \mathcal{L}^{\mathcal{M}}$ associated wit the smallest model of RT_{EC}^{-} is isomorphic to O_6. But this model is explicitly ruled out by A12 (or condition (viii)). From this example of an orthocomplemented non-orthomodular lattice we showed that the class of lattices $\mathcal{L}_{RT}$ is not isomorphic to the class of orthocomplemented non-orthomodular lattices, but forms a proper subset thereof.

Corollary 4.34. *Each lattice in $\mathfrak{L}_{RT}$ is in the class of orthocomplemented, non-orthomodular lattices. I.e. there exist non-orthomodular ortholattices that cannot be associated to a model of RT.*

We remarked before that orthomodularity is just a weaker form of modularity. By showing that the lattices are strictly not orthomodular it is implied that they are strictly non-modular - confirming the result from the previous subsection. To demonstrate that, we use the equational characterizations of modular and orthomodular lattices:

Modularity [**Grä98**]:

$$x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z) \tag{EQ9}$$

Orthomodularity [**Kal83**]:

$$x \vee \left(x^{\perp} \wedge (x \vee z)\right) = x \vee z \tag{EQ10}$$

Since modularity is a dual lattice property, we can rewrite equation EQ9 as $x \vee (y \wedge (x \vee z)) = (x \vee y) \wedge (x \vee z)$, and consider the special case $y = x^{\perp}$. It follows: $x \vee \left(x^{\perp} \wedge (x \vee z)\right) = (x \vee x^{\perp}) \wedge (x \vee z)$ and consequently the orthomodular equation $x \vee \left(x^{\perp} \wedge (x \vee z)\right) = (x \vee z)$.

5.3. Semimodularity. Semimodularity is yet another weakening of modularity. Semimodular lattices are covered extensively in [**Ste99**]. Semimodular lattices were first analyzed by Birkhoff who considered affine incidence geometries that are no longer modular, but retain some properties of modular lattices. Wilcox then showed a slightly stronger property for the axiomatization of affine geometry that are atomless (pointless). The ensuing lattices are M-symmetric [**Wil39**], which is equivalent to upper semimodularity with an additional condition, referred to as Wilcox condition [**Ste99**]. For finite lattices, upper semimodularity and M-symmetry coincide.

Affine geometry (see for a lattice theoretic account e.g. [**Ben83**]), a generalization of Euclidean geometry, is a natural way to represent e.g. spatial models (incidence geometries) of mereotopology. Therefore without modularity holding in the models, the geometric interpretation of semimodular lattices points to their applicability to mereotopology. Though, we show that even this weaker property does never hold in the models of RT^{-}_{EC}. Notice further that semimodular lattices are a special case of geometric lattices [**Ste99**].

DEFINITION 4.35. [**Ste99**] A lattice is called (upper) semimodular if it satisfies the following condition

$a \wedge b \prec a \rightarrow b \prec a \vee b$

A lattice is lower semimodular if it satisfies

$a \wedge b \prec a$ and $a \wedge b \prec b$ together imply $a \prec a \vee b$ and $b \prec a \vee b$.

THEOREM 13. *No lattice in $\mathfrak{L}_{RT}$ is upper or lower semimodular.*

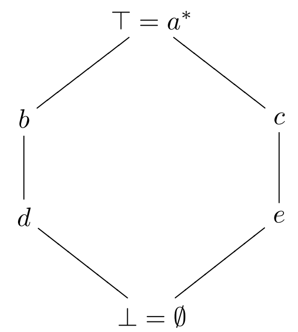
PROOF. Recall the existence of $\mathcal{L}_6^{\mathcal{M}}$, figure 5.1(a), in any model $\mathcal{M}$ of RT from lemma 4.29. The elements of $\mathcal{L}_6^{\mathcal{M}}$ do not satisfy the conditions for semimodularity. In particular, $\mathbf{c_2} \wedge \mathbf{b_2} \prec \mathbf{b_2}$ but $\mathbf{c_2} \not\prec \mathbf{c_2} \vee \mathbf{b_2}$; and $\mathbf{b_2} \wedge \mathbf{c_2} \prec \mathbf{b_2}$ and $\mathbf{b_2} \wedge \mathbf{c_2} \prec \mathbf{c_2}$ but $b_2 \not\prec \mathbf{c_2} \vee \mathbf{b_2} = \mathbf{a}$. □

REMARK 6. We also deduce that the lattices $\mathfrak{L}_{RT}$ are not geometric or matroid lattices.

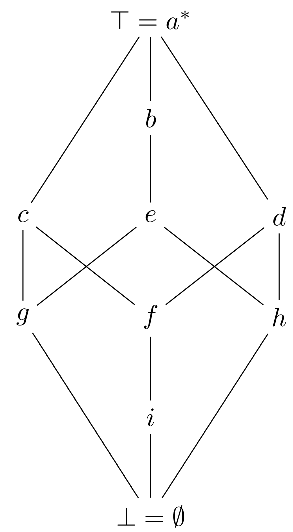
DEFINITION. [**Grä98**] A lattice L is called geometric if and only if L is semimodular, L is algebraic, and the compact elements of L are exactly the finite joins of atoms of L.

5.4. Weak modularity. Weakly modular lattices are not always semimodular, but all semimodular lattices are weakly modular. Weak modularity is defined by a rather complicated condition, but it can be simplified for finite lattices [**Grä98**]. We use the following definition in terms of modular pairs (introduced by Wilcox) that make the relationship to M-symmetric and thus semimodular lattices visible.

DEFINITION 4.36. [**MM70**] Let a and b be elements of a lattice L. We say (a, b) is a modular pair, in short $(a, b)M$, if $\forall c \leq b\,[(c \vee a) \wedge b = c \vee (a \wedge b)]$. If a pair is not weakly modular, we write $(a, b)\bar{M}$.



(a) weakly modular



(b) non weakly modular: $(c, b)\bar{M}$

FIGURE 15. Examples of lattices associated to models of RT^-

A lattice L with zero element 0 is weakly modular if in L, $(a \wedge b \neq 0)$ implies $(a, b)M$.

A lattice L with 0 is called $\bot$-symmetric if in L, $(a, b)M$ and $a \wedge b = 0$ together imply $(b, a)M$.

A lattice L is called M-symmetric if $(a, b)M$ implies $(b, a)M$.

We just give an example, compare figure 5.4(b), of a model of RT^- whose lattice $\mathcal{L}^{\mathcal{M}}$ is not weakly modular. But there also exist lattices in $\mathfrak{L}_{RT}$ that result in weakly modular lattices, e.g. figure 5.4(a). Hence we cannot state something about weak modularity as strong as we did for modularity, orthomodularity, and semimodularity for the lattices in $\mathfrak{L}_{RT}$.

NOTE 4.37. Not all lattices $\mathfrak{L}_{RT}$ are weakly modular.

EXAMPLE 4. *Consider the lattice in figure 5.4(b). It is a model of* RT^-_{EC} *with the following extensions: the extension* $P^{\mathcal{M}}$ *of parthood is given by the lattice with its transitive closure and reflexivity implied. Further* $EC^{\mathcal{M}} = \{\langle \mathbf{b}, \mathbf{f} \rangle, \langle \mathbf{f}, \mathbf{b} \rangle\}$, $OP^{\mathcal{M}} = \{\mathbf{a}^*, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{g}, \mathbf{h}, \mathbf{i}\}$, $CL^{\mathcal{M}} = \{\mathbf{a}^*, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g}, \mathbf{h}\}$, *and* $TP^{\mathcal{M}} = \{\langle \mathbf{b}, \mathbf{b} \rangle, \langle \mathbf{f}, \mathbf{f} \rangle\}$. *All other extension can be easily deduced from these.*

Now consider $\mathbf{b}$ *and* $\mathbf{c}$. *Because* $\mathbf{b} \wedge \mathbf{c} = \mathbf{g} \neq \emptyset$, $(\mathbf{b}, \mathbf{c})M$ *and* $(\mathbf{c}, \mathbf{b})M$ *must be satisfied for* $\mathcal{L}^{\mathcal{M}}$ *to be weakly modular. But then the equivalence of weak modularity*

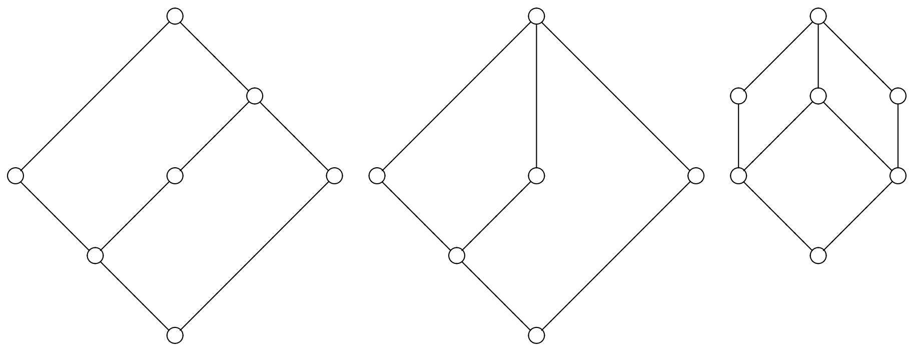


FIGURE 16. Hasse diagrams of forbidden sublattices in semi-distributive lattices

must hold, $(\mathbf{c}, \mathbf{b})M \iff \forall x \leq \mathbf{b}\,[(x \vee \mathbf{c}) \wedge \mathbf{b} = x \vee (\mathbf{c} \wedge \mathbf{b})]$*. Checking* $x = \mathbf{e}$ *(notice* $\mathbf{e} \leq \mathbf{b}$*), we get the following:* $(\mathbf{e} \vee \mathbf{c}) \wedge \mathbf{b} = \mathbf{e} \vee (\mathbf{c} \wedge \mathbf{b}) \iff \mathbf{a}^* \wedge \mathbf{b} = \mathbf{e} \vee \mathbf{g} \iff \mathbf{b} = \mathbf{e}$*. This is obviously false, therefore* $\mathcal{L}^{\mathcal{M}}$ *is not weakly modular.*

On the other side, considering the lattice in figure 5.4(a), this is weakly modular, since we only need to check for each totally ordered chain in the lattice ($(\mathbf{b}, \mathbf{d})M$ and $(\mathbf{c}, \mathbf{e})M$), which are trivially weakly modular.

However, the example $(c, b)M$ does not violate $\perp$-symmetry for the lattice in figure 5.4(b). We need to show that $(b, c)M$ does not hold as well: $(b, c)M \iff \forall x \leq c\,[(x \vee b) \wedge c = x \vee (b \wedge c)]$. As possible cases for x only g, f, i need to be checked. In fact, f makes the condition for $(b, c)M$ fail $(f \vee b) \wedge c = f \vee (b \wedge c) \iff a = f \vee g \iff a = c$. This particular example suggests that the lattices might be $\perp$-symmetric, we some interesting consequences [**MM70, BJ72**]. A complete proof or counterexample remains open. Since we already know that the lattices are not M-symmetric, $\perp$-symmetry would define the finest possible of the models with respect to symmetry/modularity properties.

CONJECTURE 1. *All lattices in* $\mathfrak{L}_{RT}$ *are* $\perp$*-symmetric.*

5.5. Semidistributivity. Semidistributivity is a lattice property weaker than distributivity but independent of modularity. Every distributive lattice is also meet- and join-semidistributive. Like orthomodularity is a stricter property of orthocomplementation, semidistributivity is stricter than pseudocomplementation as observed by [**Ste99**].

LEMMA 4.38. [**Ste99**] *(Meet-/join-) semidistributive lattices are (meet-/join-) pseudocomplemented.*

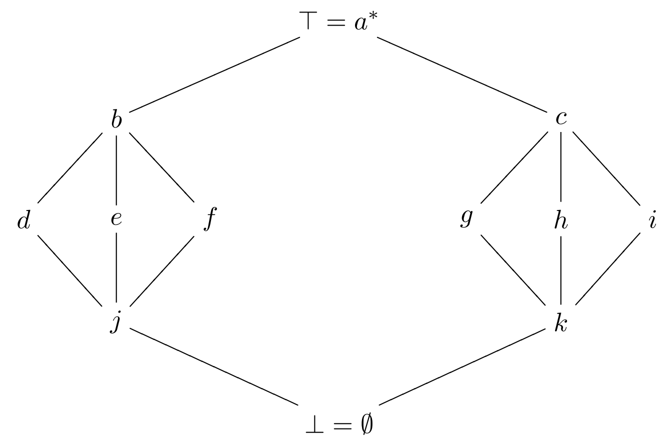


FIGURE 17. Example of a lattice associated to a model of RT^-_{EC} that is not semidistributive

Proving semi-distributivity for all lattices in $\mathfrak{L}_{RT}$ would then verify the property of pseudocomplementedness. But we show that the lattices in $\mathfrak{L}_{RT}$ are not always semidistributive. That draws a clear boundary between pseudocomplementedness and the stronger property of semidistributivity for the class $\mathfrak{L}_{RT}$. Similar to modularity or distributivity, failure of semidistributivity can be detected through a set of sublattice. Most importantly, this set contains the diamond, thus admitting a straightforward example of a lattice in $\mathfrak{L}_{RT}$ that is not semidistributive.

DEFINITION 4.39. [**Grä98**] A lattice is called meet-semidistributive if and only if

$(SD_\wedge)$ $u = x \wedge y = x \wedge z \rightarrow u = x \wedge (y \vee z)$

A lattice is called join-semidistributive if and only if

$(SD_\vee)$ $u = x \vee y = x \vee z \rightarrow u = x \vee (y \wedge z)$

A lattice that is both meet-semidistributive and join-semidistributive is called semidistributive.

THEOREM 4.40. [**DPR75**] *A lattice L of finite length is semi-distributive if and only if it contains no sublattice isomorphic to M_3, L_2, L_3, L_3^d, L_4, and L_4^d.*

EXAMPLE 5. *Take the example of figure 5.5, it is a model of RT^-_{EC}, but it contains the diamond M_3 as sublattice, e.g. the sublattice formed from $\mathbf{b}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{j}$ is a diamond. In this diamond it holds that $\mathbf{d} \wedge \mathbf{e} = \mathbf{d} \wedge \mathbf{f} = \mathbf{j}$ but $\mathbf{j} \neq \mathbf{d} \wedge (\mathbf{e} \vee \mathbf{f}) = \mathbf{d} \wedge \mathbf{b} = \mathbf{d}$. An similar example violating join-semidistributivity can be made using $\mathbf{d}, \mathbf{e}, \mathbf{f}$.*

NOTE 4.41. Not all lattices in $\mathfrak{L}_{RT}$ are semi-distributive (i.e. join- and meet-semi-distributive).

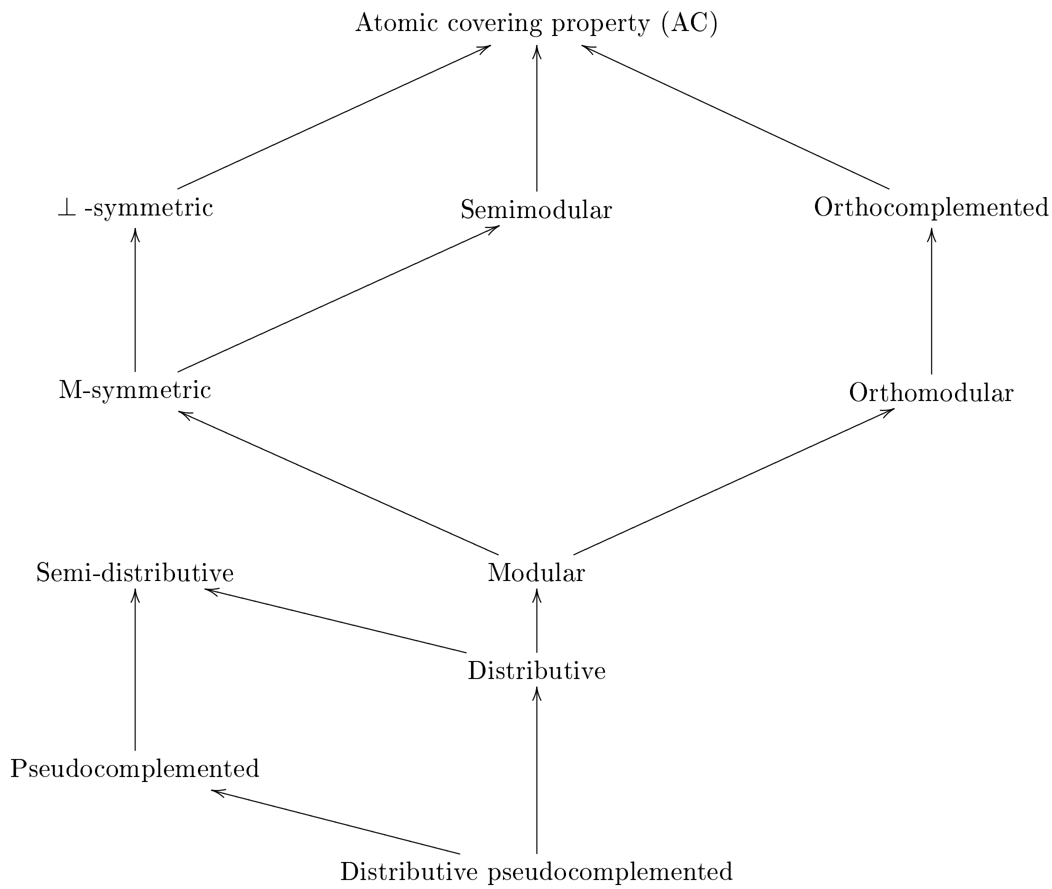


FIGURE 18. Relationship between lattice properties (modularity and distributivity) used in this section
The relations are only captured for complete lattices.

An open question here is whether there exist models that are semidistributive. We know that the 6-element lattice $\mathcal{L}_6$ which corresponds to a model of RT^-_{EC} is semidistributive. However, it is not clear whether the combination with weak contact yields strictly non-semidistributive lattices.

5.6. Summary. This section shows that any common kind of modularity or distributivity either completely fails for the lattices in $\mathfrak{L}_{RT}$ or we cannot make a precise statement forcing certain kinds of modularity or ruling it out. Failure of orthomodularity draws a clear boundary of lattices in $\mathfrak{L}_{RT}$ being always orthocomplemented, but never orthomodular. Similarly for pseudocomplementation: we know all lattices in $\mathfrak{L}_{RT}$ are doubly pseudocomplemented, but we know of examples that are not semidistributive. Another borderline exists at the most general level in figure 18: we know that the models are not semimodular, but we conjectured that they are $\perp$-symmetric. Because any kind of modularity and distributivity fails, we consequently ask whether all of the models of RT^-_{EC} are isomorphic to the atomic non-modular p-ortholattices. In chapter 6 it will turn out that this is true and it is also isomorphic to the not uniquely complemented atomic p-ortholattices.

REMARK 7. The class of complete atomic orthocomplemented lattices (without pseudocomplementation) has been investigated by [**Mac64**] with respect to models in Hilbert space. "The lattice of all closed subspaces of a separable Hilbert space has the following properties. It is complete, atomic, irreducible, semi-modular, and orthocomplemented." [**Mac64**]. Our characterization of the class $\mathcal{L}_{RT}$ is closely related: complete atomic orthocomplemented pseudocomplemented lattices. However, by non-semimodularity, they form a disjoint class with the models of separable Hilbert space (which are strictly semimodular). Nevertheless we can apply some of the results obtained by MacLaren. In particular the following theorem about the center of such a lattice holds in the lattices in $\mathcal{L}_{RT}$ as well.

THEOREM 4.42. [**Mac64**] *If P is a complete atomic orthocomplemented lattice, the center of P is a complete atomic Boolean algebra.*

6. Models of RT^- as Lattices

So far, in this chapter we established that the lattices in $\mathfrak{L}_{RT}$ are complemented lattices (by theorems 3, 4, 5). Every lattice in $\mathfrak{L}_{RT}$ is further double pseudo-complemented and orthocomplemented (theorems 7 and 8), which together impose quite restrictive constraints on the class $\mathfrak{L}_{RT}$. Moreover, we excluded lattices with stronger properties of orthomodularity and semimodularity (theorems 12 and 13), and hence modularity and distributivity. We also ensured that the lattices are never atomistic, uniquely complemented, nor section-semicomplemented. With regard to weak modularity and semidistributivity we can only say that there exist models that do not satisfy these properties, but on the other side some of the lattices in $\mathfrak{L}_{RT}$ do (or might, in the case of semidistributivity) satisfy it. The following corollary summarizes the results of this chapter.

COROLLARY 4.43. *All models of RT^-_{EC} are orthocomplemented double-p-lattices (with the empty set removed) that are neither orthomodular nor semimodular.*

In short we call these lattices non-modular p-ortholattices. Notice that since finite lattices are always complete and atomic, we do not explicitly state this. Nevertheless, we focus on the finite models of RT^-_{EC} which we can assume to be complete and atomic.

From non-distributivity we learned that the lattice structures corresponding to models of RT are not a subclass of distributive pseudocomplemented lattices, hence neither Boolean algebras, Brouwerian lattices, Heyting algebras, nor Stone lattices. This result distinguishes the structures from the characterization of Clarke's models by Biacino and Gerla [**BG96**]. In [**BG96**], it was demonstrated that Clarke's *Calculus of Individuals* is isomorphic to the atomless Boolean algebras with the empty set removed (called a mereological field in [**Ger95**]) and that the *connection structures* (R, C) of Clarke's theory (satisfying the axioms A1-A4 of [**BG96**]) are equivalent to complete orthocomplemented lattices.

A note on mereological fields

REMARK 8. Different authors called the mereology in the sense of Leśniewski (see [**Lus62**]) and Leonard and Goodman [**LG40**] a Boolean algebra with the zero element removed. The term *mereological field* was coined for such a structure, apparently first used by Gerla in [**BG96, Ger95**], but it lacks a clear definition. It is implied that it does not make a difference whether to consider Boolean algebras or the *mereological field*s . However, they are remarkably different considering following perspective: a Boolean algebra is uniquely complemented and orthocomplemented. Supremum and infimum of a Boolean algebra are each others complement: $1' = 0$ and $0' = 1$. By removing the infimum (the empty set in mereotopology) the supremum lacks a complement. Suddenly, the resulting structure is not complemented anymore. If we choose an awkward variation of complementation: $1' = 1$, the complementation law $x \cap x' = 0$ is not satisfied. The point here is that describing mereological (an mereotopological) structures in terms of Boolean algebras with the infimum removed is rather problematic. What is required here is an exact (re)definition of *mereological field* to accommodate the fact that in a *mereological field* the supremum does not have a complement defined in the field itself.

PROPOSITION 4.44. *A mereological field is a Boolean algebra without infimum and the supremum being not complemented.*

This chapter proves that the mereotopological structures RT are indeed a proper subclass of Clarke's connection structures formed by A1-A4 of [**BG96**]. The lattices corresponding to models of RT are not only orthocomplemented but also pseudocomplemented and thus a specialization of Clarke's *connection structures*. However, the lattices in $\mathfrak{L}_{RT}$ are more general than the Boolean lattices corresponding to Clarke's complete *Calculus of Individuals*. These two classes are in fact disjoint:

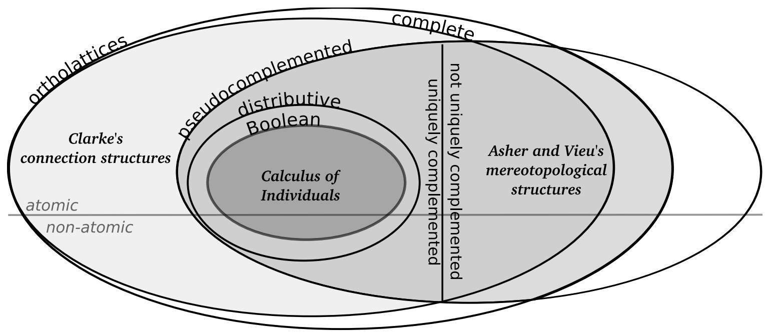


FIGURE 19. Clarke's models and the models of RT interpreted as lattices

the Boolean lattices being uniquely complemented, whereas the models of RT are strictly not uniquely complemented.

7. Isomorphic Characterization of the Models of RT^{-}

The previous description of the models of RT^{-}_{EC} as lattices is too weak for a characterization. Instead, we are aiming for a characterization up to elementary equivalence as achieved by Asher and Vieu for RT_0 in terms of the structures RT_T. We only know that all the models of RT_0, RT^{-}, and RT^{-}_{EC} give rise to p-ortholattices where the models of RT^{-}_{EC} are characterized by a more restricted set of p-ortholattices as summarized in corollary 4.43. The models of RT_0 must be again be a subset thereof.

