Multidimensional Mereotopology with Betweenness

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Outline

- Motivating Examples
- The Basic Multidimensional Mereotopology
- The Multidimensional Mereotopology with Betweenness
- Some Relationships to Other Theories of Qualitative Space

Goal: Theory for Qualitative Maps

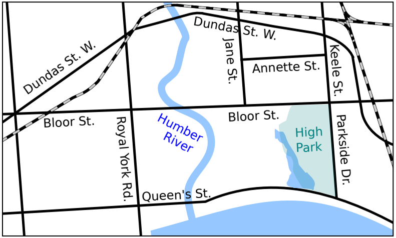
Qualitative representations of maps (cities, buildings, events)

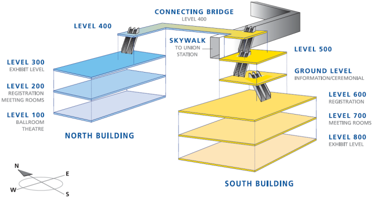
- Extract easily memorable directions from digital maps

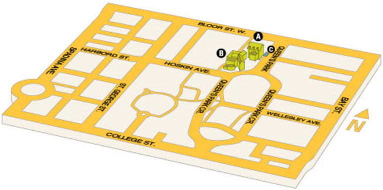
  At the next intersection turn right into Main St. After you cross the river turn left, go through the entire park and then you will see the hotel on your left.

- On-demand navigation in buildings or on event sites (in emergencies such as fires or earthquakes when some routes may be blocked)
- Navigation without GPS (e.g. using standard cell phones)
- Compact map representations (e.g. for low-bandwidth networks)

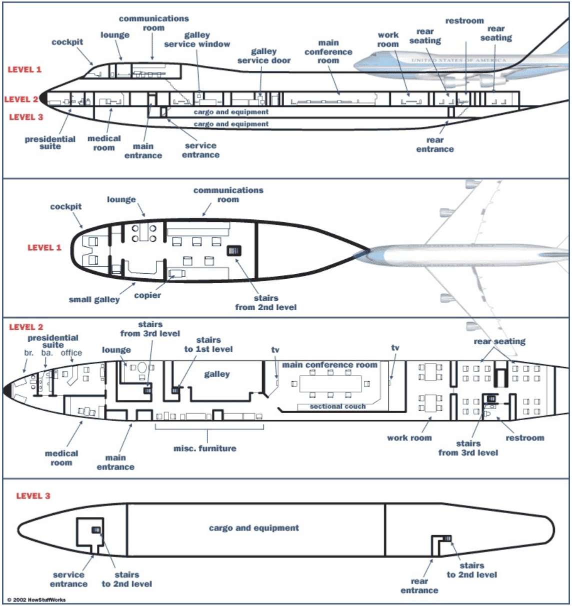
Multidimensional Qualitative Maps: Some Examples







Multidimensional Qualitative Maps: Air Force One



Qualitative Multidimensional Space

Mereological and topological relations fundamental for most qualitative representations of space.

Start: Design a weak spatial theory that:

- can define mereotopological relations of contact and parthood;
- allows models with entities of various dimensions;
- but is not restricted to a fixed set of dimensions (2D, 3D, or 4D);
- **Later:** Include a notion of order (betweenness).

Our choice: Unsorted hybrid approach

All entities (independent of their dimension) are first-class domain objects

Gotts 1996: INCH calculus good start though rather unintuitive primitive

Key Concept: Dimension

- Usually defined quite complicated in terms of other relations:
    - Vector space dimension (independent set of vectors necessary to represent any vector)
    - Inductive dimension (recursive definition: boundaries have exactly one dimension lower; points have dimension 0)
    - Hausdorff dimension (real numbers, e.g. for fractals),
    - Minkowski (box-counting) dimension
    - Lebesgue covering dimension (minimum number of overlapping open sets in some refinement for every open cover)
- Humans easily recognize (idealized) objects of various dimensions
- Many spatial relations have implicit dimension constraints
- **Idea: Why not take it as primitive?**

Relative Dimension as Primitive

Axiomatization of relative dimension using $<_{dim}$ as primitive:

