On the Skeleton of Stonian p-Ortholattices

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Setting: Mereotopologies

- Mereotopology: qualitative representation of space based on regions instead of points as primitives (point-free topology)
- Whiteheadean assumption: only equi-dimensional regions
- Mereology: binary parthood relation P
  - reflexive, anti-symmetric, and transitive
- Topology: connection (or contact or proximity) relation C
  - reflexive, symmetric
- Monotonicity: $P(x, y) \Rightarrow \forall z(C(x, z) \Rightarrow C(y, z))$

Different axiomatizations of Whiteheadean mereotopology

Region-Connection Calculus

- All regions are regular closed, i.e. $x = \mathrm{cl}(x) = \mathrm{cl}(\mathrm{int}(x))$
- Distributivity of set-theoretic union and intersection are preserved
- Models are atomless

RT: a mereotopology with the open-closed distinction

- All regions are regular , i.e. $\mathrm{cl}(x) = \mathrm{cl}(\mathrm{int}(x))$ and $\mathrm{int}(x) = \mathrm{int}(\mathrm{cl}(x))$
- Set-theoretic complementation is preserved
- Models can be atomistic

Motivating question

- How are these axiomatizations related?
- *More precise:* Does there exist a mapping between the models of the two theories?
- Approach: use the algebraic representation of the theories to obtain a mapping between them

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Next steps:

1. Review Boolean contact algebras as representation of the RCC
2. Review Stonian p-ortholattices as representation of RT^{-}

Contact algebras

$\cong$ Algebraic representation of Whiteheadean mereotopology
(usually logical axiomatizations are used)

Definition

A contact algebra is a structure (A, C) consisting of a bounded lattice $A = \langle A; 0; 1; +; \cdot \rangle$ and a binary connection relation C satisfying at least (C0)-(C3).

In a CA, the partial order $\leq$ of the lattice defines the parthood relation P.

Boolean contact algebras (BCAs)

Definition

A Boolean contact algebra (BCA) is a contact algebra (A, C) where A is a Boolean algebra and C satisfies (C0)-(C4).

(C0) $\neg C(0, x)$ (Null disconnectedness)

(C1) $x \neq 0 \Rightarrow C(x, x)$ (Reflexivity)

(C2) $C(x, y) \iff C(y, x)$ (Symmetry)

(C3) $C(x, y) \wedge y \leq z \Rightarrow C(x, z)$ (Monotonicity)

(C4) $C(x, y + z) \Rightarrow C(x, y) \vee C(x, z)$ (Topological sum)

- BCAs are a generalization of the RCC axiomatization

BCAs as representation of the RCC

Additional axioms for BCAs:

(C5)	$\forall z(C(x,z) \Leftrightarrow C(y,z)) \iff x = y$	(Extensionality)
(C5')	$x \neq 0 \Rightarrow \exists y(y \neq 0 \wedge \neg C(x,y))$	(Disconnection)
(C6)	$C(x,z) \vee C(y,z') \Rightarrow C(x,y)$	(Interpolation)
(C6')	$\neg C(x,y) \Rightarrow \exists z(\neg C(x,z) \wedge \neg C(y,z')$	
(C7)	$(x \neq 0 \wedge x \neq 1) \Rightarrow C(x,x')$	(Connection)

Theorem (Stell, 2000; Düntsch & Winter, 2004[a])

Models of the (strict) RCC correspond to BCAs satisfying (C5) and (C7).

[a] Düntsch, I. & Winter, M.: Algebraization and Representation of Mereotopological Structures. In JoRMiCS 1, 161–180, 2004.

Topological representation of BCAs

Standard topological models of BCAs

The regular closed sets $\mathrm{RC}(X) = \{a \subseteq X \mid a = \mathrm{cl}(a) = \mathrm{cl}(\mathrm{int}(a))\}$ of a topological space $\langle X, \tau \rangle$ with the following operations:

$$
\begin{aligned}
x + y &:= x \cup y \\
x \cdot y &:= \mathrm{cl}(\mathrm{int}(x \cap y)) \\
x^* &:= \mathrm{cl}(X \setminus x)
\end{aligned}
$$

Theorem (Dimov & Vakarelov, 2006)

For each Boolean contact algebra $\langle B, C \rangle$ there exists an embedding $h : B \to \mathrm{RC}(X)$ into the Boolean algebra of regular closed sets of a topological space $\langle X, \tau \rangle$ with $C(a, b)$ iff $h(a) \cap h(b) \neq \emptyset$. h is an isomorphism if B is complete.

Representation Theory for RT^-

Theorem (Hahmann et al., 2009)

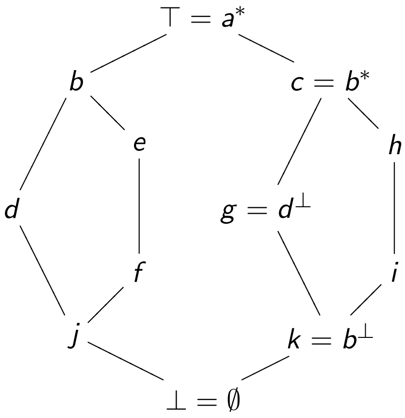
Let U be a model of RT^-.
Then $\langle U \cup \{0\}, +, \cdot, ^*, ^\perp, 0, 1\rangle$ is a Stonian p-ortholattice.