Here, we take a step backwards in order to reach the desired isomorphic characterization of RT^{-}_{EC} and ultimately RT_0. We take advantage of the fact that a lot of the proofs, in particular for failure of unique complementation, orthomodularity and semimodularity rely on the existence of at least two non-open individuals that are externally connected according to proposition 3.5. If we remove axiom A12 from the theory and consider the less restricted variant RT^{-}, the consequences proved in the theorems 6, 11, 12, and 13 suddenly do not hold anymore. We conjecture that each finite model of RT^{-} is isomorphic to a finite (complete atomic) p-ortholattice without any restrictions on modularity and distributivity and prove it in this subsection. In particular, this means that any finite p-ortholattice can be associated to a model of RT^{-} regardless of semi- and orthomodularity and regardless of unique complementation. If it is uniquely complemented or distributive, then it is automatically a Boolean lattice. Therefore any specific p-ortholattice, i.e. a Boolean lattice, gives a model of RT^{-} although it might not yield a model satisfying a more restricted set of axioms such as RT^{-}_{EC} or RT_0.

THEOREM 14. ***The lattices in $\mathfrak{L}_{RT^-}$ are isomorphic to p-ortholattices.***

PROOF. The proof contains two directions; $\mathfrak{L}_{RT}$ $\rightarrow$*p-ortholattices* is already done, since we showed that each lattice in $\mathfrak{L}_{RT}$ for a model $\mathcal{M}$ is a p-ortholattice (notice the proofs for pseudocomplementedness and orthocomplementedness do not rely on finiteness of the models).

COROLLARY 4.45. ***The finite lattices in $\mathfrak{L}_{RT^-}$ are isomorphic to finite p-ortholattices with an interior function*** int ***defined on it so that*** $int(a) \neq 0$ ***holds for all*** a.[3]

Recall that we can only use this alternative definition of the finite models of RT because we proved that every finite model of RT_0 can be associated to a finite structure RT_T, satisfying conditions (i) to (vi) of the intended models. The completeness proof in [**AV95**] showed elementary equivalence of the models of RT_0 and the structures RT_T. However, that also gives infinite models of RT_T for some models of RT_0. Nevertheless, for RT^- we can use RT_0 and RT_T synonymously, since we proved by theorem 2 that every finite model of RT^- (expressed in terms of the axioms of RT_0) gives rise to at least one finite model as expressed by conditions (i) to (vi) on the topological space (Σ_U, Σ_U^T). In the unrestricted case (theorem 14) the completeness proof itself is sufficient for the elementary equivalence between models of RT^- and models in the respective subset of conditions of a structure RT_T.

Now we focus on the direction *p-ortholattices*$\rightarrow$ $\mathfrak{L}_{RT}$ of the theorem: we show that all pseudocomplemented ortholattices can be associated to structures satisfying the conditions (i) to (vi) of RT_T. For the purpose of the proof, conditions (ii) and (iii) are split up into more manageable subconditions. Except for the conditions (ii.3) and (iii.2) the proofs are straightforward and shortly outlined here. The proofs for (ii.3) and (iii.2) are separately treated afterward. Moreover, we can derive the following corollary about the finite models of RT^-.

(i) $Y \subseteq \mathcal{P}(X)$ and $X \in Y$
Satisfied if we map the top element of the lattice (guaranteed by boundedness and completeness of the lattices) to the set X.

[3]We need the last restriction to non-empty interiors because it is not clear whether every p-ortholattices gives some possibility to define the interior function in such a way. Moreover, we deal here with special p-ortholattices, since we earlier proved that the meet- and join-pseudocomplements are also complements.

(ii.1) $\forall x\,[int(x) \in Y]$

We know each element x' is the orthocomplement of some other element x (by involution property of orthocomplementation). Setting the unique join-pseudocomplement of x' to $int(x)$ (theorem 7) gives us an interior for all elements.

(ii.2) $\forall x\,[int(x) \neq \emptyset]$

With $jpc(x) \vee x = 1$ it follows that the join-pseudocomplement $jpc(x)$ of x cannot be $\emptyset$, unless $x = 1$. However, then top element always has itself as interior.

(ii.3) $\forall x\,[int(x) = int(cl(x))]$ - *See separate proof further down.*

(iii.1) $\forall x\,[cl(x) \in Y]$

We know each element x' is the orthocomplement of some other element x (by involution property of orthocomplementation). Setting the unique meet-pseudocomplement of x' to $cl(x)$ (theorem 7) gives us a closure for all elements.

(iii.2) $\forall x\,[cl(x) = cl(int(x))]$ - *See separate proof further down.*

(iv) $int(\sim x) \neq \emptyset \rightarrow \sim x \in Y$

Forced by orthocomplementation each element in the lattice has a unique orthocomplement. If the orthocomplement is the empty set (follows from the interior being empty), then the element is not in Y.

(v) $\forall x, y\,[int(x \cap y) \neq \emptyset \rightarrow (x \cap^* y) \in Y]$

We simply use the element representing the greatest lower bound (meet) of x and y to map to $x \cap^* y$. Since the lattice must be complete, this greatest lower bound exists for any pair of elements x, y. If the meet is $\emptyset$, then $x \cap^* y$ is not an element of Y.

(vi) $\forall x, y\,[(x \cup^* y) \in Y]$

We choose the element representing the least upper bound of x and y to map to $x \cup^* y$. Completeness again ensures the existence of this element for every pair x, y.

To prove (ii.3) and (iii.2) we restate these conditions in purely lattice-theoretic terms. To achieve that, we use the notions of orthocomplementation as well as the meet- and join-pseudocomplements. In particular, we claim that replacing the standard topological operations interior and closure by their corresponding lattice-theoretic formulations (using join- and meet-pseudocomplements) as stated in the theorem 7, leads to a theorem setting orthocomplementation and the two forms of pseudocomplementation in relation. We express $int(x)$ as the join-pseudocomplement of $x^{\perp}$ (the orthocomplement of x). $int(cl(x))$ is then the join-pseudocomplement of the orthocomplement of the meet-pseudocomplement of $x^\perp$. I.e. (ii.3) can be rewritten as $jpc(x^\perp) = jpc\left[\left(mpc(x^\perp)\right)^\perp\right]$. Equally (iii.2) can be rewritten as $mpc(x^\perp) = mpc\left[\left(jpc(x^\perp)\right)^\perp\right]$ where $x^\perp$ represents the orthocomplement of x. Removing the innermost orthocomplements from the formulae simplifies them to

$$jpc(x) = jpc\left[(mpc(x))^\perp\right] \tag{EQ11}$$

$$mpc(x) = mpc\left[(jpc(x))^\perp\right] \tag{EQ12}$$

For proving these we need the following general theorem for complete p-ortholattices (applies not just for the finite ones, but also infinite complete p-ortholattices).

THEOREM 15. *In every complete p-ortholattice $jpc(p) \leq p^\perp \leq mpc(p)$ holds for any lattice element p where $^\perp : Y \to Y$ is the orthocomplement, $jpc : Y \to Y$ the join-pseudocomplement, and $mpc : Y \to Y$ the meet-pseudocomplement operation in the lattice.*

PROOF. Assume the contrary, i.e. $p^\perp \ngeq jpc(p)$: with $p \vee p^\perp = 1$ this violates the definition of the join-pseudocomplement in 4.15. The same applies for $p^\perp \nleq mpc(p)$ and $p \wedge p^\perp = 0$. □

COROLLARY 4.46. *In every complete p-ortholattice $mpc(p)^\perp \leq p \leq jpc(p)^\perp$ holds for any lattice element p where $^\perp : Y \to Y$ is the orthocomplement, $jpc : Y \to Y$ the join-pseudocomplement, and $mpc : Y \to Y$ the meet-pseudocomplement operation.*

PROOF. Follows directly from the order-reversing law of orthocomplementation: $a \leq b \to (a^\perp \geq b^\perp)$. See e.g. [**Ber85**]. □

Using the notation of MacLaren [**Mac64**], one could restate the theorem and corollary in a more elegant way.

DEFINITION 4.47. [**Mac64**] Two elements a and b are said to be orthogonal if $a \leq b^\perp$. In this case we write $a \perp b$. The relation of being orthogonal is obviously symmetric.

COROLLARY 4.48. *In a complete p-ortholattice $p \perp mpc(p)$ and $p \perp jpc(p)$ holds for any lattice element p, where $jpc : Y \to Y$ the join-pseudocomplement, and $mpc : Y \to Y$ the meet-pseudocomplement operation.*

Finishing the proof of theorem 14. Now we can prove the equations EQ11 and EQ12 from above. In the following we show that these are immediate consequences in any finite p-ortholattice where the orthocomplementation $^\perp$ is the topological complement. The proof (we show it for EQ12) is by contradiction: we show that if for any element p of the lattice, $m = mpc(p)$ and $m' = mpc\left[(jpc(p))^\perp\right]$ are satisfied, then $m = m'$. We distinct the following cases depending on the relative position of m and m' in the lattice:

(a) assume m incomparable to m',
(b) assume $m > m'$ and
(c) assume $m < m'$.

All three cases lead to a contradiction, thus the only valid solution being $m = m'$. Note hereby that the corollary 4.46 can be restricted to: $p < jpc(p)^\perp$ and $p > mpc(p)^\perp$, otherwise $m = m'$ would follow immediately from $p = jpc(p)^\perp$ and $p = mpc(p)^\perp$, respectively. In the following we do the proof for equation EQ12 only, for equation EQ11 it is analogous.

Case (a) assume that m and m' are incomparable.

We know that $jpc(p)^\perp \geq p$: $jpc(p) \leq p'$ (the join-pseudocomplement is the smallest of all complements of p), in particular it then holds $jpc(p) \leq p^\perp$ and thus by the order-reversing law of ortholattices $jpc(p)^\perp \geq p$ follows. Then naturally it follows that $mpc(jpc(p)^\perp) \leq mpc(p)$. Hence $m' \leq m$ and m and m' are comparable - contrary to the assumption.

Case (b) assume that $m > m'$ holds.

Assuming $m \wedge jpc(p)^\perp = \emptyset$ with $m > m'$ would lead to a contradiction because m' can no longer be the meet-pseudocomplement of $jpc(p)^\perp$. Hence, $m \wedge jpc(p)^\perp > \emptyset$ must hold. By completeness of the lattice, this intersection results in some element, let us denote it by z, s.t. $m \wedge jpc(p)^\perp = z$. Since (i) $m' \wedge jpc(p)^\perp = \emptyset$ and (ii) $m \wedge p = \emptyset$, the element z must further satisfy following properties: (iii) $z \wedge m = z$, (iv) $z \wedge jpc(p)^\perp = z$, (v) $z \wedge p = \emptyset$, and (vi) $z \wedge m' = \emptyset$ (because $jpc(p)^\perp \wedge m' = \emptyset$ and $z < jpc(p)^\perp$). Now let us consider the element $(z \vee m)$: From (ii) and (v) it follows $(z \vee m) \wedge p = \emptyset$ (by DeMorgan laws which apply for all orthocomplemented lattices, see [**Kal83**]) with $(z \vee m) \geq m$, hence m cannot be the meet-pseudocomplement of p unless $z = m$ and we derive a contradiction again: no such z can exist that is distinct from m, hence $m = m'$.

Case (c) assume that $m < m'$ holds.

Again $m' \wedge p = \emptyset$ would lead to a contradiction because m can no longer be the meet-pseudocomplement of p. Hence, $m' \wedge p > \emptyset$. We know $jpc(p)^\perp \geq p$ from corollary 4.46, and therefore $m' \wedge jpc(p)^\perp \geq m' \wedge p$ (notice that $a \geq b \to c \wedge a \geq c \wedge b$ holds in any complete lattice, thus in the finite lattices). With $m' \wedge jpc(p)^\perp = \emptyset$ we obtain $m' \wedge p = \emptyset$, which is contradictory to our previous assumption $m' \wedge p > \emptyset$.

From the cases (a), (b), and (c) all resulting in a contradiction, $m = m'$ must hold. Thus $mpc(p) = mpc\left[jpc(p)^\perp\right]$ for any element p in the lattice. Analogous one can prove $EQ11$. Together, $EQ11$ and $EQ12$ finish the proof for (ii.3) and (iii.2) and thus for theorem 14. □

Theorem 14 then leads to the following representation theorem for RT^-.

THEOREM 16. ***(Representation Theorem for RT^-) The lattices arising from models of RT^- are isomorphic to p-ortholattices.***

So far, our lattice-theoretic results can be summarized as:

1. The lattices arising from models of RT^- are isomorphic to p-ortholattices;
2. The lattices arising from finite models of RT^- are isomorphic to the set of finite (complete atomic) p-ortholattices;
3. The lattices arising from models of RT_0 are not atomistic, semimodular, orthomodular, nor uniquely complemented.

Since we have no tools to deal with external connection or weak contact in lattices directly, we need a different approach to characterize the models of full RT_0 or even RT^-_{EC}. We extend the characterization given here to the finite models of RT^-_{EC} in chapter 6 by providing a simple way of finding a suitable extension of EC. However, we first need a graph-theoretic representation of the models of RT_0 to fully account for external connection. Recall that EC cannot be represented appropriately in purely lattice-theoretic terms. Most importantly, we show that every finite (complete atomic) p-ortholattice can be extended by a non-empty extension EC in order to form a model of RT^-_{EC}.

Isomorphism to models of convex regions

REMARK 9. We use special properties of convex regions to satisfy the above properties. From [**Lin06**], p. 24 we know that convex regions satisfy (ii.3) unconditionally and (iii.2) if the interior is non-empty (always satisfied by (ii.2)). Thus if all elements of an arbitrary complete atomic p-ortholattices can be associated to convex regions, this will result in a topological space of convex regions satisfying the conditions (i) to (vi) of RT_0. Vice versa if we have a topological space defined as in RT_0, then each non-convex region can be transformed to an isomorphic convex regions. However, convexity is not a characteristic property of topological operations: think of a piece of stretch material: if it can be stretched to cover a convex region, but it can always be stretched to cover a non-convex region. In the mereotopology of Borgo et al. [**BGM96**] convexity is explicitly models through a predicate. Hence, the observation that the models of RT^- seem to always correspond to models with only convex regions causes the question whether for some applications the theory RT is not powerful enough and one should consider instead the theory presented in [**BGM96**]. First this theory needs to be characterized model-theoretically as well.

CONJECTURE 2. *Any model satisfying the axioms RT^- (or the equivalent conditions (i) to (vi) of RT_0) is isomorphic to a model that satisfies these axioms but in which every region is convex.*

CHAPTER 5

Graph-Theoretic Characterization

In the introduction we mentioned the possibility of different approaches when characterizing mereotopological models. After first attempting a characterization via topological spaces, we turned in the previous chapter to a lattice-theoretic approach. Although this approach was much more successful than the use of topological spaces in capturing essential structural properties, in particular of the parthood relation, it only resulted in an isomorphic characterization of the models of RT^- as complete atomic p-ortholattices. We were able to prove additional properties such as non-modularity for the more restrictive models of RT^-_{EC}, but eventually failed to characterize these models up to isomorphism. In particular, the constructed lattices fail to capture the essential relation external connection. This prompts taking a different perspective on the models of RT^-_{EC} by means of graphs. Graphs have a long tradition in discrete mathematics and are successfully applied in different contexts in computer science. In particular they are useful for characterizing finite structures based on binary relations between elements in a domain of interest. Because the theory RT relies on a single dyadic primitive *connection,* C, a look at the models of RT^-_{EC} as graphs is suggested. Since C is a reflexive and symmetric relation, all the models can be represented as undirected simple graphs. Note that throughout the remaining chapters of the thesis we only consider finite models of the theory, as the infinite models are more difficult to capture by graphs and are of less practical importance considering reasoning with RT_0.

Again, we will first consider subsets of the axioms that account for the theory RT_0 and characterize the corresponding models. But instead of weakening the theory, we separate first its topological core, the *connection structures* (or *contact structures*), and characterize its graphs in terms of modules. Afterward we regard the mereological structures which include concepts such as parthood and overlap and characterize the resulting posets in terms of chordal (and presumably comparability) graphs. Finally we look again at the theory RT^-_{EC} including external connection. It turns out to be problematic, since the resulting graphs are not even perfect anymore. We can give a classification in terms of dually chordal graphs but at the same time we show how maximum neighborhood orderings are too weak to capture essential inheritance properties of the parthood structures underlying the graphs. Instead we define in section 4.4 a stronger vertex ordering: a maximum neighborhood inclusion ordering. The graphs in $\mathfrak{G}_{RT^-_{EC}}$ that admit such orderings have the striking property that always a partial order on the neighborhoods of the vertices can be found. In fact, we prove in section 4.5 how the search algorithm Cardinality LexBFS (CLBFS) can find such an ordering in linear time if it exists. In the appendix we demonstrate that these orderings have additional nice properties on the graphs in $\mathfrak{G}_{RT^-_{EC}}$, e.g. for determining topological complements of the models.

1. Representing Models of RT_0 as Graphs

We adhere to the terminology for graphs as used in [**BLS99**]. Open neighborhoods denoted by $N(v) = \{u | uv \in E(G)\}$ do not include vertex v itself whereas closed neighborhoods denoted by $N[v] = N(v) \cup \{v\}$ do include it.

PROPOSITION 5.1. *Each model $\mathcal{M}$ of RT_0 (or a subset of the axioms) can be associated to a graph $G(\mathcal{M}) = (V, E)$ where $V_G = Y^{\mathcal{M}}$ and $\mathbf{xy} \in E_G \iff \langle \mathbf{x}, \mathbf{y} \rangle \in C^{\mathcal{M}}) \iff [\![x]\!]_g \cap [\![y]\!]_g \neq \emptyset$.*

The last part of the equivalence uses the notation defining the intended models of RT. By $\mathfrak{G}_{RT^-_{EC}}$ we denote the class of graphs containing all graphs that can be associated to a model of RT^-_{EC}. All graphs in $\mathfrak{G}_{RT^-_{EC}}$ are simple, i.e. contain no parallel edges and no self-loops, and are directed and connected graphs. n is the number of vertices and m the number of edges for any graph. Further concepts are introduced as needed.

Throughout this chapter we use the example model $\mathcal{M}$ of RT^-_{EC} as shown in figure 20.

2. Graphs of the Connection Structures

2.1. Connection structures. The axioms A1 to A3 of RT_0 define an extensional ground topology $\mathbf{T}$ in the sense of [**Var96, CV99**], with axiom A3 forcing extensionality. At the same time this theory is effectively a Strong Mereotopology [**CV99**]. All properties we prove about the connection structures here generally apply to all strong mereotopologies and extensional ground topologies.

Alone from the definition of the graphs in $\mathfrak{G}_{RT^-_{EC}}$, we derive the following simple lemma.

LEMMA 5.2. *In a model $\mathcal{M}$ of RT_0 for two elements $\mathbf{x}, \mathbf{y} \in Y^{\mathcal{M}}$, $\langle \mathbf{x}, \mathbf{y} \rangle \in C^{\mathcal{M}}$ holds if and only if $xy \in E_G$ in the graph $G(\mathcal{M})$ arising from $\mathcal{M}$.*

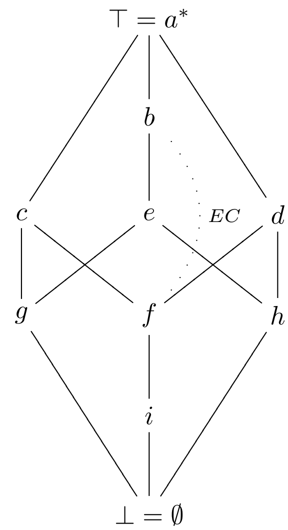


FIGURE 20. Example of a model $\mathcal{M}$ of RT^{-}_{EC}, represented as lattice $PP(\mathbf{e}, \mathbf{b})$, $PP(\mathbf{f}, \mathbf{d})$, $PP(\mathbf{f}, \mathbf{c})$, etc. Additionally, **b** and **f** are externally connected in this model, i.e. $EC(\mathbf{b}, \mathbf{f})$. In this example **b** is the orthocomplement of **i**, **e** of **f**, **c** of **h**, and **d** of **g**.

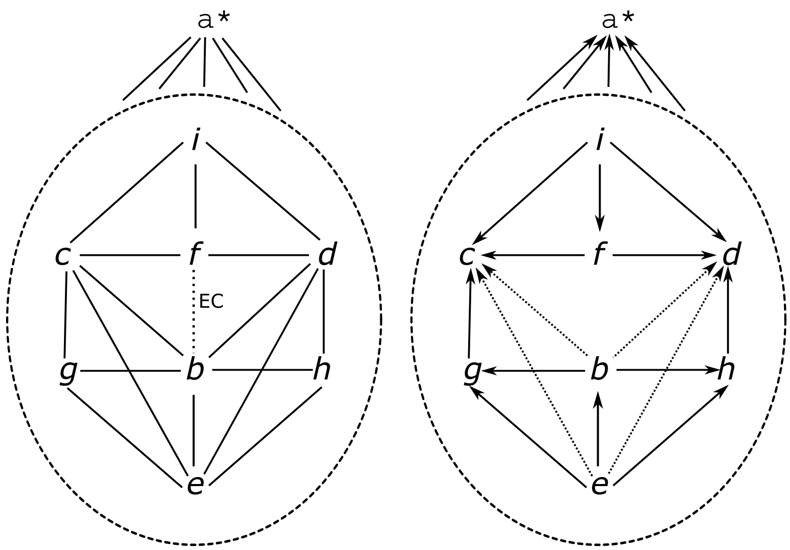
From a graph-theoretic perspective, axiom A1 is superfluous in a simple graph and axiom A2 allows to only consider undirected graphs. Thus any undirected simple graph immediately satisfies A1 and A2. Hence we only need to capture A3 through the notion of twins as we show next.

2.2. Twin modules.

DEFINITION 5.3. [**BLS99**] Let $G = (V, E)$ be a graph. The subset $M \subseteq V$ is a module (or homogeneous set) in G if for all vertices $u, v \in M$ and $w \in V \setminus M$, $uw \in E$ if and only if $vw \in E$. M is a trivial module in G if $M = V$, $M = \emptyset$, or $|M| = 1$. If G contains only trivial modules, it is a prime graph.

Two vertices $x, y \in V$ are twins if $\{x, y\}$ is a module in G. Twins x, y are true twins if $xy \in E$, otherwise x, y are false twins.

A graph is twin-free if it contains no module of size two. Note that deviating definitions excluding only false twins exist, e.g. in [**KL96, Kot97, FK02**]. An alternative characterization of twin-free graphs can be given using neighborhoods, where two separate cases cover (a) false twins and (b) true twins. Note that neither of the conditions itself is sufficient alone. In the case of true twins x, y , $y \in N(x)$ but $y \notin N(y)$ and in the case of false twins $y \notin N[x]$ but $y \in N[y]$.



(a) connection graph $G(\mathcal{M})$ for the model $\mathcal{M}$ from 20

(b) transitive orientation $G'(\mathcal{M})$ of $G(\mathcal{M})$. The dashed edges represent proper overlap, so **b** and **e** must "separate" their non-connected neighborhoods $\{\mathbf{c}, \mathbf{g}\}$ and $\{\mathbf{d}, \mathbf{h}\}$ from transitivity.

FIGURE 21. Connection graph $G(\mathcal{M})$ of the model $\mathcal{M}$ from figure 20 and an exemplary transitive orientation of the graph

LEMMA 5.4. *A graph G is twin-free if and only if G has no pair of vertices $x, y \in G$ such that*

(a) $N(x) = N(y)$ *or*

(b) $N[x] = N[y]$

Surprisingly, twin-free graphs have only been considered from a combinatorial point of view in the context of "identifying codes" [**CHHL07**]. There, twin-free graphs guarantee the existence of a unique code for each vertex. However, twin-free in [**CHHL07**] is limited to being free of false twins. Apparently, there is no common characterization of twin-free graphs comparable to the separation axioms in topological spaces that guarantee different degrees of unique ("separated") neighborhoods for every pair of points. For an example take figure 2.2(a), there no two vertices have a common open or closed neighborhood (notice that $bf \in E_G$).

3. Graphs of Mereological Structures

Throughout this section, we consider the parthood structures while ignoring the connection structures for the moment. We already know that the connection structures are isomorphic to the class of twin-free graphs. So if a given parthood structure is twin-free, it will satisfy the axioms imposed on the connection structures. Notice that by choosing undirected graphs, axiom A2 is implicitly assumed, whereas we still choose axiom A1 to be satisfied, although without further implications. However, axiom A3 is not assumed to be satisfied in the mereological structures. On the opposite, the axioms A5 to A8 characterizing mereological notions are assumed to hold. From T0.9, we can derive the following relation between proper parthood in a model $\mathcal{M}$ of RT_0 and the neighborhoods in $G(\mathcal{M})$.

LEMMA 5.5. *In a model $\mathcal{M}$ of RT_0 for two elements $\mathbf{x}, \mathbf{y} \in Y^{\mathcal{M}}$, $\langle \mathbf{x}, \mathbf{y} \rangle \in PP^{\mathcal{M}}$ holds if and only if $N[\mathbf{x}] \subset N[\mathbf{y}]$ in the graph $G(\mathcal{M})$ arising from $\mathcal{M}$.*

Since PP puts a partial order on the elements of a model, the neighborhoods in the corresponding graphs will be partially ordered as well. This leads us directly to the next subsections, defining the graphs of the mereological structures as chordal (and presumably comparability) graphs.