- $<_{dim}$ ... strict partial order (irreflexive, asymmetric, transitive) [more details in our paper that appeared at CommonSense 2011]
- Bounded: some minimal and maximal dimension
- Discrete: next highest and next lowest dimension exist: $\prec_{dim}$, $\succ_{dim}$
- Linear: any two entities are dimensionally comparable

$\Rightarrow$ Similar to inductive dimension

$ZEX(x)$ ... unique zero entity of lowest dimension

No commitment about its existence, though two extensions feasible:

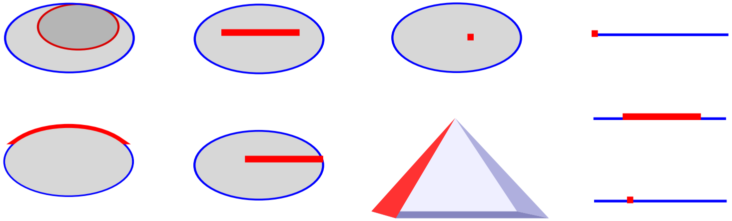
- With unique zero entity: $\exists x\ ZEX(x)$
- Without zero entity: $\forall x\ \neg ZEX(x)$

Containment as Spatial Primitive

Dimension-independent spatial relations: Containment (Mereological)

- *Cont* ... non-strict partial order (reflexive, antisymmetric, transitive)
- Interaction with relative dimension: $Cont(x, y) \rightarrow x \leq_{dim} y$ (CD-A1)

Intended point-set interpretation:
$Cont(x, y)$ iff every point in space occupied by x is also occupied by y



Parthood and Contact As Definable Spatial Relations

Equidimensional Parthood (Mereological)

- Equidimensional version of containment = *Parthood* is definable:
- $P(x, y) \leftrightarrow Cont(x, y) \wedge x =_{dim} y$ (EP-D)

Contact (Topological)

- Definable: $C(x, y) \leftrightarrow \exists z(Cont(z, x) \wedge Cont(z, y))$ (C-D)
- $\forall z(C(z, x) \rightarrow C(z, y)) \rightarrow Cont(x, y)$ (C-A5: C monotone implies *Cont*)

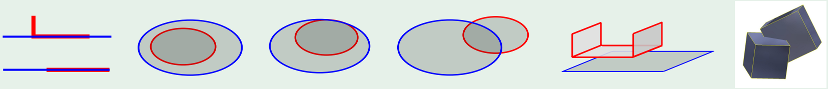
Classify contact $C(x, y)$ by the relative dimension of:

- x,y, and the entities contained in $x \cdot y$.

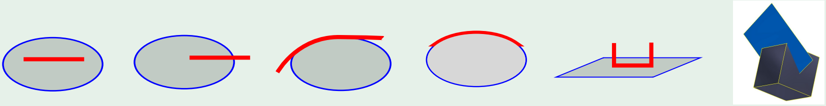
$\rightarrow$ **complete classification by three types of contact.**

Three Types of Contact

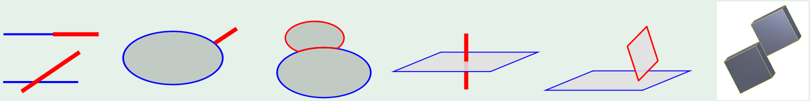
Strong Contact: **(Partial) Overlap** $x \Leftrightarrow x =_{dim} x \cdot y =_{dim} y$



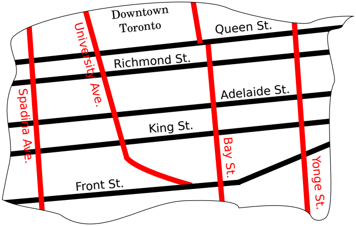
Strong Contact: **Incidence** $\Leftrightarrow x =_{dim} x \cdot y <_{dim} y$ or vice versa



Weak Contact: **Superficial Contact** $\Leftrightarrow x >_{dim} x \cdot y <_{dim} y$



What is Missing for Adequate Representations of Maps?



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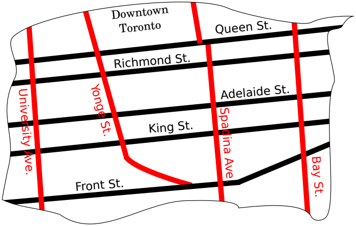

?

So far, we can place disconnected entities arbitrarily as long as contact and containment are preserved;
e.g. we can permute ‘parallel’ streets such as all vertical streets.

