A Naïve Theory of Dimension for Qualitative Spatial Relations

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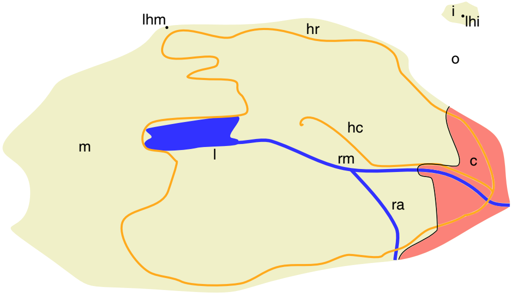
Multidimensional Qualitative Space

Qualitative spatial reasoning is a promising approach for more human-like reasoning about space.

Often requires coping with idealized features of various dimensions:



2D: **o**cean, **m**ain island, **s**mall island, **c**ity, **l**ake;
1D: **r**iver (**m**ain), **r**iver **a**rm, **h**ighway (**r**ing), **h**ighway **c**entral;
0D: **l**ight**h**ouse (**m**ain island), **l**ight**h**ouse (**s**mall island)

Multidimensional Qualitative Space

Mereological and topological relations lie at the heart of most qualitative representations of space.

Want to design a multidimensional mereotopology that:

- allows models with entities of multiple dimensions;
- defines an intuitive set of spatial relations,
- but is as general (as **weak**) as possible: *dimension-independent*.

→ Basis for a more expressive, but still intuitive logical theory

→ Basis for 'next-generation qualitative spatial reasoning'

→ Semantically integrate the large variety of spatial theories including mereotopologies and geometries

How to design such a theory?

Main decisions

- Representation language:
- How to treat the entities of the various dimension?
- What primitives to choose?

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- Representation language: **first-order logic**
  - Capable of integrating the various existing spatial theories
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How to Axiomatize Multidimensional Space?

Point-based:	Region-based:
points as only domain objects	a class of higher-dimensional entities as only domain objects
all others defined as sets thereof	lower-dimensional entities as approximations thereof (ultrafilters)

Our choice: **Hybrid approach**

All entities (independent of their dimension) are first-class domain objects

- Sorted or non-sorted → prefer **unsorted**: one basic theory that can be restricted to certain number of dimensions as desired

Gotts 1996: INCH calculus good start, but never fully developed, rather unintuitive primitive

Galton 1996, 2004 focus on boundaries as lower-dimensional entities, dimension defined in terms of boundaries

Key Concept: Dimension

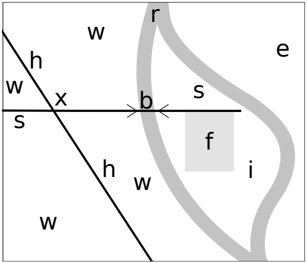
- Humans easily recognize (idealized) objects of various dimensions
- Many spatial relations have implicit dimension constraints
- Usually defined, but not easy:
  - Vector space dimension (independent set of vectors necessary to represent any vector)
  - Inductive dimension (recursive definition: boundaries have exactly one dimension lower; points have dimension 0)
  - Hausdorff dimension (real numbers, e.g. for fractals),
  - Minkowski (box-counting) dimension
  - Lebesgue covering dimension (minimum number of overlapping open sets in some refinement for every open cover)
- **Idea: Why not take it as primitive?**

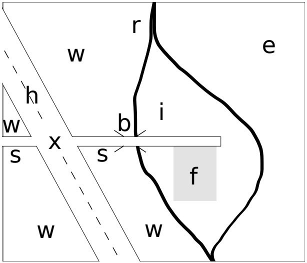
Dimension does not need to be uniquely determined

Different relations imply different dimension constraints:

- The river r contains an island i. $\dim(I) \leq \dim(R)$
- The island i is bounded by the river r. $\dim(R) \leq \dim(I)$

$\rightarrow$ Allow different interpretations of the same description:





Intended objects: regular closed regions of space of **uniform dimension** (no lower-dimensional entities missing or added)

Relative Dimension as Primitive

Axiomatization of relative dimension using $=_{dim}$, $<_{dim}$ as primitives:

- $=_{dim}$ ... equivalence relation (reflexive, symmetric, transitive)
- $<_{dim}$ ... strict partial order (irreflexive, asymmetric, transitive)
- $=_{dim}$ and $<_{dim}$ incompatible: $x <_{dim} y \rightarrow \neg x =_{dim} y$
- $ZEX(x)$ ... possible unique zero entity of lowest dimension
- some unique lowest dimension (apart from ZEX)