3.1. Overlap Cliques (O-clique). For further explanations we need one crucial concept in the graphs in $\mathfrak{G}_{RT^-}$ and also $\mathfrak{G}_P$: a O-clique. An O-clique arises when two or more lattice elements have a meet in the lattice that is not the zero element, i.e. a non-empty intersection of individuals in a model $\mathcal{M}$ of RT^- or the axioms in $RT_P = \{A1, A2, A4, A5, A6, A7, A8\}$. Then the overlapping elements together with all their parents form a clique in the corresponding graph $G_P(\mathcal{M})$. This is an important observation for these models, since every pair of elements in such a clique that are not connected to distinct elements outside the clique forms a module of size two in the graph. Hence the graph is not twin-free. We call such an O-clique maximal if no other vertex can be added without maintaining the clique property.

DEFINITION 5.6. For a model $\mathcal{M}$ of RT^- or RT_P, the elements in a set V_O form an O-clique if and only if $\forall x, y \in V_O\, [O(x, y)]$. An O-clique is maximal if $\neg\exists z \notin V_O, \forall x \in V_O\, [O(x, z)]$.

EXAMPLE 6. *Reconsider the lattice in figure 20, the sets $\{\mathbf{a}^*, \mathbf{b}, \mathbf{c}, \mathbf{e}, \mathbf{g}\}$, $\{\mathbf{a}^*, \mathbf{c}, \mathbf{d}, \mathbf{f}, \mathbf{i}\}$, $\{\mathbf{a}^*, \mathbf{b}, \mathbf{d}, \mathbf{e}, \mathbf{h}\}$ are all maximal O-cliques. If two elements x, y are exactly in the same maximal O-cliques, then their neighborhoods are identical: $N[x] = N[y]$ and the graph is not twin-free. In the example this applies to the pair $\mathbf{b}, \mathbf{e}$ and the pair*

$\mathbf{f}, \mathbf{i}$ *(see figure 2.2(a) without the external connection). Making the graph $G(\mathcal{M})$ twin-free while maintaining the partial order given by PP of the model $\mathcal{M}$ requires to add unique extensions of external connection for every such pair of elements. We will discuss the construction of such an extension in the next chapter. Notice that a graph G_p can contain cliques that are not O-cliques, for instance $\{\mathbf{a}^*, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}\}$ is such a clique in figure 2.2(a).*

3.2. Comparability graphs. Posets can be represented as comparability graphs. Define the graph $G_P(\mathcal{M}) = (V(G(\mathcal{M})), E(G(\mathcal{M})) \setminus E_{EC^{\mathcal{M}}})$ that contains edges between any two individuals overlapping in the model $\mathcal{M}$. The set of edges $E_{EC^{\mathcal{M}}}$ represents the extension of external connection in $\mathcal{M}$. All such graphs $G_P(\mathcal{M})$ make up the class $\mathfrak{G}_P$.

If we further restrict this graph to a subgraph $G_P^{PP}(\mathcal{M}) \subseteq G_P(\mathcal{M})$ containing only edges for parthood relations between individuals (without overlap), we immediately get the *underlying graph* (see e.g. [**MS91**]) of the poset (V, P) with P being the parthood relation over a set V . Every such graph G_P^{PP} is hence a comparability graph.

Comparability graphs are defined through the existence of a transitive orientation. By an orientation $G' = (V, E')$ of a graph $G = (V, E)$, we mean that for each $xy \in E$, either $xy \in E'$ or $yx \in E'$ holds. An orientation $G' = (V', E')$ is transitive if E' is a transitive relation on V. Ghouila and Houri proved the following equivalence, see [**BLS99**]. This theorem is now commonly conceived as definition of a comparability graph.

Theorem 5.7. [**BLS99**] *A graph is a comparability graph if and only if it has a transitive orientation.*

Any graph $G_P^{PP}(\mathcal{M})$ is transitively orientable simply by being the underlying graph of the poset. It might be possible to even create transitive orientations for all graphs in $\mathfrak{G}_P$. Notice that for every O-clique in a graph in $\mathfrak{G}_P$ there exists a set of transitive orientation independent. In principle every edge between two vertices can be directed in such an O-clique arbitrary, but with direct consequences on the other vertices. In general it is far from obvious whether all graphs contained in $\mathfrak{G}_P$ this class are comparability graphs. We can easily verify from the definition D4 that the theorem $\forall x, y\, [EC(x, y) \rightarrow \neg O(x, y)]$ always holds. Without external connection relations in the graphs in $\mathfrak{G}_P$, the extension of C is then equivalent to the extension of O, thus the edges in a graph in $\mathfrak{G}_P$ represent exactly the overlap relations between elements x, y that have a part z with $z \leq x$ and $z \leq y$ in common. Of course either $z = y$ or $z = x$ can hold (equivalent to $\langle y, x\rangle \in P$ or $\langle y, x\rangle \in P$), but equality $x = y = z$ is not reflected in the graph. When referring to $\mathfrak{G}_P$ we mean the class of graphs associated with models of $RT_P = \{A1, A2, A4, A5, A6, A7, A8\}$ that additionally satisfy the axiom $\forall x, y\, [C(x, y) \equiv O(x, y)]$ and each contained graph has a (irreflexive) partial order defined on it by P (PP).

For instance, each of the sets of vertices $\{\mathbf{a}^*, \mathbf{b}, \mathbf{c}, \mathbf{e}, \mathbf{g}\}$, $\{\mathbf{a}^*, \mathbf{c}, \mathbf{d}, \mathbf{f}, \mathbf{i}\}$ in figure 20 forms an O-clique. However, two vertices that are complements in the model $\mathcal{M}$ cannot be connected, but some vertices can be in the neighborhood of both. For example is $g = \sim d$ and $\langle \mathbf{g}, \mathbf{d} \rangle \notin C^{\mathcal{M}}$, but $\mathbf{b}, \mathbf{e} \in N[\mathbf{g}]$ and $\mathbf{b}, \mathbf{e} \in N[\mathbf{d}]$. For the transitive orientation of any graph G_P in $\mathfrak{G}_P$ this leads to a more general requirement: every vertex x that both a vertex v and its complement $\sim v$ are connected to cannot be on a directed path from v to $\sim v$, i.e. $vy_1, y_1y_2, \ldots, y_ix, xy_{i+1}, y_{i+1}y_{i+2}, \ldots, y_k \sim v$ cannot form a directed path in the extension of $E_{G'_P}$. For example $\mathbf{ge}, \mathbf{ed} \in E_{G'_P}$ is not allowed. Otherwise by transitivity v and $\sim v$ would have to be connected. We claim that for every model such a transitive orientation can be found taking the following approach: direct the vertices in general against the parthood order, i.e. $\langle x, y \rangle \in PP^{\mathcal{M}}$ leads to $yx \in E_{G'_P}$. But for all overlapping O-cliques, it has to be ensured that the elements in both cliques do not lead to new transitive edges that are not part of the connection relation of the original model $\mathcal{M}$. To guarantee that, we can use each such overlapping element only as source of outgoing edges or only as sink for incoming edges. Exceptions are only acceptable and necessary between two or more elements that are in the intersection of two O-cliques: they can be arbitrarily directed while violating the only-source or only-sink principle. The universal edge a^* can be either completely sink or completely source in any model of RT without effecting any other directed edges in G'_P.

CONJECTURE 3. *The graphs in* $\mathfrak{G}_P$ *are comparability graphs.*

Notice that the converse is definitely not true: there are comparability graphs that are not satisfying the axioms, i.e. not all posets are parthood structures.

EXAMPLE 7. *Take figure 2.2(b) as example, it gives a transitive orientation of 2.2(a) with both* $\mathbf{b}$ *and* $\mathbf{e}$ *being purely sources of outgoing edges except for* $\mathbf{eb}$*. In this sense, these two elements separate the cliques* $\{\mathbf{a}^*, \mathbf{b}, \mathbf{c}, \mathbf{e}, \mathbf{g}\}$ *and* $\{\mathbf{a}^*, \mathbf{b}, \mathbf{d}, \mathbf{e}, \mathbf{h}\}$ *from each other. The same applies for the O-cliques* $\{\mathbf{a}^*, \mathbf{b}, \mathbf{c}, \mathbf{e}, \mathbf{g}\}$ *and* $\{\mathbf{a}^*, \mathbf{c}, \mathbf{d}, \mathbf{f}, \mathbf{i}\}$ *that intersect in* $\mathbf{c}$*.* $\mathbf{c}$ *is purely a source in* G'_P*. It can only be source because there are already outgoing edges* $\mathbf{cb}, \mathbf{ce} \in E_{G'_P}$ *from* $\mathbf{c}$*. The only exception is* $\mathbf{cd}$ *which can be directed arbitrarily because* $\mathbf{c}$ *and* $\mathbf{d}$ *are only in the O-clique* $\{\mathbf{a}^*, \mathbf{c}, \mathbf{d}, \mathbf{f}, \mathbf{i}\}$ *together.*

However, it remains unclear whether all graphs G_P have such a transitive orientation. Nevertheless, comparability graphs capture structural properties of the models of RT_P.

3.3. Chordal graphs. The graphs in $\mathfrak{G}_P$ are furthermore chordal graphs. Besides the usual definition as graphs without induces k-cycles of for $k \geq 4$, chordal graphs can be defined in terms of the vertex orderings they admit.

DEFINITION 5.8. [**BLS99**] Let $G = (V, E)$ be a graph. The vertex $v \in V$ is simplicial in G if $N(v)$ is a clique in G.

The ordering $(v_1, v_2, \ldots, v_n)$ of the vertices in V is a perfect elimination ordering of G if for all $i \in \{1, 2, \ldots, n\}$, the vertex v_i is simplicial in $G_i = G(\{v_i, \ldots, v_n\})$.

THEOREM 5.9. [**FG65**] *A graph is chordal if and only if it has a perfect elimination ordering (peo).*

Notice that for every chordal graph, lexicographic breadth-first search (LexBFS) [**RTL76**] and maximal cardinality search (MCS) [**TY84**] are guaranteed to yield the reverse of a perfect elimination ordering (*peo*) [**RTL76**]. Using either search algorithm we can recognize chordal graphs in linear time.

PROPOSITION 5.10. *The graphs in $\mathfrak{G}_P$ are chordal (triangulated).*

PROOF. A graph is chordal if and only if it has no induced chordless cycle of four or more vertices. Assume the contrary, i.e. there exists a induced chordless cycle C_n of four or more vertices $x_0, x_1, \ldots, x_{n-1}$ with $x_i x_{i+1} \in E$ (assume + to be $mod(n-1)$). For any pair of adjacent vertices x_i, x_j it holds that $O(x_i, x_j)$. Consider all the P_3 induced on the C_n with x_k, x_{k+1}, and x_{k+2} and $x_k x_{k+1}, x_{k+1} x_{k+2} \in E_P$.

If (1) they are transitively oriented as $PP(x_k, x_{k+1})$ and $PP(x_{k+1}, x_{k+2})$ or vice versa, then by transitivity $PP(x_k, x_{k+2})$ (resp. $PP(x_{k+2}, x_k)$) follows immediately and C_n has in $x_k x_{k+2}$ a chord.

Otherwise (2) for no $i \leq n-1$ such induced P_3 with transitive proper parthood exists, then w.l.g. one has to orient the vertices as $PP(x_{k+1}, x_k)$ and $PP(x_{k+1}, x_{k+2})$ for all P_3 induced on the C_n. But then x_k and x_{k+2} overlap in x_{k+1} according to $D4$, i.e. $O(x_k, x_{k+2})$ and, by T0.19, $C(x_k, x_{k+2})$. Hence x_k and x_{k+2} are adjacent in the corresponding graph G_P and the induced cycle C_n has a chord. □

3.4. Chordal comparability graphs and strongly chordal graphs. From the last two subsection it follows immediately $\mathfrak{G}_P \subseteq chordal \cap comparability$, provided the conjecture that all the graphs in $\mathfrak{G}_P$ are comparability graphs holds.

CONJECTURE 4. *The graphs in $\mathfrak{G}_P$ are chordal comparability graphs.*

Chordal comparability graphs can be recognized by a linear-time algorithm of complexity $O(n + m)$ [**HM99, MS91**]. The recognition algorithm first transitively orients the graph in $O(n + m)$ according to a modular decomposition approach from McConnell and Spinrad [**MS94**] and then uses the algorithm proposed in [**MS91**] to test directed chordal graphs for transitivity with linear complexity.

Chordal comparability graphs not only admit a *peo*, but also a simple elimination ordering (short *seo*, or simple elimination scheme as they are called in [**BS99**]). The authors in [**BS99**] utilize the algorithm Cardinality LexBFS (CLBFS), which is a deviation of a traditional LexBFS (see section 4.5 for details), to generate a *seo* for any chordal comparability graph. However, not all graphs that admit a *seo* are chordal comparability graphs. Instead, Farber proved much earlier that a graph admits a *seo* (as well as a strong elimination ordering) if and only if it is strongly chordal.

DEFINITION 5.11. [**Far83**] A chord $x_i x_j$ in a cycle $C = (x_1, x_2, \ldots, x_{2k})$ of even length $2k$ is an *odd chord* if the distance in C between x_i and x_j is odd.

A graph G is strongly chordal if G is chordal, i.e. G has no induced cycle C_k of length $k \geq 4$, and each cycle of even length $k \geq 6$ has an odd chord.

An abundance of equivalent characterizations of strongly chordal graphs exist:

THEOREM 5.12. [**Far83, BCDV98, McK00**] *The following are equivalent for a graph G:*

(a) G is strongly chordal;

(b) G is chordal and every k-cycle with $k \geq 6$ has an odd chord;

(c) G admits a strong elimination ordering;

(d) every induces $H \trianglelefteq G$ has a simple vertex;

(e) every induced $H \trianglelefteq G$ is the clique graph of a chordal graph;

(f) G has a strong closed neighborhood tree;

(g) G contains no induced sun (trampoline).

A simple, linear-time conversion algorithm to obtain a strong elimination ordering from a *seo* was presented in [**SS03**]. Since the reverse is trivial (every strong elimination ordering is a *seo*), we can go back and forth between the two orderings in linear time. Since *seo*s are more general than strong elimination orderings, finding a linear time algorithm for producing a *seo* on strongly chordal graphs would immediately give a linear-time algorithm for recognition of strongly chordal graphs. Current recognition algorithms are based on the strong elimination orderings of strong chordal graphs with a best time complexity of $O(m \cdot log(n))$ or $O(k^2 n)$ where k is the size of the largest minimal vertex separator [**PK04**]. Strongly chordal graphs themselves are a subclass of doubly chordal graphs - the graphs that are both chordal and dually chordal. The dually chordal graphs are dealt with in the next section.

4. Graphs of Models of RT_{EC}^{-}

This section considers the intersection of connection structures with mereological structures, extended by axiom A11 requiring the existence of two externally connected individuals. These are exactly the graphs of the models that satisfy all axioms of RT_{EC}^{-} . However, the properties of the graphs of the mereological structures cannot be transferred to this more restricted class of models. The resulting graphs are not perfect anymore and thus cannot be a subclass of chordal nor comparability graphs.

LEMMA 5.13. *In a model $\mathcal{M}$ of RT_0 for two elements $\mathbf{x}, \mathbf{y} \in Y^{\mathcal{M}}$, $\langle \mathbf{x}, \mathbf{y} \rangle \in EC^{\mathcal{M}}$ is satisfied if and only if in the graph $G_{RT}(\mathcal{M})$ the following holds: $\mathbf{xy} \in E_G$ and no vertex $\mathbf{z}$ exists so that $\mathbf{z} \in (N[\mathbf{x}] \cap N[\mathbf{y}])$ and $N[\mathbf{z}] \subseteq N[\mathbf{x}] \cap N[\mathbf{y}]$.*

PROOF. $\Rightarrow$: One can easily see that if $\langle \mathbf{x}, \mathbf{y} \rangle \in EC^{\mathcal{M}}$ then $\mathbf{xy} \in E_G$. Assume the contrary for the second condition, i.e. there exists a $\mathbf{z}$ such that $\mathbf{z} \in (N[\mathbf{x}] \cap N[\mathbf{y}])$ and $N[\mathbf{z}] \subseteq N[\mathbf{x}] \cap N[\mathbf{y}]$. Then by lemma 5.5, $\langle \mathbf{z}, \mathbf{x} \rangle, \langle \mathbf{z}, \mathbf{y} \rangle \in P^{\mathcal{M}}$ and thus $\langle \mathbf{x}, \mathbf{y} \rangle \in O^{\mathcal{M}}$ would follow. That is an obvious contradiction to D4 because the extensions $O^{\mathcal{M}}$ and $EC^{\mathcal{M}}$ must be disjoint for any model $\mathcal{M}$.

$\Leftarrow$: By $\mathbf{xy} \in E_G$ it follows $\langle \mathbf{x}, \mathbf{y} \rangle \in C^{\mathcal{M}}$ must hold. If no $\mathbf{z}$ with $\mathbf{z} \in (N[\mathbf{x}] \cap N[\mathbf{y}])$ and $N[\mathbf{z}] \subseteq N[\mathbf{x}] \cap N[\mathbf{y}]$ exists, then $\mathbf{x}$ and $\mathbf{y}$ share no sub-neighborhood (i.e. have no common part), and then with $\langle \mathbf{x}, \mathbf{y} \rangle \in C^{\mathcal{M}}$, $\langle \mathbf{x}, \mathbf{y} \rangle \in EC^{\mathcal{M}}$ follows from D4 and D3. □

The most important property is the preservation of the parthood ordering given by P (respectively PP) in the models of RT_{EC}^{-}, thus lemma 5.5 still applies.

4.1. Forbidden subgraphs. In graph theory, many well-known graph classes can be characterized by a set of forbidden induced subgraphs. For example are the chordal graphs exactly the C_{n+4}-free graphs whereas the comparability graphs are described by a large set of forbidden induced subgraphs (see [**BLS99**]).

Although external connection follows strict rules regarding "inheritance" of parthood by lemma 5.5, it is impossible to capture the resulting graphs by a set of forbidden induced subgraphs. This observation is due to the simple fact that we can always take an arbitrarily large set of elements to construct a model in which these elements are externally connected. Taking exactly the set of vertices representing these elements in the corresponding graph, we induce a subgraph containing only these vertices and the external connections between them. Therefore, we can obtain induced subgraphs with any kind of properties. With arbitrary graphs allowed, no forbidden induced subgraph can be defined. Hence no set of forbidden induced subgraphs characterizes these graphs. Moreover, we are assured that no hereditary property on the graphs in $\mathfrak{G}_{RT^-_{EC}}$ can be discovered. On the contrary, many known graph classes can be characterized by a set of forbidden subgraphs. This significantly narrows the set of potential characteristic classes that have been thoroughly analyzed in graph theory. For example the strongly chordal graphs can be characterized as $\{(C_{n+4}, sun)\}$-free graphs [**Far83**]. However, some graph classes purely characterized through vertex orderings are not effected by this result. Since we already showed for the parthood structures that they have strong orderings (since they were chordal, they have perfection elimination orderings), we try to characterize or at least classify the graphs of models of RT^-_{EC} through vertex orderings they admit.

The lack of a set of forbidden induced subgraphs also leads to the following simple but momentous conclusion. It immediately allows us to rule out all classes of graphs that are subclasses of perfect graphs.

NOTE 5.14. Some graphs in $\mathfrak{G}_{RT}$ are not perfect.

PROOF. Recall that a graph is perfect if for all induced subgraphs $H \trianglelefteq G$, $\chi(H) = \omega(H)$. We know that for a cycle C_5, $\chi(C_5) \neq \omega(C_5)$. Since we can construct a cycle C_5 from a set of vertices $V = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5}\}$ by external connections, i.e. $\mathbf{v_1v_2}, \mathbf{v_2v_3}, \mathbf{v_3v_4}, \mathbf{v_4v_5}, \mathbf{v_5v_1} \in E$ with no other edges between vertices in V. We can construct a model $\mathcal{M}$ having C_5 as induced subgraph: take the individuals represented by $\mathbf{v_i}$ as smallest externally connected individuals, i.e. for all $\mathbf{x}$ in $\mathcal{M}$, $\langle \mathbf{x}, \mathbf{v_i} \rangle \in PP^{\mathcal{M}}$ implies $\langle \mathbf{x}, \mathbf{v_{i-1}} \rangle, \langle \mathbf{x}, \mathbf{v_{i+1}} \rangle \notin EC^{\mathcal{M}}$. Complete $\mathcal{M}$ by adding necessary complements and interiors for all individuals and sums for all pairs of individuals. Then $G(\mathcal{M})$ is not perfect. □

Graphs that cannot be classified as perfect graphs are generally assumed to have not many "nice" properties that can be exploited for standard (domination-like) graph problems [**BCDV98, BCD98**]. However, Brandstädt et al. noticed that perfection does not help solving domination-like problems. As example they remind us that for split graphs, which are a subclass of chordal graphs, the domination problem remains NP-complete [**BCD98**]. Consequently, it is also NP-complete for chordal graphs. However, for strongly chordal graphs, and hence also for chordal comparability graphs, there are known efficient algorithms to solve the domination problem, see e.g. [**Far84**]. Unfortunately, the graphs in $\mathfrak{G}_{RT_{EC}^{-}}$ are not completely subsumed by the class of strongly chordal graphs, since strongly chordal graphs are characterized by a forbidden induced subgraph, the sun. However, as we will see now, the graphs in $\mathfrak{G}_{RT_{EC}^{-}}$ share a lot of properties with strongly chordal graphs. In particular, a characterization through vertex orderings is promising. In the following subsection we look at the more general class of dually chordal graphs that completely contains the strongly chordal and the doubly chordal graphs. Since doubly chordal graphs must be chordal as well and such have the hereditary property of no chordless cycle of length ≥ 4, they do not contain the graphs $\mathfrak{G}_{RT_{EC}^{-}}$. Contrary, dually chordal graphs are not necessarily perfect and our results in fact prove that no set of forbidden induced subgraphs can exist.

4.2. Dually chordal graphs. Dually chordal graphs have like strongly chordal graphs a wide range of equivalent definitions, see [**BCDV98**]. From an alternative definition of a graph G being chordal if and only if its clique hypergraph $\mathcal{C}(G)$ is a dual hypertree, the naming of *dually chordal* graphs becomes clear.

DEFINITION 5.15. [**BCDV98**] A graph $G = (V, E)$ is dually chordal if and only if its clique hypergraph $\mathcal{C}(G)$ forms a hypertree.

Another way to define dually chordal graphs is through clique graphs of chordal graphs. One of the most formal definitions involves the Helly-property: dually chordal graphs are clique-Helly and clique chordal. One of the more common definitions uses maximum neighborhood orderings (*mno*), see 5.16(b). We cover *mno*s in-depth in the next subsection. Others again use duality in hypergraphs by representing a chordal graph as a hypergraph and taking its dual by switching vertices and hyperedges. The following equivalences have been proved for dually chordal graphs.

THEOREM 5.16. [**BCD98**] *Let $G = (V, E)$ be a graph. Then the following conditions are equivalent*

(a) G is a dually chordal graph;

(b) G has a maximum neighborhood ordering;

(c) $\mathcal{N}(G)$ is a hypertree;

(d) $\mathcal{D}(G)$ is a hypertree;

(e) $\mathcal{C}(G)$ is a hypertree;

(f) G is the underlying graph of a hypertree.

Most importantly, dually chordal graphs are not closed under any induced subgraphs and one can construct any kind of graph as induced subgraph by adding new universally connected vertices to an arbitrary graph [**BCD98**].

Although dually chordal graphs generalize chordal, doubly chordal, strongly chordal, and chordal comparability graphs which are all perfect, dually chordal graphs are not necessarily perfect. Dually chordal graphs have been proposed primarily as a more general class of graphs in which domination problems are still easily solvable. Like chordal comparability graphs, dually chordal graphs can be detected in linear time with $O(|E|)$ for connected graphs [**BCD98**]. An even more general class of graphs is defined through the existence of a homogeneous elimination ordering [**BDN97**]. These homogeneously orderable graphs include besides dually chordal graphs also distance-hereditary, and homogeneous graphs.

4.3. Maximum neighborhood orderings. Chordal graphs have been characterized by *peo*s, strongly chordal graphs by *seo*s, and their generalization to dually chordal graphs can be characterized by a vertex ordering as well, the so-called maximum neighborhood orderings (*mno*). Moreover, if a *mno* exists for a graph, it can be constructed in linear time [**BCD98**].

Definition 5.17. [**BCD98**] A vertex ordering $(v_1, \ldots, v_n)$ is a maximum neighborhood ordering (*mno*) of G if for all $i \in \{1, \ldots, n-1\}$, the vertex v_i has a maximum neighbor $u_i \in N_i[v_i]$ in $G_i = G \setminus \{v_1, \ldots, v_{i-1}\}$, i.e. for all $w \in N_i[v_i]$, $N_i[w] \subseteq N_i[u_i]$.