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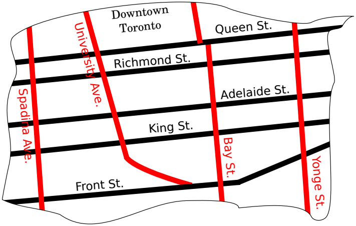

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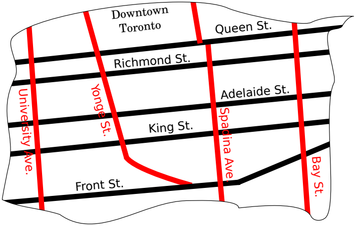

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So far, we can place disconnected entities arbitrarily as long as contact and containment are preserved;
e.g. we can permute 'parallel' streets such as all vertical streets.

$\Rightarrow$ **Require some notion of order**

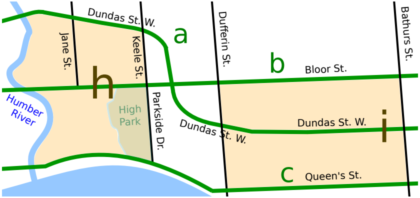
- works independent of concrete dimensions

Multidimensional Betweenness as Order Primitive

Our choice: **Multidimensional Quaternary Betweenness**

$Btw(r, a, b, c)$ ... within r, b is strictly in between a and c

- $Cont(a, r) \wedge Cont(b, r) \wedge Cont(c, r)$
- $r \succ_{dim} a =_{dim} b =_{dim} c$
- For any fixed $r = R$, $Btw(R, a, b, c)$ is a standard strict betweenness relation (acyclic, irreflexive, outer-symmetric, transitive) but without the orderability axiom [cf. Huntington & Kline 1917]



$Btw(h, a, b, c)$ and $Btw(i, b, a, c)$ – a, b, c are *not* totally orderable in $(\mathrm{h} + \mathrm{i})$.

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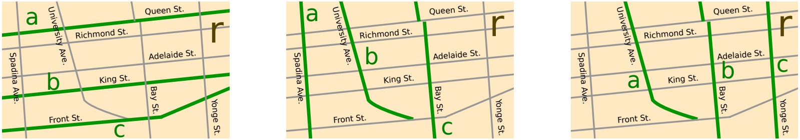


Intended point-set interpretation (similar to Jordan Curve Theorem):
$Btw(r, a, b, c)$ iff within r, every entity connecting a and c intersects b

Theoretical Results

Explore the Relationships to Existing Spatial Theories

Method: show that extensions of our theories are interpreted by the existing spatial theories.

- The extension axioms show what aspects our theories 'abstract away'
- The basic mereotopology T_{ldc} generalizes other mereopologies and incidence geometries
- The mereotopology with betweenness T_{bmt} generalizes ordered incidence geometries

The Hierarchy of Spatial Theories: Our basic MT T_{ldc}

Mereotopologies that interpret extensions of T_{ldc}

Equidimensional: Region Connection Calculus [RCC: Randell et al. 1992]

Multidimensional: INCH Calculus [Gotts 1996]

Needs strengthening in only two aspects:

- Boolean closure operations (sums, complements)
- Extensionality (under what conditions are entities identical)

Incidence structures and geometries interpret extensions of T_{ldc}

- Define points, lines, planes, etc. as classes of uniform dimension that are all maximal entities (not part of anything else)
- *Inc* is then a incidence relation in the traditional sense
- Extends to incidence geometry as shown for bipartite geometries: near-linear, linear, and affine spaces