Conversely, let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a Stonian p-ortholattice.
Then $L^+ = \{x \in L \mid x \neq 0\}$ with the relation $C(x, y) \iff x \nleq y^\perp$ is a model of RT^-.

Stonian p-ortholattices (SPOLs):

- Double p-algebra (pseudocomplemented & quasicomplemented)
- Orthocomplemented
- Satisfies Stone identity (De Morgan laws for pseudocomplementation)

Pseudo- & Orthocomplementation



Example of a p-ortholattice

Consider bounded lattices

Pseudocomplement b^* of b is
largest element b^* s.t. $b \wedge b^* = \bot$

E.g. $\{b, d, e, f, j\}^* = c$

Orthocomplement $b^\perp$ of b is

(1) Complement:
$b \wedge b^\perp = \bot$ and $b \vee b^\perp = \top$

(2) Involution: $b = b^{\perp\perp}$

(3) Order-reversing:
$d \leq b \iff d^\perp \geq b^\perp$

E.g. $b^\perp = k$, $k^\perp = b$, $e^\perp = i$

Stone identity for p-ortholattices

Definition

A p-ortholattice $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ is called Stonian iff for all $x, y \in L$, $(x \cdot y)^* = x^* + y^*$.

Theorem

Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a p-ortholattice. Then the following statements are equivalent:

1. $(x \cdot y)^* = x^* + y^*$ *for all* $x, y \in L$;
2. $(x + y)^+ = x^+ \cdot y^+$ *for all* $x, y \in L$;
3. $(x \cdot y)^{++} = x^{++} \cdot y^{++}$ *for all* $x, y \in L$;
4. $(x + y)^{**} = x^{**} + y^{**}$ *for all* $x, y \in L$;
5. *the skeleton* $S(L)$ *is a subalgebra of* L.
6. *the dual skeleton* $\bar{S}(L)$ *is a subalgebra of* L.

Partial topological representation of SPOLs

Theorem (Standard topological models of RT^-)

Let $\langle X, \tau \rangle$ be a topological space.
Let $\mathrm{RT}(X) = \{a \subseteq X \mid \mathrm{int}(a) = \mathrm{int}(\mathrm{cl}(a)) \wedge \mathrm{cl}(a) = \mathrm{cl}(\mathrm{int}(a))\}$ be the regular sets of X and define the following operations:

$$\begin{aligned} x \cdot y &:= x \cap^* y = \quad x \cap y \cap \mathrm{cl}(\mathrm{int}(x \cap y)), \\ x + y &:= x \cup^* y = \quad x \cup y \cup \mathrm{int}(\mathrm{cl}(x \cup y)), \\ x^* &:= \mathrm{cl}(X \setminus x), \\ x^\perp &:= X \setminus x. \end{aligned}$$

Then $\langle \mathrm{RT}(X), \cup^*, \cap^*, {}^*, {}^\perp, \emptyset, X \rangle$ is a Stonian p-ortholattice.

- An topological embedding theorem for Stonian p-ortholattices is still outstanding (current work)

Key to the mapping

Skeleton

$S(L) = \{a^* | a \in L\}$ is the **skeleton** of a pseudocomplemented semilattice $\langle L, \cdot, ^*, 0\rangle$.

- maintains the order relation of L
- meet $a \wedge b = a \cdot b$ and union $a \vee b = (a^* \cdot b^*)^*$
- (Glivenko-Frink Theorem) $S(L)$ is Boolean

Corollary

If $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ is a Stonian p-ortholattice, then the skeleton $S(L)$ is a Boolean subalgebra of L.
In fact, the dual $\bar{S}(L) = \{a^+ | a \in L\}$ is also a Boolean subalgebra of L.

The relationship between BCAs and SPOLs

Theorem (SPOLs $\Rightarrow$ BCAs)

Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a Stonian p-ortholattice, then $S(L)$ together with $C(a, b) \iff a \not\leq b^\perp$ is a Boolean contact algebra.

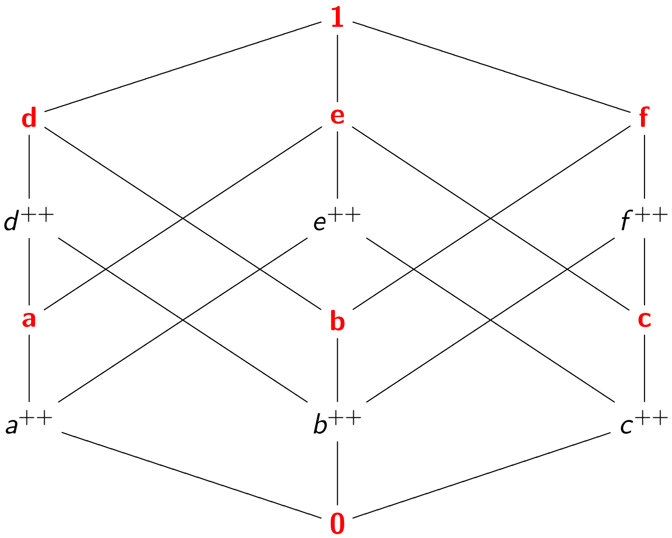
- Every Stonian p-ortholattice has a unique Boolean skeleton $S(L)$