Theorem 5.18. [**BCDV98**] *A graph G is dually chordal if and only if it admits a maximum neighborhood ordering.*

In [**BCDV98, BCD98**] it is also shown that recognizing whether a graph admits a *mno* can be done in linear time with $O(|V|+|E|)$. It uses an algorithm for detecting α-acyclicity of a hypergraph. A *mno* can also be found on a dually chordal graph in linear time using a slightly modified Maximum Cardinality Search (MCS) algorithm (the search paradigm itself is not altered).

DEFINITION 5.19. [**DHMO94**] A vertex ordering $(v_1, \ldots, v_n)$ is a domination elimination ordering (*deo*) of G if for all $i \in \{1, \ldots, n-1\}$, there is a $j > i$ such that the vertex v_i is dominated by v_j in $G_i = G \setminus \{v_1, \ldots, v_{i-1}\}$, i.e. $N_i(v_i) \subseteq N_i[v_j]$.

Just by the definition it seems that every *mno* is also a *deo*. In particular, in a *deo* every vertex v_i is dominated by some vertex v_j with $j > i$ whereas in a *mno* each vertex v_i is dominated by a vertex v_j that is in the neighborhood of v_i. Notice however that a vertex can dominate itself in a *mno*, but not in a *deo*. Therefore not every *mno* is also a *deo*.

COROLLARY 5.20. *A mno is a deo if each vertex v_i in the ordering is dominated by some vertex in $N_i(v_i)$.*

Moreover, notice the close relation to *cop-win* orderings as defined in [**NW83**], they have for each vertex v_i a dominating vertex v_j in $N[v_i]$. In a *cop-win* ordering, every vertex v_i with $i < n$ must have some vertex v_j with $j > i$ such that $N_i[v_i] \subseteq N_i[v_j]$. Obviously, v_j must be in the neighborhood of v_i. A *cop-win* ordering is a generalization of a *mno*: in a *mno* every vertex v_i with $i < n$ must have a vertex v_j in its neighborhood so that $N_i[v_i] \subseteq N_i[v_j]$ and the neighborhoods of all other vertices in $N_i[v_i]$ must be subsumed by $N_i[v_j]$. Hence, every *mno* is automatically also a *cop-win* ordering.

4.4. Maximum neighborhood inclusion orderings. With respect to the class $\mathfrak{G}_{RT^-_{EC}}$, each graph yields a set of vertex orderings with $v_n = a^*$, which all are *deo*s and *mno*s. This is for the reason mentioned in [**BCD98**]: a^* is a universally connected vertex, so any graph extended by it will be a dually chordal graph. It is easy to see why: a^* is in the neighborhood of all other vertices and subsumes the neighborhoods of all other vertices. Hence every vertex has in a^* a trivial maximum neighbor that dominates it in G_i. But we are looking for a more restricted kind of vertex ordering in which the vertices are at least partially ordered by their neighborhoods, we call it maximum neighborhood inclusion ordering or short *mnio*.

DEFINITION 5.21. A vertex ordering $V_G = (v_1, \ldots, v_n)$ on G is a maximum neighborhood inclusion ordering (*mnio*) if and only if

(1) for all $i \in \{1, \ldots, n-1\}$, there is a $j > i$ such that vertex v_i has a maximum neighbor $v_j \in N_i[v_i]$ in $G_i = G \setminus \{v_1, \ldots, v_{i-1}\}$ and

(2) for all $i \in \{1, \ldots, n-1\}$, there exists no j with $j > i$ such that $N(v_j) \subset N[v_i]$.

As one can easily see, condition (1) defines an *mnio* to be an *mno* as well which leads to the following immediate consequence.

PROPOSITION 5.22. *Graphs with a mnio are dually chordal.*

Condition (2) gives the desired ordering of neighborhoods: any vertex v_i with a neighborhood $N([v]$ subsuming the neighborhood $N(v_j)$ of v_j must appear after v_j in the ordering, i.e. $i > j$. Moreover, this condition captures implicitly that a graph with an mnio must be twin-free. In this way we merge the connection structures that correspond to twin-free graphs with the mereological structures that we generalized from $\mathfrak{G}_P$ being chordal graphs with *peo*s (and maybe also *seo*s) to $\mathfrak{G}_{RT_{EC}^-}$ being dually chordal graphs with *mnio*s. Hence, *mnio*s seem to capture the essential properties of the graphs in $\mathfrak{G}_{RT_{EC}^-}$.

PROPOSITION 5.23. *Graphs with mnios are twin-free (free of false and true twins).*

PROOF. Assume G is not twin-free, i.e. there exist at least two vertices $x, y \in V$ s.t. either (a) $N[x] = N[y]$ (true twins) or (b) $N(x) = N(y)$ (false twins). Assume (a) holds, then $N(x) \subset N[y]$ and $N(y) \subset N[x]$ so no matter whether we order x before or after y, condition (2) of the definition of a *mnio* is violated. Assume (b) holds, then $N[x] \supset N(y)$ and $N[y] \supset N(x)$ and condition (2) can again not be satisfied independent of how x and y are ordered relatively to another. □

A main motive for the definition of *mnio*s is that the proper parthood ordering is preserved by a *mnio*. We can easily see from lemma 5.5 holding for all graphs in $\mathfrak{G}_{RT_{EC}^-}$ that an individual of a model $\mathcal{M}$ of RT has all its parts ordered before itself in any *mnio*.

THEOREM 5.24. *Any mnio σ on a graph $\mathfrak{G}_{RT_{EC}^-}$ preserves the parthood ordering of all pairs of vertices $x, y \in Y^{\mathcal{M}}$, i.e. $PP(x, y)$ implies $x <_\sigma y$.*

Note that this property applies even when external connections are present. Moreover, we remark that for any chordal comparability graph that arises from an acyclic poset, the algorithm gives an order-preserving vertex ordering. That means that any so-found *mnio* preserves the partial ordering of all potential underlying orders of a chordal comparability graph (or a graph in $\mathfrak{G}_{RT_{EC}^-}$). Interestingly, that may help generating all potential partial orders or e.g occurrence trees embedding all potential partial orders.

It is not clear whether twin-free graphs with a *peo* (i.e. twin-free chordal graphs) automatically also yield a *mnio*. We know they have a *mno*, but condition (2) of an *mnio* is not trivially satisfied. If every twin-free graph with a *peo* also yields a *mnio*, one could ask whether each *peo* of such a graph itself is a *mnio*. The same questions apply to the relation to *seo*, and strong elimination orderings. We leave these as open problems for future research. Notice that reversely, the graphs in

Algorithm 1 Cardinality Lexicographic Breadth-First-Search (CLBFS)

input: graph $G(V, E)$
output: an ordering σ of G
create one partition class and place all vertices in it
order the vertices decreasingly by their cardinality
for $i \leftarrow 1$ **to** n **do**
 choose and remove the first vertex v from the first class
 $\sigma(i) \leftarrow v$
 foreach class C **do**
 split class C into two classes, $C' = C \cap N(v)$ followed by $C'' = C \setminus C'$
 (maintain the relative order of vertices in C within C' and C'')
 if C' is empty, delete it; if C'' is empty, delete it
 end for
end for

$\mathfrak{G}_{RT_{EC}^{-}}$ are not forced to have any *peo*, *seo*, or a strong elimination ordering, since all of these are restricted to some subset of perfect graphs. Notice further, that every graph in $\mathfrak{G}_{RT}$ must also have such a characteristic *mnio*.

4.5. CLBFS for finding *mnio*s on graphs in $\mathfrak{G}_{RT}$. Apart from their definition by some elimination scheme, vertex orderings such as *peo* or *seo* are used to characterize a certain class of graph. We want to show that every graph in $\mathfrak{G}_{RT}$ yields an *mnio*. We achieve this by giving a concrete algorithm, Cardinality LexBFS, that always finds such an *mnio* for any graph in $\mathfrak{G}_{RT}$.

Cardinality LexBFS (CLBFS) is a slightly changed version of the traditional LexBFS algorithm as proposed by Rose, Tarjan, and Lueker [**RTL76**]. In LexBFS whenever the next vertex is selected, it is chosen randomly from those with lexicographically largest label. In CLBFS the vertices are then ordered in decreasing order of their cardinality (i.e. their vertex degree), and a vertex of highest lexicographic label and highest cardinality is chosen (not to be confused with selecting vertices with highest visited cardinality as applied in MCS). Notice that if two or more vertices have the same lexicographic label and equivalent degrees, one of it is chosen at random again.

Practically, the algorithm applies a degree ordering before conducting the search using traditional LexBFS to ensure efficiency. Counting the degrees of all vertices and sorting them (i.e. bucket sorting) has linear-time complexity, and since LexBFS has also linear-time complexity, we know that Cardinality LBFS is also a linear-time search algorithm. In algorithm 1 we use pseudo-code to sketch out CLBFS in the style of [**Kru05**].

Inspired by the use of CLBFS for the linear time recognition of chordal comparability graphs and the fact that a lot of the assumptions underlying the modular decomposition algorithm can be extended to the graphs in $\mathfrak{G}_{RT}$ although these are not chordal comparability graphs (in fact neither chordal nor comparability graphs), we conduct CLBFS on the graphs in $\mathfrak{G}_{RT}$. Notice that twin-freeness as defined in lemma 5.4 is derived as a special property of prime graphs in the recognition algorithm for chordal comparability graphs [**HM99**]. Moreover, we strongly belief that we can characterize the graphs in $\mathfrak{G}_{RT}$ by a stronger vertex ordering than *mno*s. As we saw in the previous subsection, *mno*s characterize the complete class of dually chordal graphs, whereas we can intuitively define stronger orderings on graphs in $\mathfrak{G}_{RT}$. Here we make the relationship between CLBFS and maximum neighborhood inclusion orderings (*mnio*) for the graphs in $\mathfrak{G}_{RT}$.

THEOREM 5.25. *On every graph in $\mathfrak{G}_{RT}$, CLBFS yields the reverse of an mnio.*

PROOF. Condition (1) of an *mnio* holds: As one easily sees, the universal vertex a^* is selected first by the CLBFS algorithm: initially, no vertex is labeled, but the a^* is the vertex with highest degree in the graph and no other vertex can have the same degree, unless it is equal to a^* or violates A3. Hence, the reverse of any possible vertex ordering σ in a graph of a model of RT generated by CLBFS contains a^* as vertex with largest index and thus is a *mno*. Consequently, it satisfies condition (1) of the definition of a *mnio*.

Condition (2) of an *mnio* holds: Assume there exists an ordering σ that is generated by CLBFS and a pair of vertices $v_i, v_j \in \sigma$ with $j <_\sigma i$ such that $N(v_j) \subset N[v_i]$. Since the graph must be twin-free, $|N(v_j)| < |N(v_i)|$. Every vertex v_k with $k < i$ in the neighborhood of v_j must also be in the neighborhood of v_i. So if v_j has the lexicographic largest label at any point k of the search while $k < i$, then v_i has the same lexicographic label. Since v_i has the greater cardinality of the two, it will always be preferred to v_j and thus $i <_\sigma j$, contrary to the assumption. Hence the assumption was wrong and for the reverse $\pi = \sigma^{-1}$ of any vertex ordering of resulting from CLBFS, $j <_\pi i$ holds for $N(v_j) \subset N[v_i]$, and condition (2) of the definition of an *mnio* is satisfied.

With both conditions (1) and (2) always satisfied, the reverse $\pi = \sigma^{-1}$ of a CLBFS ordering on a graph in $\mathfrak{G}_{RT}$ is always an *mnio*. □

As outlined before, CLBFS can be conducted in linear time. That means if an CLBFS on a given graph G does not obtain the reverse of a *mnio*, we definitely know that the graph is not a model of RT^{-}_{EC}. However, this process involves checking a given order for the *mnio* properties. We know from section 4.3 that any *mno* can be recognized at least in quadratic time, maybe even more efficient. Hence property (1) of a *mnio* can be tested with the same efficiency. In a similar fashion

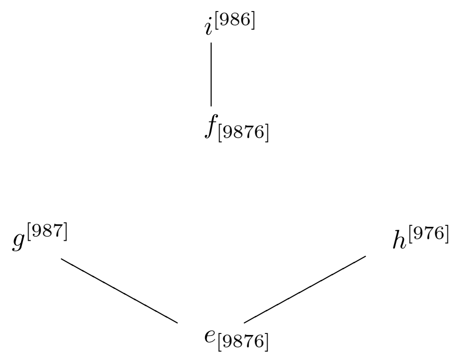


FIGURE 22. Example of graph $G_{RT}(\mathcal{M})$ after selecting four vertices using CLBFS

one can test for property (2) with at worst $O(n^2)$, so the overall recognition of a *mnio* is at worst bounded by $O(n^2)$ and hence polynomial.

However, it is not difficult to see that the reverse of the theorem above is not true: there are graphs on which CLBFS yield the reverse of a *mnio*, but that are not in $\mathfrak{G}_{RT}$. This is mainly due to the axioms A5 to A8 that require the existence of unique complements and interiors for vertices and unique sums for every pair of vertices.

EXAMPLE 8. *Consider the graph $G_{RT}(\mathcal{M})$ from figure 2.2. It has the following cardinality ordering that will be used for the CLBFS: $\{\mathbf{a}^*, \{\mathbf{c}, \mathbf{d}, \mathbf{b}\}, \mathbf{e}, \mathbf{f}, \{\mathbf{h}, \mathbf{i}, \mathbf{g}\}\}$ with the degrees (cardinalities) $\{8, \{7, 7, 7\}, 6, 5, \{4, 4, 4\}\}$ in the same order. The algorithm must start by selecting $\mathbf{a}^*$ because initially not vertex has a lexicographic label. Afterward it is free to choose amongst $\mathbf{b}, \mathbf{c}, \mathbf{d}$. But must choose all of those subsequently, because they are included in their appropriate neighborhoods. After four steps we get e.g. $\sigma = \{\mathbf{a}^*, \mathbf{c}, \mathbf{b}, \mathbf{d}\}$ and the graph as shown in 22 with the labeling l remains. Notice that we use labels instead partitions for better illustration. At this point $\mathbf{e}$ and $\mathbf{f}$ have the same lexicographically largest label, namely "9876". However, the cardinality of $\mathbf{e}$ is greater than that of $\mathbf{f}$, therefore $\mathbf{e}$ must be chosen next. In a standard LexBFS, either vertex could have been chosen. Afterward, the vertices $\mathbf{f}, \mathbf{g}, \mathbf{i}, \mathbf{h}$ must be selected in this order due to their distinct labels. Here we already see that the second half of the vertices will always be chosen in a predefined order depending on the order of the first half set of vertices; that motivates the complement discussion in the next subsection. The vertex ordering $\sigma = \{\mathbf{a}^*, \mathbf{c}, \mathbf{b}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g}, \mathbf{i}, \mathbf{h}\}$ results, which is the reverse of an* mnio *for the graph G_{RT} of the model $\mathcal{M}$. The reverse σ^{-1} is a mno because a^* dominates all other vertices as previously explained. Moreover, one can easily verify that the neighborhood of no vertex subsumes the neighborhood of any of the vertices appearing later in σ^{-1}.*

5. Conclusion

From the proof of theorem 5.25 and the knowledge that each *mnio* is always an *mno* just from their definitions, we know that the graphs in $\mathfrak{G}_{RT}$ are a subclass of the dually chordal graphs. Furthermore, the mnio of such a graph has certain special properties which can be exploited to find complements and the partial ordering on neighborhoods in linear time as demonstrated in the appendix "Complements as Dominating Pairs in Graphs in $\mathfrak{G}_{RT}$". We summarize our results in the following theorem.

THEOREM 17. *Each graph in $\mathfrak{G}_{RT}$ has an mnio. Moreover, the class $\mathfrak{G}_{RT}$ is a subclass of the dually chordal graphs.*

The restrictions on the connection structures are fully embodied in the graphs with mnio, i.e. every graph with a *mnio* is twin-free and therefore a *connection structure* (satisfying the axioms of a *Strong Mereotopology*). In the reverse, not every connection structure necessarily yields an *mnio*. Moreover we accounted for the neighborhood inclusion in the graphs in $\mathfrak{G}_{RT}$: every graph with an *mnio* is guaranteed to have a neighborhood inclusion ordering that is a partial ordering. However, the characterization through *mnio*s is not strong enough either; it does not give a isomorphic characterization of the graphs in $\mathfrak{G}_{RT}$. Dually chordal graphs are a very general class of graphs that contains graphs that are not corresponding to any model of RT_0. The *mnio* of the graphs provide a more restricted characterization, but we doubt that the graphs with an mnio are exactly the graphs in $\mathfrak{G}_{RT}$. The graphs with an mnio do not in general ensure that a sum for every pair of individuals exists and that every individual has a unique complement and interior.

But recall the motivation for this chapter: we wanted to provide adequate means to represent the models of the full theory RT_0 in a mathematical way and found simple undirected graphs appropriate. Nevertheless, we have not considered the edges caused by external connection in any detail. This was mainly caused by the lack of adequate graph classes apart from the graphs with maximum neighborhood inclusion orderings that we defined ourselves. Since even these orderings are not sufficient for an isomorphic characterization, we try a combined graph- and lattice-theoretic approach in the next chapter. This combination of lattices and graphs enables us to give a full characterization (up to isomorphism) of the models of RT^-_{EC} and extend that to a characterization of the models of RT_0.

CHAPTER 6

Characterization of RT^-_{EC} and RT as Graphs of Lattices

Chapter 4 shows that a complete characterization up to isomorphism even of the finite models of RT^-_{EC} solely in terms of lattices is difficult to achieve. Nevertheless, we already obtained an isomorphic characterization of the finite models of RT^-. Together with the models of RT^-_{EC} being representable as non-modular p-ortholattices, we aim for a construction that guarantees to find a non-empty extension of EC for any of these lattices. Notice that we need to overcome an intrinsic limitation of lattices: they can only model a partial order, not arbitrary connections. We represent the partial order given by the proper parthood relation PP of a model as a lattice, but there is no room for modeling additional relations in the lattice. For this reason the last chapter uses graphs in an attempt to characterize the finite models of RT^-_{EC}. Although we were able to extract some interesting properties about the connection structures and the mereological structures, we failed to completely characterize the finite models of RT^-_{EC}. Graphs are in general not suitable to deal with mereotopological operations as interior, closure, complement, union, etc. To overcome this problem and to profit from the peculiar advantages of graphs and lattices, we combine the two approaches: characterizing the models of RT^-_{EC} in terms of lattices that give rise to a certain class of graphs. Again, we restrict ourselves to the finite models of RT^-_{EC}. In particular, we can choose the class of finite p-ortholattices, represent them as graphs and find a way to extend them to models of RT^-_{EC}. This leads us to a result of the following format: each graph obtained from a non-semimodular, non-orthomodular, and non-uniquely complemented finite p-ortholattice extended by additional edges (representing external connection) that satisfies some graph properties is a model of RT^-_{EC} where the vertices are individuals and the edges define the extension $C^{\mathcal{M}}$. We formalize the approach by giving a construction that describes how to obtain a model of RT^-_{EC} from any such restricted finite p-ortholattice. Then we can eventually say that the finite models of RT^-_{EC} are isomorphic to non-modular (or not uniquely complemented) finite p-ortholattices.

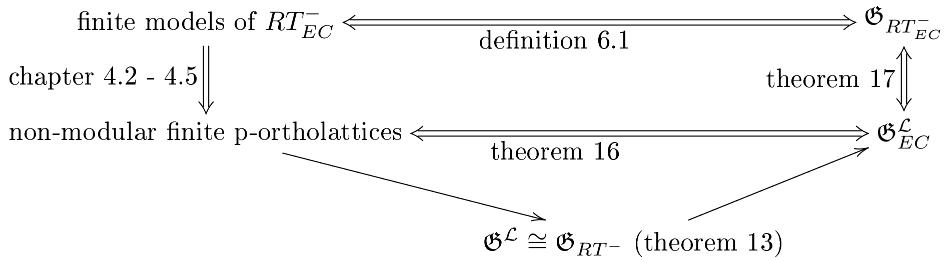


FIGURE 23. Proof that graphs of non-modular finite p-ortholattices yield models of RT^-_{EC}
The isomorphism between the class of finite models of RT^-_{EC} and the class of non-modular finite p-ortholattices is proved indirectly through the equivalence on the right. The equivalence at the bottom is used as foundation for the proof of theorem 18. The graphs in $\mathfrak{G}^{\mathcal{L}}$ are not twin-free, but the graphs $G^{\mathcal{L}}_{EC} = \big(V(G^{\mathcal{L}}), E(G^{\mathcal{L}}) \cup E_{EC}\big)$ in $\mathfrak{G}^{\mathcal{L}}_{EC}$ are twin-free and E_{EC} is guaranteed to be non-empty.

1. Characterization of P-ortholattices as Graphs

1.1. Notation. With $G^{\mathcal{L}}$ we refer to a graph that results from a finite p-ortholattice $\mathcal{L}$ (not semimodular, orthomodular, nor uniquely complemented). It is the graph of the overlap relation O of a model of RT^-_{EC}. The class of such graphs is referred to as $\mathfrak{G}^{\mathcal{L}}$. In $\mathfrak{G}^{\mathcal{L}}$ each graph $G^{\mathcal{M}}$ is also a graph of the connection relation C (but not necessarily a *connection structure*, i.e. it is not twin-free) of a model $\mathcal{M}$ of RT^-_{EC} where each edge xy represents the two incident vertices x, y to be connected, i.e. $\langle x, y\rangle \in C^{\mathcal{M}}$, in the corresponding model of RT^-.

$G^{\mathcal{L}}_{EC}$ refers to a graph that can be constructed from a graph $G^{\mathcal{L}}$ in a later defined way. $G^{\mathcal{L}}_{EC}$ is an extension of $G^{\mathcal{L}}$ in the sense that it includes edges for the external connection relation EC of a model of RT^-_{EC}. All the graphs $G^{\mathcal{L}}_{EC}$ together constitute the class $\mathfrak{G}^{\mathcal{L}}_{EC}$. The objective of this chapter is to prove that for finite models $\mathfrak{G}_{RT^-_{EC}} \cong \mathfrak{G}^{\mathcal{L}}_{EC}$, i.e. the class of graphs constructed from the non-modular finite p-ortholattices with uniquely-defined, non-empty sets E_{EC} (disjoint from $E_{G^{\mathcal{L}}}$) and the class of all graphs representing the connection relation C for some finite model of RT^-_{EC} are isomorphic.

1.2. Graphs of lattices of RT^-_{EC}. Let $\mathcal{L}$ be a non-modular finite p-ortholattice over the elements $Y' = Y \cup \emptyset$. We construct a graph $G^{\mathcal{L}}$ from it in the following way. Note that the empty set $\emptyset$ is discarded in the graph. The first rule defines the vertices of $G^{\mathcal{L}}$ as all elements of the lattice $\mathcal{L}$ except the infimum of the lattice (the empty set). The second step introduces edges for all overlap relations, i.e. the cliques represented by maximal O-cliques are connected.

DEFINITION 6.1. Given a p-ortholattice $\mathcal{L}$, define the undirected graph $G^{\mathcal{L}} = (V, E)$ with $y \in Y \iff y \in V(G^{\mathcal{L}})$ (or simplified $V \cong Y$) and $x, y, z \in Y\,[z \leq x \wedge z \leq y] \iff xy \in E(G^{\mathcal{L}})$. Notice that $\emptyset \notin Y$. $G(\mathcal{L})$ is simple and finite if $\mathcal{L}$ is finite.

We can give an alternative definition in terms of the overlap relation that defines the set of edges in $G^{\mathcal{L}}$ as $\langle x, y\rangle \in O^{\mathcal{M}} \iff xy \in E(G^{\mathcal{L}})$. Remember that two individuals overlap if they have a nonempty meet. Then each O-clique (defined in subsection 3.1 of the previous chapter) is a clique in the graph and no other edges are added. Notice that a O-clique is only defined by the number of vertices it contains and by the relation position in the lattice to the other vertices (given by the parthood relation that is the same for all vertices in the O-clique towards outside vertices). Therefore the sublattice associated with a O-clique is not necessarily unique and can consist of different orderings. Hence we cannot reconstruct the lattice $\mathcal{L}$ from a graph $G^{\mathcal{L}}$ and not every lattice constructed from a graph $G^{\mathcal{L}}$ is then in fact in the restricted subset of the complete p-ortholattices or even a complete atomic p-ortholattice.