The Hierarchy of Spatial Theories: MT with Btw. T_{bmt}

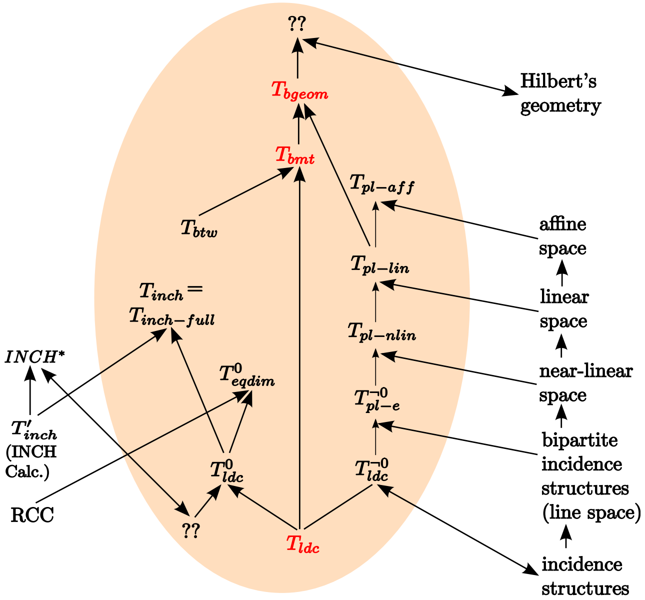
Ordered geometries interpret extensions of T_{bmt}

- Same construction as for the incidence geometries
- Need to strengthen the betweenness relation: **Totally orderable**
- Other axioms added as necessary (continuity axiom, Pasch axiom, Dedekind axiom) to obtain standard geometries such as:
  - betweenness geometry [Hashimoto 1958]
  - ordered incidence geometry
  - ordered affine geometry
  - Hilbert's geometry (incidence, order, congruence, continuity)
  - Euclidean geometry

$\Rightarrow$ T_{bmt} **Qualitatively Abstracts Ordered Incidence Geometry**

- No metric (distances, angles, ...) or congruence
- May violate near-linear axiom: Two points not always on a line

The Hierarchy of Spatial Theories: A Summary



Summary: A Theory of Qualitative Maps

Pragmatic approach: Not a single theory, but a ‘family of theories’

- Methodology: Start with a weak theory and strengthen successively as necessary (bottom-up modular design)
- Result: Hierarchy of qualitative spatial theories
- **Side effect: Semantic integration of spatial theories**

More concrete contributions: Semantic integration

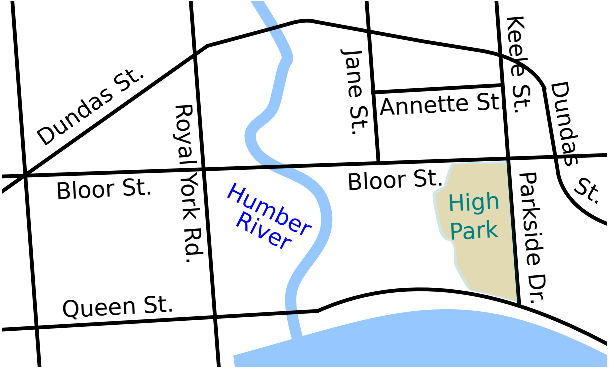
- Relationship to known mereotopologies (RCC, INCH-Calculus)
- Relationship to incidence structures and geometries
- Relationship to ordered incidence geometry

Outlook

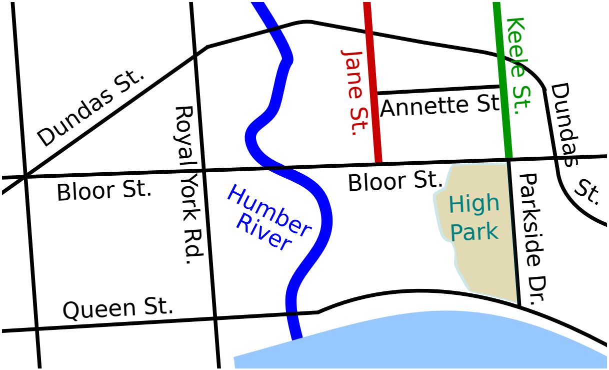
Work in progress:

- Full representation theorem for INCH Calculus [Gotts 1996]
- Distinguish interior from boundary and explore interpretations by [Egenhofer & Herring 1991, Clementini et al. 1993, McKenney et al. 2005]
- Disambiguate various interpretations of betweenness (separates, encloses, partially between, ...)