$\rightarrow$ Similar to inductive dimension

- incomparable dimensions possible: $x \not<_{dim} y \wedge y \not<_{dim} x \wedge x \neq_{dim} y$

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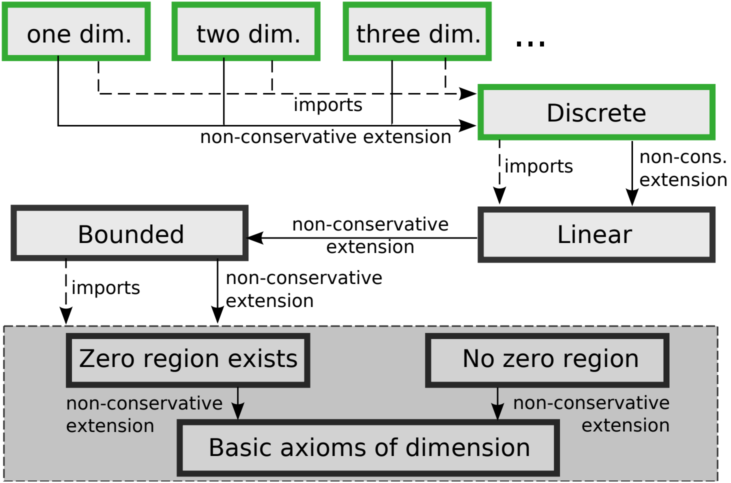
Two extensions:

- Boundedness: some maximum dimension
- Linear dimension: Any two entities are dimensionally comparable
- Discrete: next highest/lowest dimension exists

The modules of the theory of relative dimension



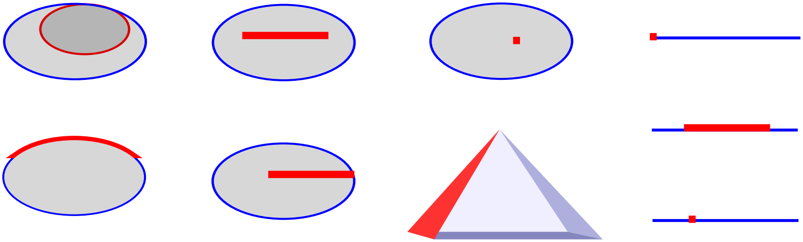
Containment as Spatial Primitive

Dimension-independent spatial relations: Containment (Mereological)

- *Cont* ... non-strict partial order (reflexive, antisymmetric, transitive)
- Interaction with relative dimension: $Cont(x, y) \rightarrow x \leq_{dim} y$

Point-set interpretation:

$Cont(x, y)$ iff every point in space occupied by x is also occupied by y



Contact As Definable Spatial Relation

Contact (Topological)

- Definable: $C(x, y) \leftrightarrow \exists z(Cont(z, x) \wedge Cont(z, y))$
- $\forall z(C(z, x) \rightarrow C(z, y)) \rightarrow Cont(x, y)$ (C monotone implies $Cont$)

Classify contact $C(x, y)$ by the relative dimension of:

- x,y, and the entities contained in $X \cdot y$.

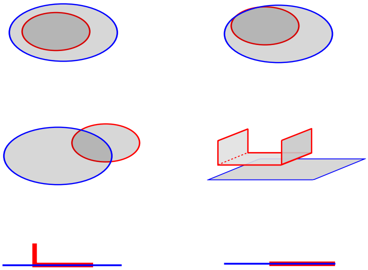
$\rightarrow$ **complete classification by three types of contact.**

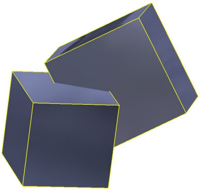
- Need equidimensional version of containment = *parthood*:
  $P(x, y) \leftrightarrow Cont(x, y) \wedge x =_{dim} y$

Three Types of Contact

Strong Contact: (Partial) Overlap

- $x =_{dim} x \cdot y =_{dim} y$
- $PO(x, y)$ ... x and y have a part in common

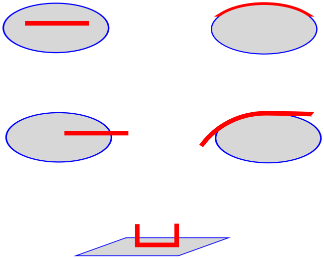


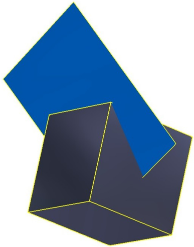


Three Types of Contact

Strong Contact: Incidence

- $x =_{dim} x \cdot y <_{dim} y$ (or vice versa)
- $Inc(x, y)$ ... x and y only share an entity that is a part of one of them