LEMMA 6.2. *For every lattice $\mathcal{L}$ the graph $G^{\mathcal{L}}$ is uniquely defined.*

PROOF. Follows directly from the unique definition of $V(G^{\mathcal{L}})$ and $E(G^{\mathcal{L}})$. □

EXAMPLE 9. *See e.g. figure 1.2, its lattice is not orthocomplemented but has the same graph $G^{\mathcal{L}}$ as the lattices in figures 5.5 and 1.2. From the graphs alone, the lattice cannot be uniquely determined. Figure 1.2 can be easily extended to a model of RT^{-}_{EC} by choosing $\mathbf{e}^{\perp} = \mathbf{i}$, $\mathbf{f}^{\perp} = \mathbf{h}$, $\mathbf{d}^{\perp} = \mathbf{g}$, $\mathbf{b}^{\perp} = \mathbf{k}$, and $\mathbf{c}^{\perp} = \mathbf{j}$ as orthocomplements and introducing $\langle \mathbf{d}, \mathbf{j}\rangle, \langle \mathbf{e}, \mathbf{i}\rangle, \langle \mathbf{f}, \mathbf{h}\rangle, \langle \mathbf{e}, \mathbf{h}\rangle, \langle \mathbf{b}, \mathbf{g}\rangle, \langle \mathbf{b}, \mathbf{h}\rangle, \langle \mathbf{b}, \mathbf{i}\rangle, \langle \mathbf{c}, \mathbf{d}\rangle, \langle \mathbf{c}, \mathbf{e}\rangle, \langle \mathbf{c}, \mathbf{f}\rangle, \langle \mathbf{b}, \mathbf{c}\rangle \in EC^{\mathcal{M}}$ (with the symmetric ones in $EC^{\mathcal{M}}$ as well).*

1.3. Unique extension of any finite $G^{\mathcal{L}}$ with a set E_{EC}. The graphs in $\mathfrak{G}^{\mathcal{L}}$ constructed from the finite p-ortholattices are elementary equivalent (while choosing adequate extensions of the parthood and overlap relations) to the finite models of RT^{-}. But for any not uniquely complemented finite p-ortholattice, the corresponding graphs in $\mathfrak{G}^{\mathcal{L}}$ are not twin-free and are missing at least one edge representing an external connection in the corresponding model of RT^{-}_{EC}. In order to make the graphs twin-free while remaining the partial ordering by neighborhood inclusion as observed in lemma 5.5, the neighborhoods of any set of elements that are in all the same maximal O-cliques need to be extended uniquely while maintaining the proper parthood ordering, i.e. if $\langle x, y\rangle \in EC^{\mathcal{M}}$ and $\langle x, z\rangle \in PP^{\mathcal{M}}$ then $\langle y, z\rangle \in C^{\mathcal{M}}$ must hold.

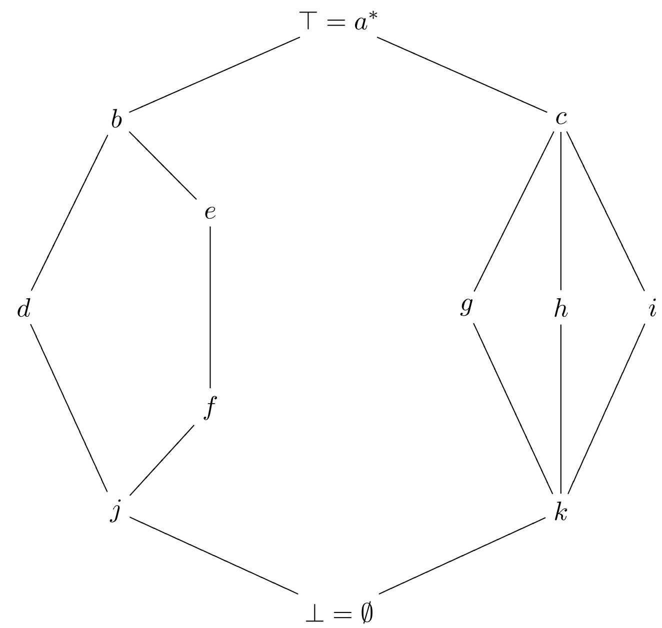


FIGURE 24. Example of an non-orthocomplemented lattice that can be constructed from the graph $G^{\mathcal{L}}$ of the lattice in fig. 5.5 that can never be extended to a model of RT^{-}_{EC}

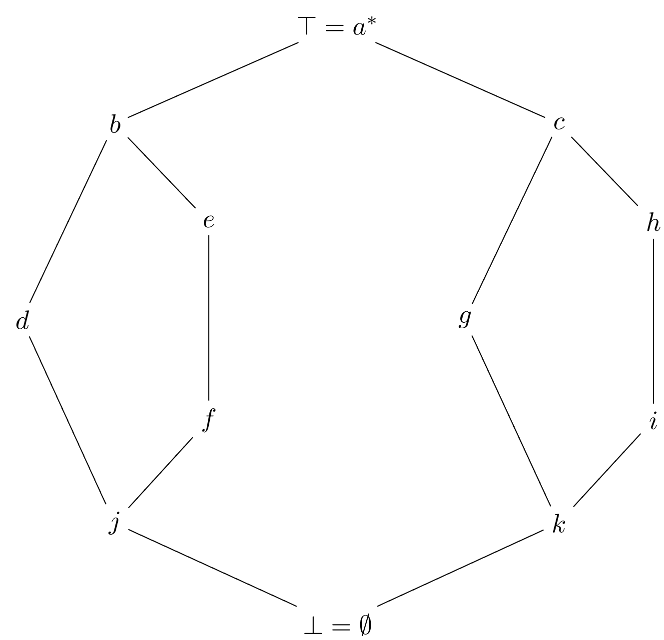


FIGURE 25. Example of an alternative p-ortholattice that can be constructed from the graph $G^{\mathcal{L}}$ of the lattice in fig. 5.5

We then construct the set of edges E_{EC} in $G^{\mathcal{M}}_{EC} = \big(V(G^{\mathcal{M}}), E(G^{\mathcal{M}}) \cup E_{EC}\big)$ where $E_{EC} \cap E(G^{\mathcal{M}}) = \{\}$. For any such graph $G^{\mathcal{M}}_{EC}$ to actually be the graph of the connection relation of a model of RT^{-}_{EC} it must be ensured that the O-cliques are treated appropriately. In particular, one element in each maximal O-clique must remain open, i.e. in the extension of $OP^{\mathcal{M}}$, to serve as interior for the smallest non-open elements in the O-clique. This is only possible, if it is not externally connected to any other element. Therefore one element of each maximal O-clique must be no endpoint of any edge in E_{EC}. Furthermore, we need a greatest element in each O-clique. This is the element that subsumes all external connection relations in the O-clique. Intuitively, this element is the common "parent" part of all the elements in the O-clique. Formalizing the existence of an interior and of a maximal element in any maximal O-clique: for all maximal O-cliques V_O, $\exists v \in V_O\,(\forall w \in V(G_P)\,[vw \notin E_{EC}])$ and $\exists v \in V_O\,[\forall w \in V_O\,(xw \in E_{EC} \rightarrow vw \in E_{EC})]$. Further, each maximal O-clique V_O must be closed under sum and intersection in the following way: $\forall x, y \in V_O \exists z \in V_O\,(N[x] \cup N[y] = N[z])$ and $\forall x, y \in V_O \exists z \in V_O\,(N[x] \cap N[y] = N[z])$. This corresponds to the axioms A5 and A6 of RT_0.

Remark 10. Here we present some properties between individuals and their complements expressed both in terms of the models of RT as well as properties of the resulting graphs. These observations lead to the definition of the extension of EC in the following section.

Proposition 6.3. *In a model $\mathcal{M}$ of RT it holds for an individual x and its complement $-x$:*
(1) $\forall x\,[\neg C(x, -x)]$
(2) $\forall x, y\,[C(x, y) \vee C(-x, y)]$ *and*
(3) $\forall x, z\,[PP(x, z) \rightarrow C(-x, z)]$
(4) $\forall x, z \neq \mathbf{a}^*\,[P(z, x) \equiv \neg P(z, -x)]$ and
$\forall x, z \neq \mathbf{a}^*\,[P(x, z) \equiv \neg P(-x, z)]$.

In the graph-theoretical expression, by (2) such pairs $x, -x$ form dominating sets, whereas the universal vertex is a dominating set by itself.

Proposition 6.4. *Any individual x and its complement $-x$ in a graph $G_{RT}(\mathcal{M})$ of a model $\mathcal{M}$ of RT satisfy the following conditions*
(1) $\forall x\,[-xx \notin E(G_{RT})]$
(2) $\forall x\,[N[x] \cup N[-x] = V(G_{RT}(\mathcal{M}))]$ *and*
(3) $\forall x, z\,[N[x] \subset N[z] \rightarrow z \in N[-x]]$
(4) $\forall x, z \neq \mathbf{a}^*\,[N[z] \subseteq N[x] \iff N[z] \nsubseteq N[-x]]$ and
$\forall z \neq \mathbf{a}^*\,[N[x] \subseteq N[z] \iff N[-x] \nsubseteq N[z]]$.

2. Definition of E_{EC} Through Orthocomplements

In the following we present a result that is based on the observation that connectedness and disconnectedness (i.e. in graph-theoretic terms non-adjacency) is primarily based on orthocomplementation. In fact, each element in a complete p-ortholattice has a unique orthocomplement. It is not always uniquely identified in the lattice, e.g. in figures 5.5, 1.2, and 1.2, the elements $\mathbf{b}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{j}$ have in $\mathbf{c}, \mathbf{g}, \mathbf{h}, \mathbf{i}, \mathbf{k}$ interchangeable orthocomplements because their neighborhoods in the respective graph $G^{\mathcal{L}}$ are identical. But we claim that in such cases, it does not matter which one is the actual orthocomplement, since the relative order to all other elements is similar. But recall that an element in a model of RT_{EC}^{-} cannot be connected to its own topological complement. By identity of a topological complement in a model $\mathcal{M}$ of RT_0 with its orthocomplement in the corresponding lattice $\mathcal{L}^{\mathcal{M}}$, any element for a model of RT_0 cannot be connected to its orthocomplement in the lattice $\mathcal{L}^{\mathcal{M}}$. Now if an element is not connected to its orthocomplement, it can obviously not be connected to the parts of its orthocomplement (i.e. all elements the orthocomplement covers in the lattice by lemma 5.5). But intuitively, anything larger than the topological complement must be connected to the element itself. We formalize this idea in the following and prove that a set of edges E_{EC} constructed for any non-modular finite p-ortholattice satisfying this condition is non-empty and gives a graph $G_{EC}^{\mathcal{L}}$ that is elementary equivalent to a finite model of RT_{EC}^{-}.

We define the set E_{EC} as

(EQ13) $$E_{EC} = \{xy \mid y \not\leq x^{\perp}\} \setminus E(G^{\mathcal{L}})$$

and hence the graph $G_{EC}^{\mathcal{L}}$ to a graph $G^{\mathcal{L}}$ of a non-modular finite p-ortholattice $\mathcal{L}^{\mathcal{M}}$ (of a model $\mathcal{M}$ of RT_{EC}^{-}) as

(EQ14) $$G_{EC}^{\mathcal{L}} = \big(V(G^{\mathcal{L}}), E(G^{\mathcal{L}}) \cup E_{EC}\big) = G^{\mathcal{L}} \cup \{xy \mid y \not\leq x^{\perp}\}$$

(EQ15) $$G_{EC}^{\mathcal{L}} = K_{|V(G^{\mathcal{L}})|} \setminus \{xy \mid y \leq x^{\perp}\}$$

where $x^{\perp}$ is the orthocomplement of an element x in the lattice $\mathcal{L}^{\mathcal{M}}$. The characterization in EQ15 describes the class of models in terms of complete graphs, but with the additional precondition that the graph $G^{\mathcal{L}}$ itself must be constructed from a non-modular finite p-ortholattice. Not all arbitrary complete graphs give models of RT_{EC}^{-} by the last characterization.

THEOREM 6.5. *The set of edges E_{EC} that extends a graph $G^{\mathcal{L}}$ resulting from a non-modular finite p-ortholattice is always non-empty.*

PROOF. Assume otherwise, i.e. $E_{EC} = \{\}$ for some graph $G^{\mathcal{L}}$. Then by the definition of E_{EC} the set of edges $\{xy|y \not\leq x^{\perp}\} \subseteq E(G^{\mathcal{L}})$. But then we claim that every element in the lattice is uniquely complemented, since $E(G^{\mathcal{L}}) \subseteq \{xy|y \not\leq x^{\perp}\}$ because no individual can be connected to its complement or parts thereof, and each element x in the graph has then a unique neighborhood $N[x] = \{xy|y \not\leq x^{\perp}\}$. This is in fact a model of RT^{-} that arises from a uniquely complemented finite p-ortholattice and hence does not satisfy the property of being in the class of the non-modular finite p-ortholattices. □

THEOREM 18. *Each non-modular finite p-ortholattice $\mathcal{L}$ resulting in a graph $G^{\mathcal{L}}$ gives in $G^{\mathcal{L}}_{EC} = G^{\mathcal{L}} \cup \{xy|y \not\leq x^{\perp}\}$ the graph of the connection relation $C^{\mathcal{M}}$ of some finite model $\mathcal{M}$ of RT^{-}_{EC}.*

COROLLARY 6.6. *Each finite p-ortholattice extendable by a non-empty set $E_{EC} = \{xy|y \not\leq x^{\perp}\} \setminus E(G^{\mathcal{L}})$ is a model of RT^{-}_{EC}.*

Both formulations can be used synonymously. The next lemmas prove that every such constructed graph $G^{\mathcal{L}}_{EC}$ satisfies all axioms of RT^{-}_{EC} and therefore is a model thereof. Assume for all the lemmas that $G^{\mathcal{L}}$ is the graph of a non-modular finite (complete atomic) p-ortholattice $\mathcal{L}$ where non-modular means non-semimodular, non-orthomodular, and not uniquely complemented. We know from the previous chapters that for any finite p-ortholattice the graph $G^{\mathcal{L}}$ is a model of RT^{-}. Surprisingly, all the graphs of non-modular lattices yield non-empty edge sets E_{EC} and thus models of RT^{-}_{EC} and all graphs of uniquely complemented finite (modular) lattices give models of RT^{-} but not of RT^{-}_{EC}. Remember the proof of theorem 14 showing the isomorphism between the finite models of RT^{-} and finite p-ortholattices relies on the implicit assumption of a predefined set E_{EC} that ensures for each such lattice that is non-modular a resulting twin-free graphs $G^{\mathcal{L}}_{EC}$ or, equivalently, a model satisfying A3. From this observation it also becomes clear that axiom A3 is not explicitly represented in the definition of a structure RT_T because the proof of theorem 14 works without worrying about the uniqueness of the extension of C for every element in the domain.

LEMMA 6.7. *Every finite graph $G^{\mathcal{L}}_{EC} = G^{\mathcal{L}} \cup \{xy|y \not\leq x^{\perp}\}$ is twin-free and hence satisfies the axioms A1 to A3.*

PROOF. Assume there exist two elements $x, y \in V(G^{\mathcal{L}}_{EC})$ such that $N[x] = N[y]$. Since $xx^{\perp} \notin E$ and thus $x^{\perp} \notin N[x]$ it follows that $x^{\perp} \notin N[y]$. The same for

$y^\perp$, i.e. $y^\perp \notin N[x]$. Then by the definition of $G^{\mathcal{L}}_{EC}$, both $y^\perp \leq x^\perp$ and $x^\perp \leq y^\perp$ must hold. This is an obvious contradiction. Hence our assumption that two elements x, y in the graph $G^{\mathcal{L}}_{EC}$ with $N[x] = N[y]$ can exists is falsified. □

LEMMA 6.8. *Every graph $G^{\mathcal{L}}_{EC} = G^{\mathcal{L}} \cup \{xy | y \not\leq x^\perp\}$ satisfies axiom A4.*

PROOF. By definition of a p-ortholattice: $\perp = \top^\perp = (a^*)^\perp$. Any element $d < a^*$ has an orthocomplement in the graph that it is not connected to and therefore is not universally connected in $G^{\mathcal{L}}_{EC}$. □

For some of the proofs it is essential to know that every non-modular finite p-ortholattice $\mathcal{L}$ can be extended to a graph $G^{\mathcal{L}}_{EC}$ by a non-empty set E_{EC}, while the extension $O^{\mathcal{M}}$ of the model $\mathcal{M}$ associated to the graph $G^{\mathcal{L}}_{EC}$ is given already by the lattice $\mathcal{L}$. In other words, the extension of the overlap relation of the resulting model of RT^-_{EC} is not altered when adding the edges E_{EC} to the graph $G^{\mathcal{L}}$. Since overlap is uniquely defined by D3 through the extension of parthood $P^{\mathcal{M}}$, it is sufficient to prove that the parthood extension remains unchanged when adding E_{EC} to the graph.

Moreover, we use a fundamental observation which can be proved from the interaction of orthocomplementation with pseudocomplementation directly from proposition 6.3(4).

$$\forall x, y \left[\langle x, y\rangle \in O^{\mathcal{M}} \vee \langle -x, y\rangle \in O^{\mathcal{M}}\right] \qquad \text{(EQ16)}$$

LEMMA 6.9. *The extension $P^{\mathcal{M}}$ of the parthood relation in $\mathcal{M}$ is given by the lattice $\mathcal{L}$, i.e. $x \leq y \iff \langle x, y\rangle \in P^{\mathcal{M}}$.*

PROOF. Assume $x \leq y$ for some pair x, y. That means $N[x] \subseteq N[y]$. Now whenever a third element z is connected to x, it will also be connected to y, since by order-reversing law, $y^\perp \leq x^\perp$ holds and if $z \not\leq x^\perp$ then $z \not\leq y^\perp$. So $N[x] \subseteq N[y]$ is preserved by adding E_{EC} to the graph and thus $\langle x, y\rangle \in P^{\mathcal{M}}$. On the reverse, if $\langle x, y\rangle \in P^{\mathcal{M}}$ in a model of RT^-_{EC}, then $N[x] \subseteq N[y]$ in the graph $G^{\mathcal{L}}_{EC}$. If now $N[x] \subseteq N[y]$ do not hold in the graph $G^{\mathcal{L}}$ of the lattice $\mathcal{L}$, then $x \not\leq y$. However, then also $y^\perp \not\leq x^\perp$. That means some z can be connected to x without being connected to y for the extension E_{EC}. In particular, either $y^\perp > x^\perp$ or $y^\perp, x^\perp$ are incomparable. In the first case $z \in N[x]$ but $z \notin N[y]$. In the second case, $y^\perp \in N[x]$ but obviously $y^\perp \notin N[y]$. □

LEMMA 6.10. *Every graph $G^{\mathcal{L}}_{EC} = G^{\mathcal{L}} \cup \{xy | y \not\leq x^\perp\}$ satisfies axiom A5.*

PROOF. Let $z \in V(G^{\mathcal{L}})$ be the sum of some pair of elements $x, y \in V(G^{\mathcal{L}})$ with $z = x \cup^* y \geq x, y$. We prove each direction of the equivalence in A5 individually.

(a) $\exists v\,[(C(v,x) \vee C(v,y)) \rightarrow C(v,z)]$
Since $z \geq x$, either $z = x$ with $zv \in E \iff xv \in E$, or $z > x$ and by lemma 5.5 it follows $xv \in E \Rightarrow zv \in E$. The same holds for y. Hence in any case, whatever x, y are connected to, it is ensured that z is connected to it as well.

(b) $\exists v\,[(C(v,x) \vee C(v,y)) \leftarrow C(v,z)]$
Assume the contrary, i.e. there exists an element v s.t. $zv \in E$ but $xv, yv \notin E$. If v is comparable to z it is not necessarily comparable to x and/or y. However, this can only occur if $v < z$ and v is disjoint with both x and y. If there is a common proper part u, i.e. $u < v, x$ or u, v, y then v and x or y are in a common O-clique and by definition connected. If no such u exists, there are three or more atoms in this subbranch of the lattice. Then the lattice is not pseudocomplemented, since any other dual-atom would not have a unique join-pseudocomplement. In all other cases if v is comparable to z, it is comparable to at least one of x or y.
If v is not comparable to z, then we consider two subcases again: $v \leq z^{\perp}$, in this case v cannot be connected to z by definition. If $v > z^{\perp}$, it is either comparable to one of $x^{\perp}$ or $y^{\perp}$ or incomparable to either one. If v is incomparable to both, there must exist three distinct dual-atoms in this subbranch of the lattice and the lattice would not be meet-pseudocomplemented. If v is comparable to only one of them, i.e. w.l.g. to x then $yv \in E$ since $v \not\leq y^{\perp}$. If v is comparable to x and y and smaller then them, i.e. $v < x^{\perp}, y^{\perp}$ then it $v = x \cap^{*} y$ and thus $v^{\perp} = x \cup^{*} y$ by the order-reversing law. Hence z is not the sum of x and y. If $v > x^{\perp}, y^{\perp}$ (note that if x and y are comparable with each other, they are ordered and z is not the sum of x and y) then $v > z^{\perp}$ and $xv, yv, zv \notin E$ would follow contrary to our assumption that $zv \in E$. □

Lemma 6.11. *Every graph $G^{\mathcal{L}}_{EC} = G^{\mathcal{L}} \cup \{xy | y \not\leq x^{\perp}\}$ satisfies axiom A6.*

Proof. Recall that by lemma 6.9, the parthood relation is predefined by the lattice. So is the overlap relation, which is entirely dependent on the parthood extension. Therefore we only have to prove that if the previously unique intersection $z = x \cap^{*} y$ given by the lattice $\mathcal{L}$ with $z < x, y$ has an additional element in its neighborhood, both x and y have it as well. This is straightforward: assume v with $zv \in E$, then $v \not\leq z^{\perp}$. Since $z^{\perp} \geq x^{\perp}, y^{\perp}$ it follows that $v > x^{\perp}, y^{\perp}$ or v is incomparable to $x^{\perp}, y^{\perp}$. The latter case also implies $v \not\leq x^{\perp}$ and $v \not\leq x^{\perp}$. Such in any case, $vx, vy \in E$. □

Lemma 6.12. *Every graph $G^{\mathcal{L}}_{EC} = G^{\mathcal{L}} \cup \{xy | y \not\leq x^{\perp}\}$ satisfies axiom A7.*

Proof. This follows naturally from the use of the orthocomplement as topological complement. The complement $x^{\perp}$ is supposed to be connected to everything that x is not connected to by A7. All the elements y with $y \leq x^\perp$, $x^\perp$ is already connected to (by parthood). Since everything else x is connected to in $G^{\mathcal{L}}_{EC}$, axiom A7 is then immediately satisfied. □

LEMMA 6.13. *Every graph* $G^{\mathcal{L}}_{EC} = G^{\mathcal{L}} \cup \left\{xy | y \not\leq x^\perp\right\}$ *satisfies axiom A8.*

PROOF. Since the underlying lattices are atomic, each element x has a smaller element y s.t. $y \leq x$ that is an atom, i.e. $y \succ \perp$. This element y is not connected to any other element beyond its overlap relation by definition: its orthocomplement $y^\perp$ is a dual-atom (by orthocomplementation) and $\forall z \left[z \not> y \rightarrow y^\perp \geq z\right]$ and $yz \notin E$ follows (and $\left\{yz | z > y^\perp\right\} = \emptyset$). Otherwise the lattice would not be pseudocomplemented. Thus y is the element that satisfies both directions of the implication in axiom A8. Notice that in particular no element can be externally connected to y and thus $\langle y, y\rangle \in NTP^{\mathcal{M}}$ and further $\langle x, y\rangle \in C^{\mathcal{M}}$ since $y \leq x$ in the lattice. □

LEMMA 6.14. *Every graph* $G^{\mathcal{L}}_{EC} = G^{\mathcal{L}} \cup \left\{xy | y \not\leq x^\perp\right\}$ *satisfies axiom A9.*

PROOF. Trivially satisfied. □

LEMMA 6.15. *Every graph* $G^{\mathcal{L}}_{EC} = G^{\mathcal{L}} \cup \left\{xy | y \not\leq x^\perp\right\}$ *satisfies axiom A10.*

PROOF. By definition an element x is in the extension $OP^{\mathcal{M}}$ of some model $\mathcal{M}$ (which can be associated to the graph $G^{\mathcal{L}}_{EC}$) if and only if $\left\{xv | v \not\leq x^\perp\right\} = \{\}$. The same applies for y. Therefore $\neg\exists v \left[v \leq x^\perp, y^\perp | xv \in E_{EC} \text{ or } yv \in E_{EC}\right]$ follows. Taking $z = x \cap^* y \leq x, y$ then $\{\langle z, v\rangle \in E_{EC}\} \subseteq \{\langle x, v\rangle \in E_{EC}\}, \{\langle y, v\rangle \in E_{EC}\}$. Because the latter two sets are empty, so is $\{\langle z, v\rangle \in E_{EC}\}$. Hence z is without any external connections. Then because $zz^\perp \notin E$ and and $z^\perp \geq x^\perp, y^\perp$, $\neg\exists v \left[v \leq z^\perp | zv \in E\right]$ follows immediately and z is in the extension of $OP^{\mathcal{M}}$ as well. If $z = x$ (or $z = y$) then $y < x$ (or $x < y$) then z is trivially in $OP^{\mathcal{M}}$. If there exists another element u s.t. w.l.g. $z^\perp > u > x^\perp$ then $u^\perp < x$ and therefore $\left\{uv | v \not\leq u^\perp\right\} = \emptyset$. But then with x and u being comparable, they would have the same closed neighborhood and violate twin-freeness. □

LEMMA 6.16. *Every graph* $G^{\mathcal{L}}_{EC} = G^{\mathcal{L}} \cup \left\{xy | y \not\leq x^\perp\right\}$ *satisfies axiom A11.*

PROOF. Remember that the lattices are not uniquely complemented, i.e. there exists some element x in the lattice so that y and z are complements of it. Since the lattice is pseudocomplemented, all the potential complements must be comparable to each other, i.e. y to z. That means y and z have initially the same closed neighborhood. But then, either (1) $y > z$ and thus $z^\perp > y^\perp$ and $yz^\perp \in E$ but $yz^\perp \notin E$. Then $yz^\perp \in E_{EC}$ because otherwise, if $yz^\perp \in E(G^{\mathcal{L}})$, then the assumption of same neighborhoods of y, z in $G^{\mathcal{L}}$ would force $zz^\perp \in E$ which is a contradiction to topological complementedness. Alternatively, (2) y and z are incomparable, then $y^{\perp}$and $z^{\perp}$are also incomparable and $yz^{\perp}, zy^{\perp} \in E$, but $yy^{\perp}, zz^{\perp} \notin E$ by definition. Then $yz^{\perp}, zy^{\perp} \in E_{EC}$ must hold to not violate the assumption of same closed neighborhoods $N[y] = N[z]$ in $G^{\mathcal{L}}$. □

LEMMA 6.17. *Every graph $G^{\mathcal{L}}_{EC} = G^{\mathcal{L}} \cup \{xy | y \nleq x^{\perp}\}$ satisfies axiom A13.*

PROOF. The axiom A13 states that for each element x in a model of RT^{-}_{EC}, there exists an open element y (not to be confused with y in the definition of $G^{\mathcal{L}}_{EC}$) so that $y \geq x$ in the lattice (equivalent to $P(x, y)$ in the model) and $\forall z\, [z \geq x \rightarrow y \geq z]$. On the reverse, the axiom is violated if and only if there exist two (or more) elements y and z that are both in the extension $OP^{\mathcal{M}}$, i.e. $y, z \geq x$ and y, z are incomparable. Note that if $\langle x \rangle \in OP^{\mathcal{M}}$, then the axiom A13 is trivially satisfied. The only case where the axiom can be violated is in an O-clique containing all of x, y, and z (compare subsection 3.1 of chapter 5). Then, by definition 6.1 it follows for the neighborhoods $N[y] = N[z]$. However, this contradicts the lemma proving that the resulting graph is twin-free. In particular y and z would form a module. Hence, axiom A13 must be satisfied in every graph $G^{\mathcal{L}}_{EC}$. □

Altogether compliance with all axioms of RT^{-}_{EC} is proved and theorem 18 follows immediately. Now we have a way to construct all finite models of RT^{-}_{EC} from the not-uniquely complemented finite p-ortholattices. Moreover, this equivalence shows that the property of a complete atomic p-ortholattice being not uniquely complemented, being not semimodular, and being non-orthomodular are all equivalent, because all ensure that the set E_{EC} becomes non-empty and thus gives a valid model of RT^{-}_{EC}. This is a specialized case of the result from [**McL56**] that every atomic uniquely complemented lattice is modular. For the isomorphism we therefore choose to characterize the lattices as not uniquely complemented (or short: not unicomplemented).