Other Usages of ‘Betweenness’



Other Usages of 'Betweenness' - I

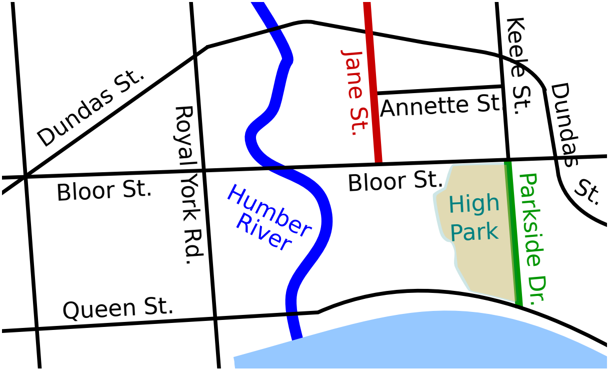


Strong Btw: Jane between Humber River and Keele?

$$\Longleftrightarrow$$

'Jane is in between the Humber River and Keele on every linear feature connecting all three'

Other Usages of ‘Betweenness’ - II

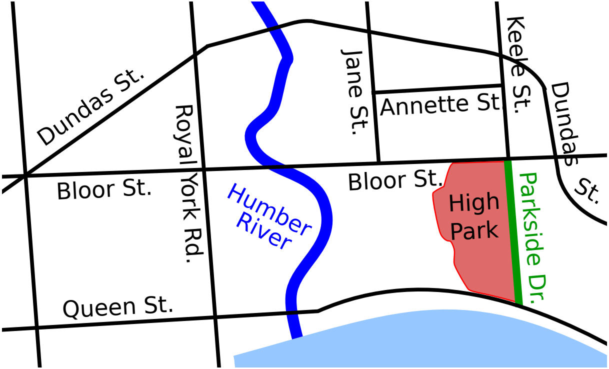


Weak Btw: Jane between Humber River and Parkside?

$$\Longleftrightarrow$$

‘Jane is in between the Humber River and Parkside on some linear feature connecting all three and on no linear feature they are ordered differently’

Other Usages of 'Betweenness' - III

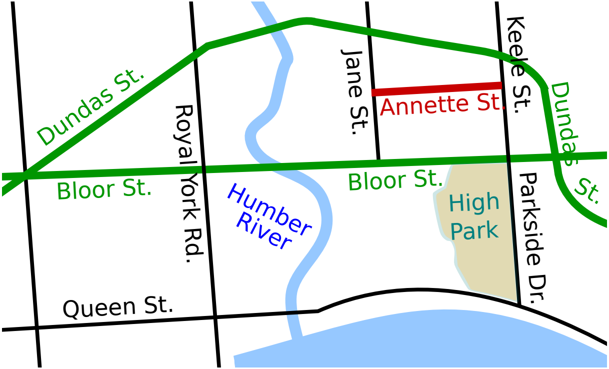


Higher-dim. Btw: High Park between Humber River and Parkside?

$$\Longleftrightarrow$$

'The boundary of High Park is strongly/weakly in between the Humber River and Parkside'

Other Usages of 'Betweenness' - IV

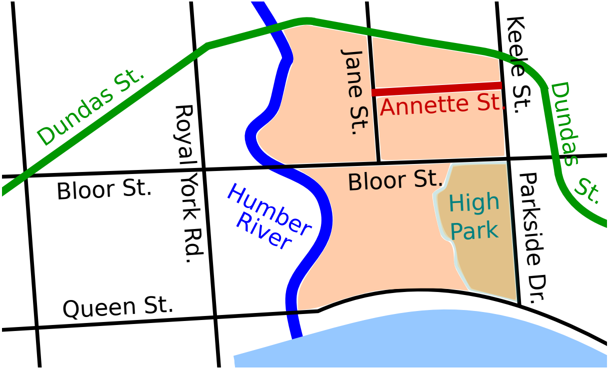


Enclosure: Dundas and Bloor enclose Annette

$$\Longleftrightarrow$$

'The region completely bounded by Dundas and Bloor contains Annette'

Other Usages of ‘Betweenness’ - V



Partial Enclosure: Humber River and Dundas partially enclose Annette

$$\Longleftrightarrow$$

‘A region partially bounded by Dundas and Bloor contains Annette’

Outlook

Future Challenge: Explore applicability

- Domains for which the current qualitative representation is sufficient
- Domains for which we need additional primitive relations
  What is not preserved, but necessary, e.g. for navigation tasks?
  - Order over intersections?

If you know of interesting applications, test scenarios, or queries;
I want to know: `torsten@cs.toronto.edu`

Poster: tomorrow 10:30 to 12:10, Rm. 124/125