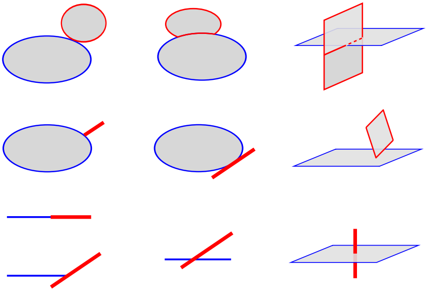


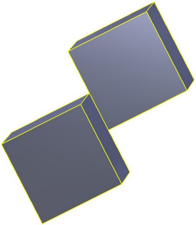


Three Types of Contact

Weak Contact: Superficial Contact

- $x >_{dim} x \cdot y <_{dim} y$
- $SC(x, y)$ ... x and y share an entity that is neither a part of x, y





Design Choices: Summary

The theories of relative dimension and containment can distinguish three types of contact; these completely classify of contact.

Main decisions

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  - **Relative dimension** $<_{dim}$ **and** $=_{dim}$ **and containment** *Cont*

Evaluation: How Adequate is the Axiomatization?

- *First step*: prove consistency and non-trivial consistency (consistency of all relations having a non-empty extension)
- Is the axiomatization sufficiently strong?
- Is the axiomatization sufficiently weak?

Evaluate using relative interpretations to known theories:

- Region Connection Calculus (Randell et al., 1992)
- INCH Calculus (Gotts 1996)

Based on the linear, bounded extension of dimension (with *ZEX*):

Needs strengthening in only two aspects:

- Boolean closure operations (sums, complements)
- Extensionality (under what conditions are entities identical)

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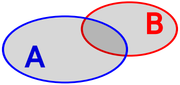
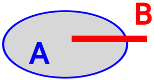
Extension interpretable by the INCH Calculus

Single primitive: $INCH(x, y)$ ... 'x includes a chunk of y' where a chunk is an equidimensional part



INCH(A,B) &
INCH(B,A)



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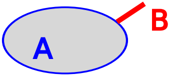

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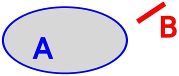

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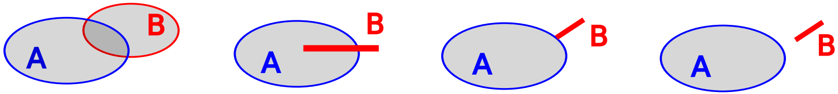

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INCH(A,B) & ¬INCH(B,A)

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1. implicit dimension constraint: $x \geq_{dim} y$
2. implicit contact relation: there exists something contained in both

We can define: $INCH(x, y) \leftrightarrow \exists z(Cont(z, x) \wedge Cont(z, y) \wedge z =_{dim} y)$

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- Shorter: $INCH(x, y) \leftrightarrow \exists z(Cont(z, x) \wedge P(z, y))$

All axioms of the INCH Calculus become provable except:

- Extensionality of INCH
- Boolean closure (we do not force closures)
- $CH(x, y) \rightarrow CS(x, y)$ (*translated:* $P(x, y) \rightarrow Cont(x, y)$)

$\rightarrow$ Every model of our theory that satisfies these additional axioms can be faithfully extended to a model of the INCH calculus

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Not every model of the INCH calculus can be faithfully extended to a model of our theory

Extension interpretable by the Region Connection Calculus

Idea: Select a set of equidimensional entities from a model

Choice: All entities of maximal dimension

The entities of the maximal dimension in a model of our theory form a model of the RCC if the following axioms are satisfied:

- Extensionality of C amongst highest dimension entities
- Boolean closure and connectedness of complements
- Infinite divisibility

$\rightarrow$ Every model of our theory that satisfies these additional axioms definably interprets some model of the RCC

Can all RCC models be extended to definably equivalent models of this extension of our theory?

Conclusion & Future Work

Elegant, minimal theory sufficient to define basic topological relations in multidimensional space

- Can be consistently extended to capture only models of the INCH Calculus or the RCC

Still very rudimentary – Useful extensions:

Boundaries: How are things connected? (boundary, interior, tangential)

Other dimension-independent relations: Betweenness

Weak closures already closed under intersections – is this sufficient? Want to avoid exponential explosion in number of entities forced by complements and sums

Applicability needs to be explored:

- E.g. revisit MapSee: can multidimensional qualitative space interpret sketch maps as in (Reiter & Mackworth 1989)?
- Will verify whether the theory is **strong enough** to be useful