THEOREM 19. *Any graph $G^{\mathcal{L}}_{EC}$ in the class of graphs $\mathfrak{G}^{\mathcal{L}}_{EC}$ containing all graphs $G^{\mathcal{L}}_{EC} = G^{\mathcal{L}} \cup \{xy | y \nleq x^{\perp}\}$, where each $G^{\mathcal{L}}$ is a graph representation of a finite non-unicomplemented p-ortholattices $\mathcal{L}$, is elementary equivalent to a graph in $\mathfrak{G}_{RT^{-}_{EC}}$ that represents the connection relation $C^{\mathcal{M}}$ of some finite model $\mathcal{M}$ of RT^{-}_{EC}.*

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3. Incorporation of Weak Contact Through Products of Lattices

Having characterized the finite models of RT^{-}_{EC} up to isomorphism in terms of graphs of lattices, a single step is left to complete the characterization of the finite models of the full axiom system RT_0 of Asher and Vieu. This requires the inclusion of the concept weak contact $WCont$ into our analysis. In particular, we are interested in how weak contact affects the models. All the examples given so far of size up to 11 elements in the domain do not allow any weak contacts. In other words, all the small models considered so far are "too trivial" for being considered a true mereotopological model by Asher and Vieu.

3.1. Basic properties on Weak Contact. Recall the definition and explanation of weak contact from chapter 2, section 2. Two elements x and y are in weak contact (connection) if their closures are not connected but any open neighborhood greater than (subsuming, i.e. strictly greater) one of x and y is always connected to both x and y.

(A12) $$WCont(x, y) \equiv_{def} \neg C(c(x), c(y)) \wedge \forall z\,[(OP(z) \wedge P(x, z)) \rightarrow C(c(z), y)]$$

Notice that if an element x is clopen, then it cannot be in weak contact because $cl(x) = x$ and $\langle cl(x), cl(y)\rangle \notin C^{\mathcal{M}}$ then translates to $\langle x, cl(y)\rangle \notin C^{\mathcal{M}}$ but since x is always open and trivially contains itself, $\langle x, cl(y)\rangle \in C^{\mathcal{M}}$ is supposed to hold as well. By similar reasoning we collected the following simple theorems for weak contact.

Lemma 6.18. *The theory RT_0 entails the following theorems*

(W1) $\forall x, y\,[WCont(x, y) \rightarrow \neg C(cx, cy)]$

(W2) $\forall x, y\,[WCont(x, y) \rightarrow \neg C(x, y)]$

(W3) $\forall x, y\,[WCont(x, y) \rightarrow \neg P(x, y) \wedge \neg P(y, x)]$

(W4) $\forall x, y\,[WCont(x, y) \rightarrow \neg EC(x, y)]$

(W5) $\forall x, y\,[WCont(x, y) \rightarrow \neg O(x, y)]$

(W6) $\forall x, y\,[WCont(x, y) \rightarrow \neg OP(x) \wedge \neg OP(y)]$

(W7) $\forall x, y\,[WCont(x, y) \rightarrow \exists u, v\,(EC(x, u) \wedge EC(y, v))]$

(W8) $\forall x, y\,[WCont(x, y) \rightarrow \exists u, v\,(ix = u \wedge iy = v)]$

W1 follows directly from the definition of $WCont$; for W2 we have given an informal argument. One can derive W3, W4, W5, W6 from W2 by applying the subsumption relations proved for the JEPD lattice. Theorems W7, W8 are direct implications of W6 considering that any open individual has no external connection and any individual that has some open element as interior must distinguish itself from it through some external connection.

(W9) $\forall x, y\, [WCont(x, y) \equiv \neg C(cx, cy) \wedge C(x, c(ny))]$ [**AV95**]

This is fact 3 of [**AV95**]. Negating it, rewrites it as W9'.

(W9') $\forall x, y\, [\neg WCont(x, y) \equiv (C(x, c(ny)) \to C(cx, cy))]$
$\forall x, y\, [\neg WCont(x, y) \equiv (\neg C(x, c(ny)) \vee C(cx, cy))]$

Observe that always $x \leq cx$ and $y \leq cy \leq ny \leq c(ny)$ must hold. The latter one follows because cy is not open, hence it is externally connected and everything greater than it (up to its closure) is externally connected as well. Moreover, the open neighborhood must be strictly greater (unless it is the universal element, i.e. $cy = ny = a^*$). Its closure is greater or equal than $n(cy)$ again.

$\forall x, y\, [WCont(x, y) \to WCont(cx, y)]$ holds then in any model of RT_0, which is easily verifiable from $cx \geq x$ substituted into W9. Then immediately with $C(x, c(ny)) \to C(cx, c(ny))$, W10 follows as a theorem.

(W10) $\forall x, y\, [WCont(x, y) \to WCont(cx, y)]$

Consequently, in order to satisfy axiom A12, we only have to consider the case where $\langle x\rangle, \langle y\rangle \in CL^{\mathcal{M}}$ with $\langle x, y\rangle \in WCont^{\mathcal{M}}$, since $WCont$ a is symmetric relation and any other weak contact for elements x, y with $x < cx$ and $y < cy$ implies then $\langle cx, cy\rangle \in WCont^{\mathcal{M}}$ directly. Substituting this knowledge into W9, we deduce W11 and we alter axiom A12 to A12'.

(W11) $\forall x, y\, [WCont(cx, cy) \equiv \neg C(cx, cy) \wedge C(cx, c(n(cy))) \wedge C(cy, c(n(cx)))]$
(A12') $\exists x, y\, [WCont(cx, cy)]$

Moreover, RT entails $\forall x, y\, [PP(-x, y) \to C(x, y)]$ and as a special case thereof (both condition and consequence are generalized): $\forall x, y\, [P(-x, y) \to C(cx, cy)]$. For a weak contact to exist between x and y, both cannot be open, therefore $-x, -y$ cannot be closed and for every $y \geq -x$ it holds $\langle x, y\rangle \in C^{\mathcal{M}}$. Assume now $y = -x$ then because neither x nor y can be open, neither can be closed as well (the complement of a closed element must be open). It would follow that $cx > x$ or $cy > y$ and then $\langle cx, cy\rangle \in C^{\mathcal{M}}$ with the consequence that $\langle x, y\rangle \notin WCont$. Hence equality cannot hold and we conclude $y > -x$ and $x > -y$ if there exists a weak contact between x and y.

(W12) $\forall x, y\, [WCont(x, y) \to PP(y, -x) \wedge PP(x, -y)]$

Finally we can express a relationship resulting from weak contact between the neighborhood ny and the orthocomplement $x^{\perp}$ (equivalent to the complement $-x$ in RT_0). This gives a unique biggest element y that contains all elements in weak contact to x. This y is the greatest element fully contained in the orthocomplement of x (remember that the neighborhood ny is uniquely defined as shown in [**AV95**]).

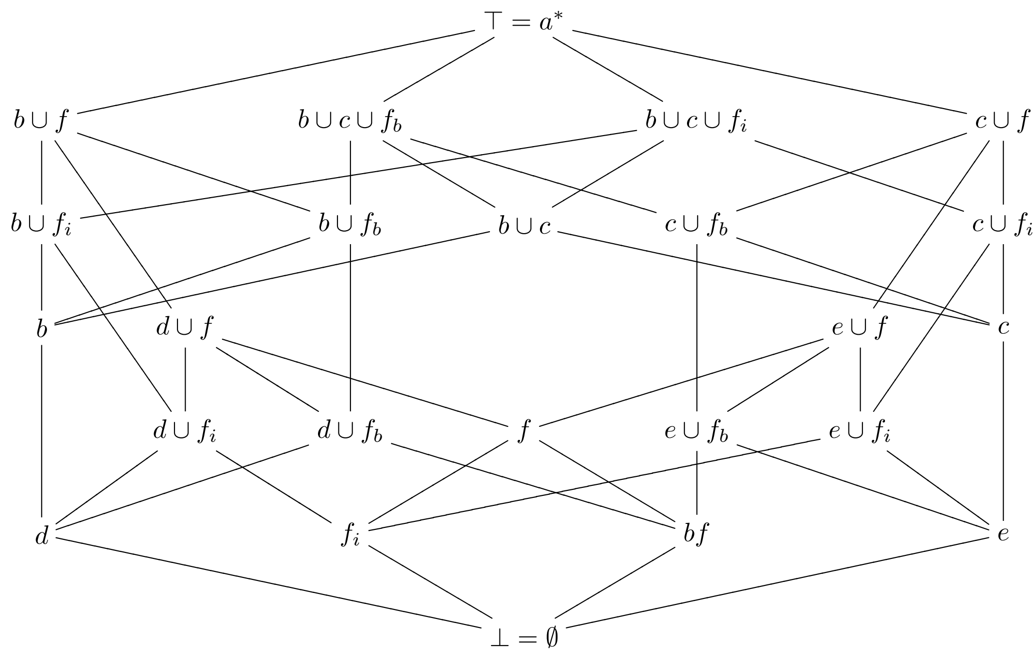


FIGURE 26. Failed attempt of an p-ortholattice satisfying all axioms of RT

$$\text{(W13)} \qquad \forall x, y \, [WCont(x,z) \rightarrow \exists y \, (P(z,y) \wedge WCont(x,y) \wedge -x = ny)]$$

W13, W11, and A12' give as a very restricted view on the existence of weak contact; it allows us to only care about this greatest element y that is in weak contact to x (and vice versa) and has the special property $-x = ny$ (and $-y = nx$). This translates back to orthocomplements as $x^{\perp} = ny$ and $y^{\perp} = nx$ and is closed. We conclude that if a weak contact exists at all, it must exist between two meet-pseudocomplements x and y, where one of them, e.g. y, is smaller (in the lattice) than the (ortho)complement of x. In fact, there must exist a maximal closed part y of $x^{\perp}$ for any weak contact to exist.

EXAMPLE 10. *A model of the full theory RT_0 needs to have the smallest model of RT_{EC}^{-} as submodel (compare figure 5.4(a)) and an additional element connected to one of the elements by weak contact. In order to satisfy the above conditions, we need an element* $\mathbf{f}$ *that is not open, thus has a specified distinct interior* $\mathbf{if} = int(\mathbf{f})$. *To maintain orthocomplementedness, we need the difference between* $\mathbf{f}$ *and* $\mathbf{if}$ *to be explicitly modeled as another element, we call it* $\mathbf{bf} = \mathbf{f} - \mathbf{if}$ *(the border of* $\mathbf{f}$*). This gives us as minimum model a p-ortholattice with 4 atoms as shown in figure 3.1. Here a complete list of the orthocomplements in the lattice:* $\mathbf{d}^{\perp} = \mathbf{c} \cup \mathbf{f}$, $\mathbf{e}^{\perp} = \mathbf{b} \cup \mathbf{f}$, $\mathbf{f_i}^{\perp} = \mathbf{b} \cup \mathbf{c} \cup \mathbf{f_b}$, $\mathbf{f_b}^{\perp} = \mathbf{b} \cup \mathbf{c} \cup \mathbf{f_c}$, $\mathbf{b}^{\perp} = \mathbf{e} \cup \mathbf{f}$, $\mathbf{c}^{\perp} = \mathbf{d} \cup \mathbf{f}$, $\mathbf{f}^{\perp} = \mathbf{b} \cup \mathbf{c}$, $(\mathbf{d} \cup \mathbf{f_i})^{\perp} = \mathbf{c} \cup \mathbf{f_b}$, $(\mathbf{e} \cup \mathbf{f_i})^{\perp} = \mathbf{b} \cup \mathbf{f_b}$, $(\mathbf{d} \cup \mathbf{f_b})^{\perp} = \mathbf{c} \cup \mathbf{f_i}$, $(\mathbf{e} \cup \mathbf{f_b})^{\perp} = \mathbf{b} \cup \mathbf{f_i}$.

In this example, $\langle \mathbf{b}, \mathbf{c}\rangle, \langle \mathbf{b} \cup \mathbf{f_i}, \mathbf{c} \cup \mathbf{f_b}\rangle, \langle \mathbf{b} \cup \mathbf{f_b}, \mathbf{c} \cup \mathbf{f_i}\rangle \in EC^{\mathcal{M}}$ *(and the symmetric equivalents are also in* $EC^{\mathcal{M}}$*). We want the extension of weak contact* $WCont^{\mathcal{M}}$ *to include the pairs* $\langle \mathbf{b}, \mathbf{f}\rangle, \langle \mathbf{b}, \mathbf{f_b}\rangle, \langle \mathbf{c}, \mathbf{f}\rangle, \langle \mathbf{c}, \mathbf{f_b}\rangle$ *and their symmetric equivalents. Note that this model arises if for instance we want only* $\mathbf{b}$ *and* $\mathbf{f}$ *to be in weak contact in a two-dimensional spatial model. Unproblematic is that then* $\langle \mathbf{b}, \mathbf{f_b}\rangle \in WCont^{\mathcal{M}}$ *follows. However, one could not conclude that* $\mathbf{c}$ *and* $\mathbf{f}$ *are in weak contact as well. The problem with this model is that* $\mathbf{b} \cup \mathbf{c}$ *has no unique interior than itself. That would mean again that it is open and by definition* $D11$ *it would follow that* $\langle \mathbf{f}, \mathbf{b} \cup \mathbf{c}\rangle \in C^{\mathcal{M}}$ *if* $\langle \mathbf{f}, \mathbf{b}\rangle \in WCont^{\mathcal{M}}$ *or* $\langle \mathbf{f}, \mathbf{c}\rangle \in WCont^{\mathcal{M}}$*. Hence we need to define a interior of* $\mathbf{b} \cup \mathbf{c}$ *that is distinct from* $\mathbf{b} \cup \mathbf{c}$*. Intuitively it would be* $\mathbf{d} \cup \mathbf{e}$*. This interior will need a complement, call it* $\mathbf{f}'$*, that is greater than* $\mathbf{f}$ *(by order-reversing law). Moreover, for* $\mathbf{f}'$ *to not be the closure of* $\mathbf{f}$*, we need an element that is their difference* $\mathbf{f}' - \mathbf{f}$*. This is intuitively the border* $\mathbf{bc_b}$ *of* $\mathbf{c} \cup \mathbf{b}$*. The border* $\mathbf{bc_b}$ *must be further split into two separate atoms:* $\mathbf{b_b}$ *and* $\mathbf{c_b}$*.*

This example demonstrates that just extending the lattices of models of RT^{-}_{EC} by additional elements to construct a model of RT is not a very successful strategy. It is more or less left to chance whether extending a given model of RT^{-}_{EC} leads to a model of RT_0 by adding some additional elements. In the next subsection, we approach the construction of models of RT_0 more systematically by using products of lattices.

3.2. Characterization of the finite models satisfying A12. The previous example demonstrates that constructing models with weak contact successively by adding more atoms leads to an exponential growth in the overall number of elements (with respect to the number of atoms) since sums of every pair of elements must exist. A similar phenomenon appears in lattice products: consider two small models of RT^{-}_{EC} represented as lattices $\mathcal{L}_1, \mathcal{L}_2$ that we join by a direct product $\mathcal{L}_1 \times \mathcal{L}_2 = \mathcal{L}_{1,2}$ where $\langle x_1, x_2\rangle \in \mathcal{L}_{1,2}$ for all combinations of $x_1 \in \mathcal{L}_1$ and $x_2 \in \mathcal{L}_2$. We show here what conditions must be satisfied to ensure that a weak contact is build instead of joining two separate, disconnected spaces just by sums. That results in the following conjecture that we approach step-by-step.

Conjecture 5. *The finite models of* RT_0 *are direct (Cartesian) products of finite p-ortholattices.*

Indeed, we only obtain a weaker result that direct products of finite p-ortholattices gives (again finite) models of RT only under certain restrictions and by applying slight changes to the original lattices. We will discuss these in more detail in the following. However, we cannot prove the forward direction of the conjecture (every finite mode of RT is a product of finite p-ortholattices), even when maintaining the restrictions. However, a more general result is suggested.

First of all, it is necessary to show that the product of two p-ortholattices is a p-ortholattice again. This is a rather obvious theorem when looking at two finite (complete) bounded lattices $\mathcal{L}_1$,$\mathcal{L}_2$: in each of them the intersection and sum of any pair of elements is defined as the greatest lower bound and the lowest upper bound. Moreover do both lattices have a unique infimum and supremum defined. Orthocomplementedness follows directly for the product lattice $\mathcal{L}_{1,2} = \mathcal{L}_1 \times \mathcal{L}_2$ since $x_1 \in \mathcal{L}_1$ and $x_2 \in \mathcal{L}_2$ result in an element $(x_1, x_2) \in \mathcal{L}_{1,2}$ which has the unique orthocomplement $(x_1^\perp, x_2^\perp) \in \mathcal{L}$. Any meet-pseudocomplement and any join-pseudocomplement is uniquely defined in the product lattice $\mathcal{L}_{1,2}$: if $mpc(x_1)$ is the meet-pseudocomplement of x_1 in $\mathcal{L}_1$ and $mpc(x_2)$ is the meet-pseudocomplement of x_2 in $\mathcal{L}_2$, then $(mpc(x_1), mpc(x_2)) \in \mathcal{L}_{1,2}$ is the uniquely defined meet-pseudocomplement of $(x_1, x_2) \in \mathcal{L}_{1,2}$. Note that it is comparable to any complement y_1 of x_1 where $y_1 < mpc(x_1)$: $(y_1, mpc(x_2)) < (mpc(x_1), mpc(x_2))$. Then same applies for the join-pseudocomplements.

THEOREM 20. *The direct product $\mathcal{L}_{1,2} = \mathcal{L}_1 \times \mathcal{L}_2$ of two finite p-ortholattices $\mathcal{L}_1$, $\mathcal{L}_2$ is a finite p-ortholattice itself.*

PROOF. The ortholattices form a variety that is closed under product taking. Hence, the product of two ortholattices is again orthocomplemented. The meet-pseudocomplemented semi-lattices and their dual, the join-pseudocomplemented semi-lattices each form a variety. Hence the product of two p-ortholattices is again meet- and join-pseudocomplemented and thus a double p-lattice. □

We already know that the finite p-ortholattices are isomorphic to the finite models of RT^-, but it is not guaranteed that there exists an external connection in the product. Although the uniquely complemented lattices do not form a variety, the finite ones are closed under product taking. Hence if $\mathcal{L}_1$ and $\mathcal{L}_2$ do not entail an external connection, then so does $\mathcal{L}_1 \times \mathcal{L}_2$ from the following lemma.

LEMMA 6.19. *The direct product $\mathcal{L}_{1,2} = \mathcal{L}_1 \times \mathcal{L}_2$ of two finite uniquely complemented p-ortholattices $\mathcal{L}_1$,$\mathcal{L}_2$ is a finite uniquely complemented lattice.*

PROOF. Consider elements $x, x' \in \mathcal{L}_1$ and $y, y' \in \mathcal{L}_2$ where x', y' are the unique complements of x and y in $\mathcal{L}_1$ and $\mathcal{L}_2$. Then (x', y') is the unique complement of (x, y) in $\in \mathcal{L}_{1,2}$. □

This further restricts the class of models to direct products of p-ortholattices, where at least one of the lattices, call it $\mathcal{L}_1$, is in the not uniquely complemented class of complete atomic p-ortholattices, i.e. $\mathcal{L}_1$ is a model of RT^-_{EC}. Notice that two lattices with their direct (Cartesian) product taken do not automatically create a weak contact, i.e. simply applying the Cartesian product assumes the space (or whatever model) represented by $\mathcal{L}_1$ being totally separate from the elements in $\mathcal{L}_2$ and vice versa. This occurs exactly when the suprema of the two lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ are *clopen* and hence cannot be in weak contact according to the earlier discussion. To create a weak contact between them, the suprema $\top_1$ and $\top_2$ of $\mathcal{L}_1$ and $\mathcal{L}_2$ must be closed but not open. Openness for $\top_1$ and $\top_2$ is guaranteed if they have unique interiors $int(\top_1) \neq \top_1$ and $int(\top_2) \neq \top_2$ in $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Generalized to arbitrary elements $x_1 \in \mathcal{L}_1$ and $x_2 \in \mathcal{L}_2$, i.e. $\langle (x_1, \bot_2), (\bot_1, x_2) \rangle \in WCont^{\mathcal{M}}$, $int(x_1) \neq x_1$ and $int(x_2) \neq x_2$ must hold. Moreover, all parents of x_1 and x_2 in $\mathcal{L}_1$ and $\mathcal{L}_2$, e.g. $y_1 > x_1$, cannot be open (trivially true for $x_1 = \bot_1$ and $x_2 = \bot_2$), since the union with the empty set, i.e. $(y_1, \bot_2)$, is still a parent of x_1, but not connected to x_2. Otherwise in this example $(x_1, \bot_2)$ would be a part of $(y_1, \bot_2)$, and thus $(y_1, \bot_2)$ would be an (open) neighborhood of $(x_1, \bot_2)$ that is not connected to $(\bot_1, x_2)$. Therefore both $\mathcal{L}_1$ and $\mathcal{L}_2$ must be extended by additional elements to lattices $\mathcal{L}'_1$ and $\mathcal{L}'_2$ that satisfy these conditions.

To achieve that we extend both lattices by explicit closures of their suprema, i.e. $cl(\top_1)$ to $\mathcal{L}_1$ and $cl(\top_2)$ to $\mathcal{L}_2$. The resulting lattices $\mathcal{L}'_1$ and $\mathcal{L}'_2$ are still pseudocomplemented, but not orthocomplemented anymore. However, we claim that the product $\mathcal{L}_{1,2} = \mathcal{L}'_1 \times \mathcal{L}'_2$ extended by another element x is orthocomplemented again. To show that we need to define the orthocomplement of each element in $\mathcal{L}_{1,2}$. To distinguish in $\mathcal{L}_{1,2}$ the neighborhood of $((cl(\top_1), e_2)$ from $((\top_1, e_2)$, and $((e_1, cl(\top_2))$ from $((e_1, \top_2)$, we introduce an element x modeling the difference between $a^* = ((cl(\top_1), cl(\top_2))$ and $((\top_1, \top_2)$. x is an atom and the orthocomplement of $((\top_1, \top_2)$ in $\mathcal{L}_{1,2}$. The union of x with any other element (e_1, e_2), with $e_1 \neq cl(\top_1) \in \mathcal{L}_1$ and $e_2 \neq cl(\top_2) \in \mathcal{L}_2$, needs to be added to $\mathcal{L}_{1,2}$ as well. With all these elements we call the lattice $\mathcal{L}'_{1,2}$. Then it is ensured that each such union $x \vee (e_1, e_2)$ also has an orthocomplement in $(-e_1, -e_2)$. Hence the resulting lattice is orthocomplemented again. Each element $(cl(\top_1), e_2)$ with $e_2 \neq cl(\top_2) \in \mathcal{L}_2$ has in $(\emptyset, e_2)$ an orthocomplement and so does each element $(e_1, cl(\top_2))$ with $e_1 \neq cl(\top_1) \in \mathcal{L}_1$ has in $(e_1, \emptyset)$ an orthocomplement. The proof that the resulting lattice $\mathcal{L}'_{1,2}$ is pseudocomplemented is lengthy, but not difficult: show that in the set of all unions of x has only pseudocomplements outside the set and vice versa. In the lattice $\mathcal{L}'_{1,2}$ the elements $(cl(\top_1), \emptyset)$ and $(\emptyset, cl(\top_2))$ are then in weak contact and we have in $\mathcal{L}'_{1,2}$ a lattice representation of a model of RT.

In the following we demonstrate how to construct in this way the smallest model of the full theory RT.

EXAMPLE 11. *Let $\mathcal{L}_1$ be the lattice of the five elements* $\mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{ic}, \mathbf{id}$ *as shown in figure 3.2(a). Previously, we demonstrated that this is the smallest model of RT^-_{EC}. Let $\mathcal{L}_2$ be a three-element p-ortholattice* $\mathbf{e}, \mathbf{ie}, \mathbf{ce}$ *of an element with its interior, its complement, and the empty set, see figure 3.2(b). A possible interpretation is as following: $\mathcal{L}_1$ is a model of two elements* $\mathbf{c}, \mathbf{d}$ *that are externally connected and have their distinct interiors* $\mathbf{ic}, \mathbf{id}$. *$\mathcal{L}_2$ is a simple region* $\mathbf{e}$ *with a specified interior* $\mathbf{ie}$ *and its complement* $\mathbf{ce}$ *where the two cannot be broken down any further (the top element is the region itself, the bottom element the empty set). Note that only two non-open elements can be in weak contact. Therefore the only feasible elements for creating a weak contact are* $\mathbf{b}$ *and* $\mathbf{e}$ *which must have a distinct interior. $\mathcal{L}_2$ is already the minimal model of RT^- without external connection whereas $\mathcal{L}_1$ is the minimal model of RT^-_{EC} with external connection.*

Now extend $\mathcal{L}_1$ and $\mathcal{L}_2$ to non-ortholattices $\mathcal{L}'_1$ and $\mathcal{L}'_2$ by adding $\mathbf{cl}(\mathbf{b})$ *and* $\mathbf{cl}(\mathbf{e})$. *The resulting lattices are depicted in figure 3.2. Most importantly, the elements* $\mathbf{cl}(\mathbf{b})$ *and* $\mathbf{cl}(\mathbf{e})$ *are not open. The product $\mathcal{L}_{1,2} = \mathcal{L}'_1 \times \mathcal{L}'_2$ then needs to be extended by an element $\mathbf{be_{diff}} = \mathbf{a}^* - (\mathbf{b}, \mathbf{e})$ and all unions thereof. We get $\mathcal{L}'_{1,2}$ as a model of RT with*

$$Y = \{(y,x) | y \in \mathcal{L}'_1, x \in \mathcal{L}'_2\} \cup \{\mathbf{be_{diff}}\} \cup \{\mathbf{be_{diff}} \vee (y,x) | y \in \mathcal{L}'_1, x \in \mathcal{L}'_2\}$$

as (finite) set of elements and the partial order satisfying the following axioms:

$$\forall x, z \in \mathcal{L}'_1 \forall y \in \mathcal{L}'_2 \left[\langle x, z \rangle \in P_{\mathcal{L}'_1} \to \langle (x,y), (z,y) \rangle \in P_{\mathcal{L}'_{1,2}} \right] \text{ and}$$

$$\forall x, z \in \mathcal{L}'_2 \forall y \in \mathcal{L}'_1 \left[\langle x, z \rangle \in P_{\mathcal{L}'_2} \to \langle (y,x), (y,z) \rangle \in P_{\mathcal{L}'_{1,2}} \right] \text{ and}$$

$$\forall x \in \mathcal{L}'_2 \left[\langle \mathbf{be_{diff}}, (\mathbf{cl}(\mathbf{b}), x) \rangle \in P_{\mathcal{L}'_{1,2}} \right] \text{ and}$$

$$\forall x \in \mathcal{L}'_1 \left[\langle \mathbf{be_{diff}}, (x, \mathbf{cl}(\mathbf{e})) \rangle \in P_{\mathcal{L}'_{1,2}} \right] \text{ and}$$

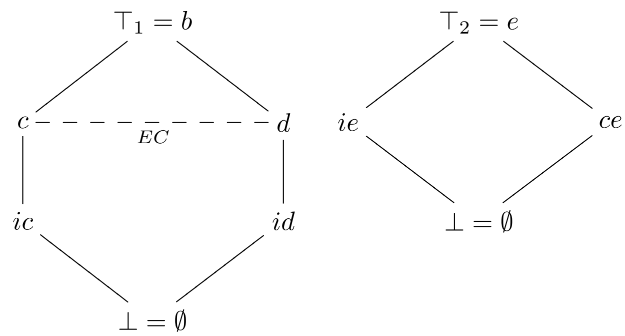
$$\forall (x,y) \in \mathcal{L}'_{1,2} \left[\langle \mathbf{be_{diff}}, \mathbf{be_{diff}} \vee (x,y) \rangle \in P_{\mathcal{L}'_{1,2}} \right] \text{ and}$$

$$\forall (x,y) \in \mathcal{L}'_{1,2} \left[\langle (x,y), \mathbf{be_{diff}} \vee (x,y) \rangle \in P_{\mathcal{L}'_{1,2}} \right] \text{ and}$$

$$\forall (x,y), (z,y) \in \mathcal{L}'_{1,2} \left[\langle x, z \rangle \in P_{\mathcal{L}'_1} \to \langle \mathbf{be_{diff}} \vee (x,y), \mathbf{be_{diff}} \vee (z,y) \rangle \in P_{\mathcal{L}'_{1,2}} \right] \text{ and}$$

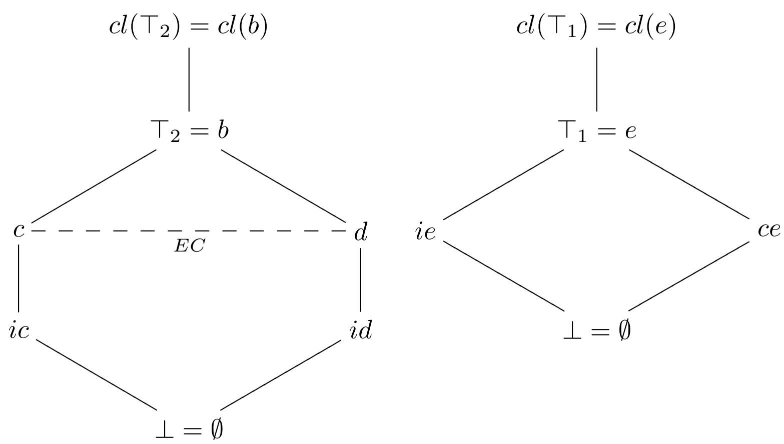
$$\forall (x,y), (x,z) \in \mathcal{L}'_{1,2} \left[\langle y, z \rangle \in P_{\mathcal{L}'_2} \to \langle \mathbf{be_{diff}} \vee (x,y), \mathbf{be_{diff}} \vee (x,z) \rangle \in P_{\mathcal{L}'_{1,2}} \right]$$

Because $\mathcal{L}_1$ and $\mathcal{L}_2$ are the smallest models of RT^-_{EC} and RT^-, respectively, $\mathcal{L}'_{1,2}$ is in fact the smallest model of RT. It contains $5 \cdot 7 - 1 + 1 + 4 \cdot 6 = 59$ elements (the product $\mathcal{L}_{1,2}$ contains $5 \cdot 7 - 1$ elements, $\mathbf{be_{diff}}$ counts as one, and the products of $\mathbf{be_{diff}}$ with any element in $\mathcal{L}_{1,2}$ that does neither contain $cl(\mathbf{b})$ nor $cl(\mathbf{e})$ add $4 \cdot 6$ elements).



(a) $\mathcal{L}_1$, a model of RT^-_{EC} (b) $\mathcal{L}_2$, a model of RT^-

FIGURE 27. Models of RT^- and RT^-_{EC} as lattices for the exemplary construction of a model of RT



(a) $\mathcal{L}'_1$ (b) $\mathcal{L}'_2$

FIGURE 28. Extended lattices of models of RT^- and RT^-_{EC} with explicit closures of the suprema

Thus products of two finite p-ortholattices with one of them being not uniquely complemented are models of RT.

THEOREM 21. *The product $\mathcal{L}_{1,2} = \mathcal{L}'_1 \times \mathcal{L}'_2$ of two arbitrary finite p-ortholattices $\mathcal{L}_1$ and $\mathcal{L}_2$, extended by explicit closures of their suprema to $\mathcal{L}'_1$ and $\mathcal{L}'_2$, is a finite model of RT if at least one of the models represented by $\mathcal{L}_1$ and $\mathcal{L}_2$ contains an external connection.*

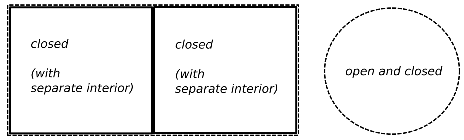


FIGURE 29. Multiple product lattice $\mathcal{L}_{1,2,3} = (\mathcal{L}'_1 \times \mathcal{L}'_2) \times \mathcal{L}'_3$
In this example, the product $(\mathcal{L}'_1 \times \mathcal{L}'_2)$ creates a weak contact, but on the top level the product of $(\mathcal{L}'_1 \times \mathcal{L}'_2)$ with $\mathcal{L}'_3$ (assume that $\mathcal{L}_3$ contains some external connection) does not need to confirm to the described conditions. Simply the product without any further extension can be taken.

The reverse cannot be stated for the following reason: We can have models with several nested sets of products, but not on the top level of the products of $\mathcal{L}'_1$ and $\mathcal{L}'_2$. But on some level we need a product that results in a weak contact, but we do not necessarily need to have an external connection in the participating submodels. The external connection can be added on a later (or earlier level), see for example figure 29.

To capture these more complicated structures, we propose the following conjecture:

CONJECTURE 6. *Every finite model of RT can be constructed recursively: take finitely many products of finite p-ortholattices (either with or without extending their suprema by separate closures). At least for one of the products one of the multiplicand must be a not uniquely complemented p-ortholattice, and at least one product is taken of two lattices that are extended by additional closures of their suprema.*

Since the result of finitely many products is again a p-ortholattice, the finite models of RT are then a proper subset of the finite p-ortholattices. Since moreover, the proves for pseudocomplementedness and orthocomplementedness do not rely on finiteness of the models, all models of RT are in the class of p-ortholattices.

CHAPTER 7

Conclusion

Let us recapitulate the line of argument throughout the thesis. We have tried three strategies for characterizing the models of increasingly larger subsets of RT. The first approach, using topological spaces, was only partly successful. We exhibited parallels to the characterizations of Clarke's *Calculus of Individuals* as well as of the *RCC*, but the characterization was far too imprecise and standard tools of topology like the separation axioms failed altogether. Only semi-regularity, which seems to correspond to the condition of *smooth boundaries* can be proved for the infinite models of RT. Even when considering only the infinite models, local connectedness fails to capture the second critical part of *regularity*, the property of *full interiors*. Most problematic is the characterization's inability to adequately capture finite models by their embedding topological spaces: the spaces reduce to discrete topologies. But we should be reminded that the models are originally defined using topological spaces, but without any restrictions upon the spaces. Further research questions arise here if we consider mereotopological models constructed using more restricted topological spaces such as Hausdorff or even regular spaces. Overall, the results were less than satisfactory.

In the next chapter, we used methods and definitions from universal algebra by interpreting the models of RT^- as lattices. The similarity between posets that underlie lattices and mereological concepts like parthood and overlap is striking. It turned out that characteristic properties of the models of RT^- can be captured by orthocomplementation and pseudocomplementation which together give an isomorphic description of the models of RT^- as p-ortholattices. However, there was no room for the distinctive mereotopological concepts of external connection and weak contact as required for representing models of RT^-_{EC} and RT. These concepts were introduced with existential axioms by [**AV95**] but the lattices alone could not account for them. Existence of external connection prohibits uniquely complemented lattices and equally any kind of modularity for models of RT^-_{EC}. The lattices of any model of RT are thus strictly not uniquely complemented which delimits the models from the *Calculus of Individuals* and from the *RCC*. The former were characterized as Boolean lattices which are equivalent to the uniquely complemented distributive pseudocomplemented lattices (distributive pseudocomplemented is not enough, this

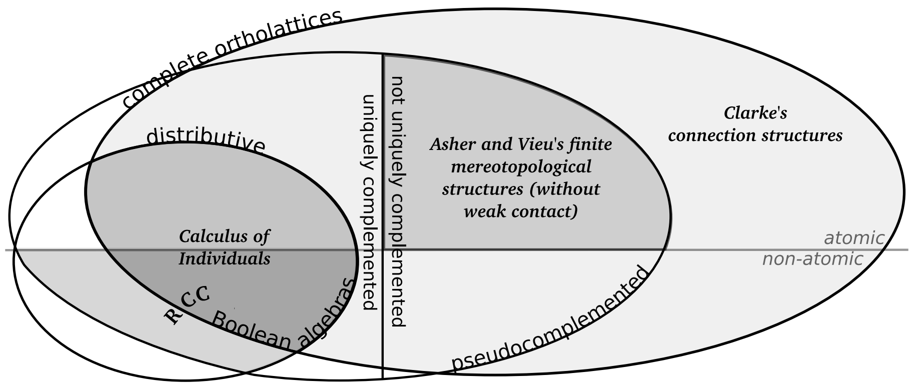


FIGURE 30. Asher and Vieu's mereotopology, Clarke's *Calculus of Individuals*, and the *RCC* as subclasses of complete lattices
All models are complete lattices because the sum of any pair of elements in required. The models of *RCC* are the atomless distributive pseudocomplemented lattices, the models of the *Calculus of Individuals* are the distributive orthocomplemented lattices, and the models of RT^{-}_{EC} are the atomic not-unicomplemented, pseudocomplemented orthocomplemented lattices.

class contains Heyting and Stone lattices as well) and models of *RCC* were characterized as inexhaustible distributive pseudocomplemented lattices, where inexhaustible corresponds to our understanding of atomless, see figure 30. Both sets of models can be represented as distributive, uniquely complemented lattices. This is partly caused in Clarke's system by the error in the definition of external connection that maps it to overlap and in *RCC* by the lack of any distinction between open and closed elements as separate individuals of the models. This greatly simplifies the models of *RCC* sacrifices a higher expressiveness offered by the system of *RT*. Empirical approaches will be necessary to evaluate in which cases this is acceptable and which applications or domains require the higher expressiveness of Asher and Vieu's theory.

In order to completely characterize the full models of *RT* (including the extensions of external connection and weak contact), we tried a third approach by representing models as undirected graphs. This is perhaps the most natural way to examine the models build from a single dyadic primitive C. Since all other extensions depend solely on the extension of C, such a graph uniquely characterizes a model. *Connection structures* which are defined by A1 to A3 and which are common throughout mereotopological systems are isomorphic to twin-free graphs. Remarkably, twin-freeness as we define it (free of true **and** false twins) has not received much attention in graph theory and therefore no additional properties are known. Separately from the connection structures, we described the mereological features of the models in terms of chordal (and presumably comparability) graphs. However, the generalization of such a description to the models of RT^-_{EC} fails. We can extract that all the models of RT^-_{EC} are dually chordal graphs, but at the same time we gave evidence that this is only a very general classification. Inspired by the use of vertex orderings for all the considered graph classes, we defined a new vertex ordering called *maximum neighborhood inclusion order* (mnio) and demonstrated that this ordering defines a class of graphs that includes all graphs of RT^-_{EC}, but itself is a much smaller subset of the dually chordal graphs. We supported our conjecture that these orderings are characteristic for the graphs of RT^-_{EC} and showed a way to find such an ordering in linear time for every model of RT^-_{EC} by the search algorithm CLBFS. However, we suspect that not all properties defined by the axioms of RT are captured, especially the existence of sums, intersections, and interiors is not properly translated to graphs withs *mnio*s. Nevertheless, the graph-theoretic characterization gives us valuable insight into the models of the mereotopology and their substructures and we collected characteristic properties that might generalize to other mereotopological theories.

The two chapters on lattices and graphs show the advantages and disadvantages of both approaches: lattices easily capture the existence of unique sums and intersections of pairs of elements as well as the underlying parthood ordering that makes up the mereological part of the models, while graphs are capable of representing the full set of specialized JEPD relations of the connection relation C. Consequently, we brought both representations together, which led to a full characterization of the models of RT^-_{EC} in terms of graphs of lattices: every finite not unicomplemented p-ortholattice $\mathcal{L}$ is equivalent to a finite model $\mathcal{M}$ of RT where $\langle x,y\rangle \in O^{\mathcal{M}} \iff \exists z\,[z \leq x \wedge z \leq y \wedge z \neq \emptyset] \iff xy \in E(G^{\mathcal{L}})$ and $\langle x,y\rangle \in EC^{\mathcal{M}} \iff \{xy \in \left(E(G^{\mathcal{L}}_{EC}) \setminus E(G^{\mathcal{L}})\right) \wedge y \not\leq x^{\perp}\}$. In this way all the lattice properties are maintained while we can easily extend the graphs to twin-free graphs by adding a uniquely defined extension $EC^{\mathcal{M}}$. This extension of external connection again reuses the orthocomplement defined in the lattice as a straightforward, constructive way to build models of RT^-_{EC}.

Eventually, we included the last missing axiom requiring the existence of a weak contact. We showed how finite models of RT can be obtained from direct products of finite p-ortholattices. Altogether, these models are too large to give any examples. We only provide two lattices whose product will be the smallest model of RT. Nevertheless, we were able to discuss what additional properties the lattices must satisfy in order to ensure that their product contains a weak contact. The existence of a weak contact is reduced to the existence of a weak contact between two closed, non-open individuals. Taking two finite p-ortholattices of which at least one must be not uniquely complemented, and extending them by separate closures of their suprema, the product of such extended lattices is a (finite) model of RT, but the proof that any model of RT can be obtained in this way is still open. Hence we cannot provide yet a proof up to isomorphism of the class of model of RT.

Apart from that, recalling the objectives of our work, we achieved our chief goals. We gave a representation theorem for the models of RT^- and a partial characterization of the models of RT. On the way we obtained valuable insights into the characteristic properties of all models. From the given characterization it is now easy to construct p-ortholattices that correspond to models and even more importantly, we can easily identify the extensions of all relations from the lattice alone. Orthocomplements in the lattices map to complements in the models, the join and meet of pairs of elements in the lattice represent the unique sum and intersection in the models. The closure and interiors are equivalent to the meet- and join-pseudocomplements of the orthocomplement. Overlap relations produce a meet distinct from the empty set and external connections for a given individual are identified by all the elements not part of the orthocomplement that the individual is not connected to by any other means. We also presented alternative properties such as non-orthomodularity and non-semimodularity for the restricted class of complete atomic, not unique complemented p-ortholattices. Where applicable, we compared our analysis to similar analyses of other mereotopological systems. Some of the characterizations generalize well to *connection structures* (as twin-free graphs) and we show that the finite models are a proper subset of Clarke's contact algebras which were characterized as complete orthocomplemented lattices. An open question here is whether it can be proved that all models of RT including the infinite ones give complete lattices. If not, the theory RT actually weakens Clarke's unrestricted fusion axiom. Otherwise, we obtain a proof that the unrestricted fusion can be replaced equivalently by the sum axiom A5 without impacting the infinite models.

Moreover, we provided a methodological outline for thorough analysis of other mereotopologies to enable a model-theoretic comparison of mereotopological theories (as opposed to a comparison by the axioms and theorems of a mereotopology). Instead of questioning underlying philosophical foundations, we take the theory itself for granted and analyze it from the perspective of their models. For further research in the same direction, a lattice-based approach is most promising since it captures essential mereological concepts such as parthood and topological concepts such as complements. All mereotopological theories that depend on a single primitive can be also modeled as graphs or graphs of lattices in a similar fashion as demonstrated here.

For further research, two major directions are possible. First of all, a model-theoretic analysis of other mereotopologies can help to understand differences and commonalities between different axiomatizations. We think in particular of the systems of Borgo, Guarino, and Masolo [**BGM96**] that explicitly distinguishes a topological (*simple region*) and a mereological primitive (*parthood*) and comprises a notion of convexity. Other interesting candidates that have not yet treated in a model-theoretic way are the mereotopologies of Smith [**Smi96**] and of Pratt and Schoop [**PS97**]. Another option for future research is the reverse perspective: choosing a promising class of lattices and showing whether it yields sufficiently expressive mereotopological systems, maybe on a generic level (subsuming other mereotopological systems) or restricted to certain clearly defined application domains. Some candidates that we identified throughout the thesis are semimodular lattices, geometric lattices, and the full class of p-ortholattices (from the isomorphism we already know that the class of p-ortholattices gives the theory defined by RT^-). However, this list can be extended arbitrarily (Stone lattices, Heyting lattices, or the full class of pseudocomplemented distributive lattices) and it might turn out selecting the most suitable lattices for modeling mereotopology is too difficult or cumbersome. However, in a similar fashion the search for a useful lattice representation of pointless topology could be conducted in the future.

Appendix

Complements as Dominating Pairs in Graphs in $\mathfrak{G}_{RT}$

The graph-theoretic analysis detected some useful properties of the *mnio* found by CLBFS that utilize the characterization of the set E_{EC} obtained in chapter 6. Here, we give more details why this vertex ordering is special for the graphs in $\mathfrak{G}_{RT}$.

The CLBFS algorithm gives a linear-time algorithm to find a *mnio* for every graph of a model of RT^{-}_{EC}. But these vertex orderings have additional useful properties, e.g. for finding all the complements of a given model in linear time. For the proof of that property, we first show that in fact every individual and its complement are connected to a fixed number of other individuals. We already know that any individual and its complement are a dominating pair just from the definition of a topological complement. But here we identify the number of other individuals that both elements are connected to in the resulting graphs.

THEOREM 22. *For every graph G_{RT} every vertex $x \in V(G_{RT})$, $x \neq a^*$ satisfies $|N[x]| + |N[-x]| = \frac{3n-5}{2}$, where $-x$ is its complement defined by A8.*

PROOF. x is in parthood relation to $\frac{n+1}{2}$ elements including itself, i.e. in G_{RT} it is adjacent to $\frac{n-1}{2}$ other vertices. The same applies for $-x$. The sum of the number of vertices adjacent to x and $-x$ (those adjacent to both count double) caused by proper parthood relations is then $n-1$. By order-reversing of the resulting lattice structures half of those relations (removing both instances of the previously double counted universal element), i.e. $\frac{1}{2}(n-1)-1$ account for the cases where x or $-x$ is proper part of some other element, and half of those account for the cases where the other element is proper part of x or $-x$. Then by (2) and (3) above exactly half of those adjacencies cause the complement to be adjacent to the same vertices. Then we get: $\frac{3}{2}(n-1)-1 = \frac{3n-5}{2}$ for the sum of the degrees of x and $-x$. □

THEOREM 7.1. *On every graph G in $\mathfrak{G}_{RT}$, the reverse $\sigma^{-1} = v_1, v_2, \ldots, v_n$ of any Cardinality LexBFS (CLBFS) ordering on G contains dominating pairs v_i, v_{n-i} where $v_i = -v_{n-i}$.*

For the proof we show first that the following claims hold. Claim 1 is used to prove claim 2, and both of them are needed for the complete proof of the theorem. The proof is recursive; starting with the elements of highest degree and working its way down along the proper parthood ordering of the lattice.

CLAIM 1. If $PP(x, y)$ holds then $|N[x]| < |N[y]|$.

Follows directly from lemma 5.5 by the assumption $PP(x, y)$ which translates in the corresponding graph to $N[x] \subset N[y]$.

CLAIM 2. Assume $|N[x]| \geq |N[-x]|$ and x and $-x$ have the same lexicographic label, then selecting x as next vertex in a CLBFS ensures that all elements $z \in N(x)$ are selected before $-x$ into the vertex ordering. In particular, all y with $PP(-x, y)$ are selected before $-x$ in the resulting ordering.

If x and $-x$ previously had the same lexicographic label, then choosing x as the next vertex for the ordering labels all vertices in the neighborhood of x, but not $-x$, since x and $-x$ are never adjacent (see proposition 6.4(1)). That all parents of $-x$ (in the meaning of proper parts) are selected before $-x$ follows directly from the fact that by the definition of E_{EC} (theorem 18) x must be externally connected to all parents of $-x$. Since the parents of $-x$ are all in the neighborhood of x, but $-x$ itself is not, all parents get a lexicographic larger label than $-x$. Hence, all such parents must be selected into the vertex ordering before $-x$ by any CLBFS algorithm.

PROOF. Consider the start of the search: CLBFS first selects $u_1 = a^*$ because all vertices are lexicographically unlabeled and a^* has the highest cardinality. Afterward, one of the vertices with the next highest cardinality is chosen. If two or more vertices with the same cardinality exist and they are adjacent, then they will be selected for the ordering consecutively. However, in any case whatever vertex was chosen first, call it u_2, its neighborhood has to be exhaustively added to the vertex ordering before any vertex in its non-neighborhood can be selected into the ordering. Moreover, by claim 2, $-u_2$ must be selected after the complete neighborhood of u_2 and all parents of $-u_2$. Notice that u_2 must be a dual-atom because of claim 1 and then $-u_2$ is an atom. With the set X containing all vertices that are not comparable to $-u_2$, and the other elements in $X \setminus \{-u_2\}$ being comparable but not proper part of $-u_2$ (because it is an atom an has no proper parts), i.e. $-u_2$ must be part of every element in $X \setminus \{-u_2\}$. By the definition of the set E_{EC} (theorem 18) u_2 is connected to all other vertices in $X \setminus \{-u_2\}$ and thus $N[u_2] = V \setminus \{-u_2\}$. Consequently, by claim 2, $-u_2$ is selected as the last vertex by the CLBFS.

Similarly, all other dual atoms are selected subsequently and their complements are selected as relative last elements, i.e. the last element before the complements of the already ordered vertices. Afterward all vertices u_k that are covered in the lattice by a dual-atom (i.e. those that satisfy for some dual-atom d the following $PP(u_k, d) \wedge \neg\exists v\,[PP(v, d) \wedge PP(u_k, v)]$) are selected and their complements are relatively last[1]. In the resulting order $\sigma = u_1, u_2, \ldots, u_n$ it holds that $-u_{i+1} = u_{n-i+1}$ for all i with $1 \leq i \leq \frac{n-1}{2}$. The reverse σ^{-1} then satisfies $v_i = -v_{n-i1}$ for all $1 \leq i \leq \frac{n-1}{2}$ and because of symmetry it also holds for $1 \leq i \leq n$. By proposition 6.3(2) each such pair v_i, v_{n-i} is then a dominating pair. □

This gives us a very simple way (linear-time $O(n)$ algorithm) for determining complements in models of RT. This can be used for practical reasoning application that would use such a mereotopology for modeling e.g. space. The orderings are furthermore degree orderings, as one can derive from the last two theorems.

COROLLARY 7.2. *On every graph G_{RT}, any ordering σ obtained through CLBFS is a descending degree ordering, i.e. $\forall j < i$, $|N(v_j)| \geq |N(v_i)|$.*

Beware that this is different from what a MCS would generate: any ordering σ produced by CLBFS can also be obtained by MCS, but the reverse is not true, since MCS only considers cardinalities for the subgraphs G_i for $i = \{1, \ldots, n\}$. Notice furthermore how easy a degree ordering is recognizable: a simple linear-time algorithm will do it.

[1]Relatively last because all the vertices u_j with $j < k$ already have a complement that must be selected later. Relatively to all the elements from which we are still free to choose.

Notation

Symbolic:

$\emptyset$	empty set, usually used as the bottom element of a lattice arising from models of RT
$\{\}$	empty set as used in set-theoretic discussions
$\bot$	bottom element (infimum) of a bounded lattice
$\top$	top element (supremum) of a bounded lattice
a^*	unique universal element of a model of RT
$-x$	set-theoretic complement of the point-set x in a topological space
x	unique topological complement of the individual x in a model of RT (or a subset thereof)
$x^\perp$	uniquely identified orthocomplement of x in an orthocomplemented lattice
$A^{\mathcal{M}}$	extension of relation A for a specific model $\mathcal{M}$ of RT
$\neg X$	unary predicate negation in FOL
$\exists x$	existential quantification in FOL
$\forall x$	universal quantification in FOL
$x + y$	unique sum of the two individuals x and y in a model of RT
$x \cdot y$	intersection of the two individuals x and y in a model of RT, usually means the non-empty intersection
$\mathcal{L}_1 \times \mathcal{L}_2$	direct (Cartesian) product of two lattices, i.e. for any tuple a, b with $a \in \mathcal{L}_1$ and $b \in \mathcal{L}_2$ the element $(a, b) \in \mathcal{L}_1 \times \mathcal{L}_2$
$a \prec b$	b covers a (or a is covered by b) in a lattice if $a < b$ and no c exists with $a < c < b$ exists
$x \subset y$	x is a proper subset of y
$x \subseteq y$	x is a subset of y
$A \rightarrow B$	A implies B, implication in FOL
$A \leftarrow B$	B implies A, implication in FOL
$A \equiv B$	logical equivalence: $A \Rightarrow B$ and $A \Leftarrow B$; also used for elementary equivalence between two (classes of) models

$A \cong B$	A is isomorphic to B
$A \iff B$	if and only if
$x \supset y$	x is a proper superset of y
$x \supseteq y$	x is a superset of y
$x < y$	x is strictly smaller than y with respect to the partial order given by P or the lattice
$x \leq y$	x is smaller than or equal to y with respect to the partial order given by P or the lattice
$x \nleq y$	$x > y$, or x and y are incomparable
$x > y$	x is strictly greater than y with respect to the partial order given by P or the lattice
$x \geq y$	x is greater than or equal to y with respect to the partial order given by P or the lattice
$x \ngeq y$	$x < y$, or x and y are incomparable
$A - B$	set-theoretic difference of A and B
$A \vee B$	A or B hold (FOL)
$x \vee y$	join of x and y in a lattice
$A \wedge B$	A and B hold (FOL)
$x \wedge y$	meet of x and y in a lattice
$x \cup y$	standard sum of two point sets
$x \cup^* y$	sum of two sets in RT
$x \cap y$	standard intersection of two point sets
$x \cap^* y$	intersection of two sets in RT
$(a, b)M$	modular pair $\langle a, b \rangle$ in a lattice L satisfying $x \leq b$ implies $x \vee (a \wedge b) = (x \vee a) \wedge b$ for all $x \in L$
$\langle x, y \rangle \in C^{\mathcal{M}}$	x and y are in the extension of C for the model $\mathcal{M}$, analog for all other predicates in the theory RT
$\langle S, \mathcal{O} \rangle$	topological space defined over the set S with $\mathcal{O}$ as set of open sets (the topology)
(X, T)	alternative notation for a topological space over X with T as topology, usually referring to the topological space underlying a model of RT_0

(Σ_U, Σ_U^T) topological space over the set Σ_U with the topology Σ_U^T

$[c_n]$ the equivalence class of a constant c_n in a set of sentences Σ consistent with RT_0, i.e. $[c_n] = \{c_j : \Sigma \vdash_{RT_0} c_j = c_n\}$

$\Omega_{[c_n]}$ the points in the topological space associated with the equivalence class of a constant c_n in a set of sentences Σ satisfying RT

Σ maximal set of sentences consistent with RT_0

Σ_C set of constants c_k occurring in the set Σ

Σ_U the set of points that forms the topological space for a model; defined as $\Sigma_U =_{def} \bigcup \{\Omega_{[c_n]} : c_n \in \Sigma_C\}$

Σ_U^T the set of open sets (the topology) $\subseteq \mathcal{P}(\Sigma_U)$ over a maximal, saturated set of sentences Σ consistent with RT_0

Non-symbolic (in alphabetical order):

C extension of connection in RT

$\mathcal{C}$ set of closed sets in a topological space

$c(x)$, cx the closure of an element as defined by RT_0

$cl(x)$ closure operation in RT_T

$Cl(x)$ closure of an element in standard topology

CL extension of the unary predicate *closed* in RT

Con extension of the unary predicate *self-connected* in RT

EC extension of external connection in RT

$G(\mathcal{M})$ graph of a model $\mathcal{M}$, which depending on the context can be a model of RT, RT^-_{EC}, or RT^-

$G^{\mathcal{L}}$ graph $G^{\mathcal{L}} = (V, E)$ resulting from a given complete atomic, not uniquely complemented p-ortholattice $\mathcal{L}$ over a poset Y by the definition $y \in Y \iff y \in V(G^{\mathcal{L}})$ and $x, y, z \in Y\, [z \leq x \wedge z \leq y] \iff xy \in E(G^{\mathcal{L}})$

$\mathfrak{G}^{\mathcal{L}}$ class of all graphs $G^{\mathcal{L}} = (V, E)$ that result from a complete atomic, not uniquely complemented p-ortholattice $\mathcal{L}$

$\mathfrak{G}_C$ class of graphs that are models of A1 to A3, identical with the twin-free graphs

$G^{\mathcal{L}}_{EC}$ graph constructed from a not uniquely complemented complete atomic p-ortholattice with a non-empty induced extension E_{EC}

$\mathfrak{G}^{\mathcal{L}}_{EC}$ class of graphs $G^{\mathcal{L}}_{EC}$ constructed from not uniquely complemented complete atomic p-ortholattices

$\mathfrak{G}_P$ class of graphs satisfying the axioms of RT_P

$\mathfrak{G}^{PP}_P$ class of graphs satisfying the axioms of RT_P where only the extension of proper parthood PP represents edges in a graph G^{PP}_P in $\mathfrak{G}^{PP}_P$

$G_{RT^-}(\mathcal{M})$ graph of the model $\mathcal{M}$ of RT^- where edges represent the extension of $C^{\mathcal{M}}$

$\mathfrak{G}_{RT^-}$ class of graphs that can be constructed from models of RT^- where edges represent the extension of $C^{\mathcal{M}}$ for each model $\mathcal{M}$

$G_{RT^-_{EC}}(\mathcal{M})$ graph of the model $\mathcal{M}$ of RT^-_{EC} where edges represent the extension of $C^{\mathcal{M}}$

$\mathfrak{G}_{RT^-_{EC}}$ class of graphs that can be constructed from models of RT^-_{EC} where edges represent the extension of $C^{\mathcal{M}}$ for each model $\mathcal{M}$

$\mathfrak{G}_{RT}$ class of graphs that can be constructed from models of RT where edges represent the extension of $C^{\mathcal{M}}$ for each model $\mathcal{M}$

$i(x)$, ix interior function in RT_0

$int(x)$ interior operation in RT_T

$Int(x)$ interior of an element in topology

$jpc(x)$ join-pseudocomplement of x in a join-pseudocomplemented lattice

$\mathcal{L}$ denoting a lattice

$\mathcal{L}^{\mathcal{M}}$ denoting a lattice associated to a finite model $\mathcal{M}$

$\mathcal{L} = (Y \cup \emptyset, \cap^*, \cup^*, \emptyset, a^*)$ denoting the lattice constructed from a model of RT with the set of individuals Y

$\mathcal{L}_6$ 6-element lattice that is neither modular nor uniquely complemented

L_2,L_3,L_4 special lattices prohibited as sublattices in semi-distributive lattices

$\mathfrak{L}_{RT}$ class of lattices arising from models of RT

$mpc(x)$ meet-pseudocomplement of x in a meet-pseudocomplemented lattice

M_3 diamond lattice prohibited in distributive lattices and required in modular non-distributive lattices

N_5 pentagon lattice prohibited in modular lattices

NTP	extension of non-tangential part in RT, something is a non-tangential part of another individual if their closures do not overlap
NTP^{-1}	inverse of the extension of non-tangential part in RT, i.e. if $\langle x,y\rangle \in NTP^{\mathcal{M}}$ then $\langle y,x\rangle \in \left(NTP^{-1}\right)^{\mathcal{M}}$ for any model $\mathcal{M}$ of RT
O	extension of overlap in RT
$\mathcal{O}$	set of open sets in a topological space
OP	extension of the unary predicate *open* in RT
$open(x)$	Boolean operation to test for open elements in RT_0
P	extension of parthood in RT
$\mathcal{P}(X)$	powerset of the set X
PO	extension of proper overlap, also used in RCC
$\langle x,y\rangle \in PO$	partial overlap defined as $\langle x,y\rangle \in PO \iff \langle x,y\rangle \in O \wedge \langle x,y\rangle \notin P \wedge \langle y,x\rangle \notin P$
PP	extension of proper parthood in a model of RT
$RT = \langle Y, f, [\![]\!]\rangle$	theory RT defined in terms of the intended models of RT_T
RT	full theory of mereotopology as defined by Asher and Vieu (RT_0)
RT_0	the axiomatic theory of RT
RT^-	theory defined by the axioms $A1$ to $A10$ and $A13$
RT_C	theory that is axiomatized by the set of axioms $A1$ to $A3$ (extensional ground topology)
RT^-_{EC}	theory defined by the axioms $A1$ to $A11$ and $A13$
RT_T	definition of an structure satisfying the conditions of Asher and Vieu's mereotopology over a topological space with topology T
T_0	separation axiom T_0 (*Kolmogorov*)
T_1	separation axiom T_1 (*Fréchet* or *accessible*), equivalent to T_0 and R_0 (*symmetric*)
T_2	separation axiom T_2 (*Hausdorff* or *separated*), implicitly implying T_0, T_1, and R_1 (*preregular*)
T_3	separation axiom T_3 (*Vietoris*), implicitly implying T_0 and *regular*
TP	extension of tangential part in RT, something is a tangential part of another individual if their closures overlap

TP^{-1} inverse of the extension of tangential part in RT, i.e. if $\langle x, y\rangle \in TP^{\mathcal{M}}$ then $\langle y, x\rangle \in \left(TP^{-1}\right)^{\mathcal{M}}$ for any model $\mathcal{M}$ of RT

$WCont$ extension of weak contact in RT

Terminology

ANTI-SYMMETRIC a binary relation R is antisymmetric if and only if for every tuple x, y, never both $\langle x, y\rangle \in R$ and $\langle y, x\rangle \in R$ hold

ATOM an element in a lower-bounded lattice that covers only $\bot = \emptyset$

ATOMLESS a theory that forces infinitesimal elements, i.e. each individual must have a proper part

BOTTOM ELEMENT the zero element $0 = \bot = \emptyset$ of a lattice

BOUNDARY the difference between an individual's closure and its interior

CARDINALITY LEXBFS same as CLBFS

CLOSURE the closure is the smallest closed set containing the individual itself (set-theoretic)

CONNECTION binary relation C of a mereotopology that expresses that two individuals are in contact

CONNECTION STRUCTURE structure equivalent to a model of $RT_C \cup \{A4\}$; can be regarded as contact algebra with universal element

CONTACT same as connection

CONTACT ALGEBRA structure equivalent to a model of RT_C, defined through various set of axioms as in [**DW06, DW05**]

DIAMOND special lattice M_3 containing four elements that is modular but not distributive

DUAL-ATOM an element d such that $d \prec 1$ in a bounded lattice

ELEMENT usually used for elements of a lattice, otherwise synonymous with *individual*

ELEMENTARY EQUIVALENCE two $\mathfrak{L}$-languages $\mathfrak{M}$ and $\mathfrak{N}$ are elementary equivalent if $\mathfrak{M} \models \Phi \iff \mathfrak{N} \models \Phi$

FOL first-order logic

FULL INTERIOR referring to condition (ii) of RT_0: no object in the n-dimension space can have "holes" of a lower dimension

INCOMPARABLE if in a lattice neither $x \geq y$ nor $y \geq x$ holds, x and y are incomparable

INDIVIDUAL element (region) of a mereotopological model that is represented by a point set

INFIMUM minimum (zero) element of a lattice

INTERIOR the smallest (open) neighborhood contained (set-theoretic) in the individual

IRREFLEXIVE a binary relation R is irreflexive if and only if for all $x, \langle x, x\rangle \notin R$

ISOMORPHISM a structure-preserving mapping between two objects; i.e. two lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic if there is a bijective function between the elements of $\mathcal{L}_1$ and $\mathcal{L}_2$ that preserves the partial order on the lattices; between different kind of objects (e.g. a mereotopological model and a lattice) it is important that there is a function f and its inverse f^{-1} that preserve the structure of either object when the function itself or the inverse is applied

JOIN the lowest upper bound of two elements in a lattice

JEPD jointly exhaustive, pairwise disjunct

CLBFS cardinality lexicographic breadth-first search algorithm for graphs, which uses the same technique as LexBFS but with an additional tie-breaking mechanism based on the cardinality of the vertices

LEXBFS lexicographic breadth-first search algorithm for graphs, see [**RTL76**]

MCS maximum cardinality search algorithm for graphs, see [**TY84**]

MEET the greatest lower bound of two elements in a lattice

MODULAR PAIR if $(a, b)M$, then a and b form a modular pair in a lattice

PARTIAL ORDER a binary relation, usually denoted by $\leq$, that is reflexive, antisymmetric, and transitive; if it irreflexive, it is explicitly noted

PARTIAL OVERLAP two elements of a model of RT partially overlap if neither is part of the other, but they share a common part

PENTAGON special lattice N_5 containing five elements that is not modular

POSET a set with a partial order defined on it

RCC Region Connection Calculus as developed by Cohn, Randell, et al.

REFLEXIVE a relation is reflexive if and only of for all x, $\langle x, x\rangle \in R$

REGULAR according to Cohn et al. [**CBGG97a**], a mereotopological model is regular if $Cl(Int(x)) = Cl(x)$ and $Cl(Int(x)) = Cl(x)$ hold for every individual x; this notion of regularity is identical with the conditions of *full interiors* and *smooth boundaries* imposed by (ii) and (iii) in RT_0; however, the topological representations of the models of both mereotopologies are not necessarily regular

REGULAR_CLOSED a set in a topological space is called regular closed if $A = Cl(Int(A))$

REGULAR OPEN a set in a topological space is called regular open if $A = Int(Cl(A))$

REGULAR SPACE topological space in which for a point in any open set O is contained in some closed subset of O; only if a regular space is Hausdorff, it is also a T_3-space

SEMI-REGULAR SPACE topological space whose regular open sets form a base of the space

SMOOTH BOUNDARIES condition (iii) of RT_0: every part of an object in an n-dimensional space must be n-dimensional as well

SUBLATTICE subset of the elements of the original lattice that is closed under join and meet

SUPREMUM top element $\top$ of a lattice s.t. for all element in the lattice $x \vee \top = \top$ holds

SYMMETRIC a relation R is symmetric if and only if for all x, y, $\langle x, y\rangle \in R$ implies $\langle y, x\rangle \in R$

TOP ELEMENT the one element $1 = \top$ of a lattice

TRANSITIVE a relation R is transitive if and only of for all x, y, $\langle x, z\rangle, \langle z, y\rangle \in R$ implies $\langle x, y\rangle \in R$

UNIVERSAL_ELEMENT the (unique) element in a mereotopology that everything else is connected to

UNIVERSE space (point set) covered by the universal element in a mereotopological model

WEAKLY REGULAR SPACE topological space (X, T) that is semiregular and for each non-empty set S_1 in T there exists a non-empty set S_2 in T so that $Cl(S_2) \subseteq S_1$ [**DW05**]

Classes of Lattices

ATOMIC every element x contains an atom, i.e. $\forall x \exists a\,[x \geq a]$ where a is an atom

ATOMISTIC each element is representable as the join of some set of atoms

BOOLEAN distributive complemented lattice; it is satisfies all stronger types of complementation (uniquely complemented, pseudocomplemented, orthocomplemented, and relatively complemented)

BOUNDED lattice with unique infimum $0 = \perp$ and supremum $1 = \top$ so that for all $a \in L$: $a \wedge 0 = 0$, $a \vee 0 = a$, $a \wedge 1 = a$, and $a \vee 1 = 1$

BROUWERIAN relatively pseudocomplemented lattice; subclass of the distributive pseudocomplemented lattices; usually it is assumed to have no zero element $\perp$, otherwise it is a Heyting lattice

COMPLEMENTED each element $a \in L$ has a complement $-a \in L$ so that $a \wedge -a = \perp$ and $a \vee -a = \top$

COMPLETE for every tuple of elements, there exists a join and meet; every finite lattice is complete

DISTRIBUTIVE (i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ is satisfied for all $a, b, c \in L$, or
(ii) the lattice contains no sublattice isomorphic to a pentagon or a diamond

DOUBLY PSEUDOCOMPLEMENTED join- and meet-pseudocomplemented lattice, i.e. for each element a there exists a meet-pseudocomplement $mpc(a)$ and a join-pseudocomplement $jpc(a)$ so that for all complements a' of a it holds that $mpc(a) \geq a'$ and $jpc(a) \leq a'$

DOUBLE P-LATTICE abbreviated name for a doubly pseudocomplemented lattice

GEOMETRIC (i) semimodular, algebraic lattice where the compact elements are exactly the finite joins of atoms
(ii) complete, atomistic, semimodular lattice where all atoms are compact

HEYTING Brouwerian lattice with a zero element; hence a subclass of the distributive pseudocomplemented lattices

INVOLUTION bounded lattice L together with an antitone mapping $^{\perp} : L \rightarrow L$ such that $a = a^{\perp\perp}$ for every $a \in L$

JOIN-SEMIDISTRIBUTIVE $d = a \vee b = a \vee c \rightarrow d = a \vee (b \wedge c)$ holds for any quadruple of elements $a, b, c, d \in L$

LOWER SEMIMODULAR $a \wedge b \prec a$ and $a \wedge b \prec b$ together imply $a \prec a \vee b$ and $b \prec a \vee b$ for any $a, b \in L$

MATROID different name for geometric lattices

MEET-SEMIDISTRIBUTIVE $d = a \wedge b = a \wedge c \rightarrow d = a \wedge (b \vee c)$ holds for any quadruple of elements $a, b, c, d \in L$

MODULAR (i) $a \geq c \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ is satisfied for all $a, b, c \in L$, or
(ii) $(a, b)M$ holds for any $a, b \in L$, or
(iii) the lattice contains no sublattice isomorphic to a pentagon

M-SYMMETRIC $(a, b)M$ implies $(b, a)M$ for any $a, b \in L$; finite upper semimodular lattices are M-symmetric lattices

NON-MODULAR used in the context of thesis to describe the non-semimodular, non-orthomodular, and not uniquely complemented lattices. In general applies only to the strong kind of modularity (see *modular*) found in lattices

NOT UNIQUELY COMPLEMENTED at least one element a in the lattice has two distinct complements a'_1 and a'_2 so that $a \wedge a'_1 = a \wedge a'_2 = \perp$ and $a \vee a'_1 = a \vee a'_2 = \top$

ORTHOCOMPLEMENTED involution lattice $(L, ^\perp)$ in which the involution is an orthocomplementation in the sense that $a \wedge a^\perp = 0$ for every $a \in L$

ORTHOLATTICE abbreviated name of an orthocomplemented lattice

ORTHOMODULAR ortholattice in which the orthomodular identity $a \leq b \rightarrow b = a \vee (b \wedge x^\perp)$ holds for all $a, b \in L$

$\perp$-SYMMETRIC $a \vee b = \perp$ and $(a, b)M$ together imply $(b, a)M$ for all $a, b \in L$

P-LATTICE abbreviated name for a doubly pseudocomplemented lattice

P-ORTHOLATTICE abbreviated name for a (doubly) pseudocomplemented and orthocomplemented lattice

PSEUDOCOMPLEMENTED short for meet-pseudocomplemented lattice, i.e. for each element a in the lattice there exists a meet-pseudocomplement $mpc(a)$ so that $mpc(a) \geq a'$ for all complements a' of a

RELATIVELY COMPLEMENTED every interval of the lattice is a complemented sublattice, or equivalently every element has a relative complement in any interval containing it

SECTION-COMPLEMENTED lattice with infimum $\perp$ so that for every $a \in L$, the interval $[\perp, a]$ is complemented

SECTION-SEMICOMPLEMENTED every interval of the lattice is a semicomplemented sublattice

SEMICOMPLEMENTED every element $a \in L$ with $a \neq \top$ has a semicomplement $b \neq \bot$ so that $a \wedge b = \bot$

SEMIDISTRIBUTIVE meet- and join-semidistributive lattice

SEMIMODULAR short name for upper semimodular lattice

STONE distributive pseudocomplemented lattices in which $mpc(a) = mpc(mpc(a))$ is satisfied for all $a \in L$, or equivalently in which $mpc(a \wedge b) = mpc(a) \vee mpc(b)$ holds for all tuples $a, b \in L$

UNICOMPLEMENTED abbreviated name for a uniquely complemented lattice

UNIQUELY COMPLEMENTED every element a in the lattice has a unique complement, i.e. $a \wedge a'_1 = a \wedge a'_2 = \bot$ and $a \vee a'_1 = a \vee a'_2 = \top$ imply $a'_1 = a'_2$

UPPER SEMIMODULAR $a \wedge b \prec a \rightarrow b \prec a \vee b$ holds for any tuple of elements $a, b \in L$

WEAKLY MODULAR $a \vee b \neq \bot$ implies $(a, b)M$ for all $a, b \in L$

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