Theorem (BCAs $\Rightarrow$ SPOLs)

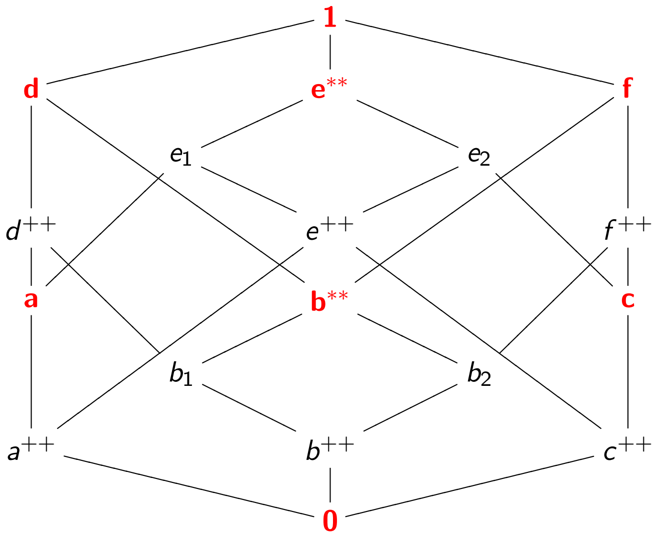
Let $\langle B, C\rangle$ be an arbitrary BCA. Then there is a Stonian p-ortholattice $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ so that the skeleton $S(L)$ is isomorphic to $\langle B, C\rangle$.

- Every BCA can be extended to infinitely many Stonian p-ortholattices

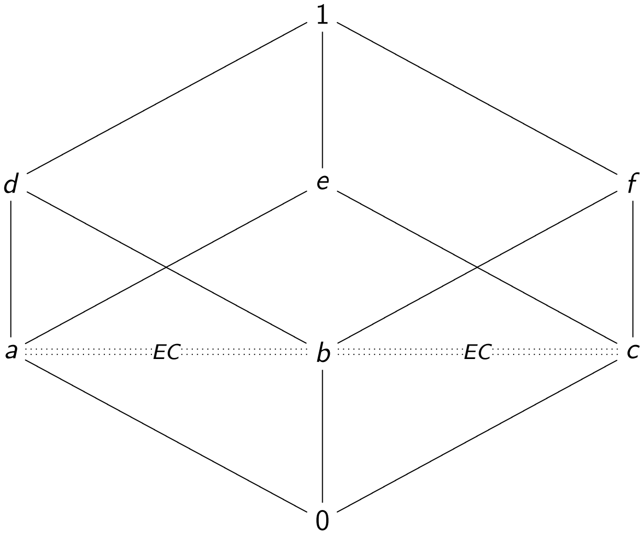
Example: The Stonian p-ortholattice C_{14}



Example: The Stonian p-ortholattice C_{18}



Example: A Boolean contact algebra



Preserving additional properties

$$\text{(C5')}\quad x \neq 0 \Rightarrow \exists y(y \neq 0 \wedge \neg C(x,y)) \qquad \text{(Disconnection)}$$
$$\text{(C6)}\quad C(x,z) \vee C(y,z') \Rightarrow C(x,y) \qquad \text{(Interpolation)}$$
$$\text{(C7)}\quad (x \neq 0 \wedge x \neq 1) \Rightarrow C(x,x') \qquad \text{(Connection)}$$

Lemma

Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a Stonian p-ortholattice and $\langle S(L), C\rangle$ its skeleton BCA. Then we have:

1. *$S(L)$ is dense in L iff C satisfies (C5').*
   - $(\forall a \in L > 0)(\exists b \in S(L) > 0) b \leq a$
2. *L is $*$-normal iff C satisfies (C6).*
   - *$\forall a, b \in L$ with $a^{**} \leq b^+$, there exists a $c \in L$ s.t. $a^{**} \leq c^{++}$ and $b^{**} \leq c^+$*
3. *L is connected iff C satisfies (C7).*
   - *0, 1 are the only clopen elements*

Conclusion

Relationship between RCC and RT^-

- Every connected model of RT^- with a dense skeleton has – in the skeleton – a corresponding model of the full RCC
  - If the RT^- model is *-normal, the RCC model also satisfies C6
- The skeleton of a RT^- model is a model of RCC $\setminus \{C5, C7\}$
- *Trivial:* every Boolean algebra is a (distributive) Stonian p-ortholattice *BUT* a RCC model is *NOT* a RT^- model (contact relations differs)

$\Rightarrow$ Demonstrates the benefit of algebraic methods for studying MT

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Future work

- Topological embedding theorem for Stonian p-ortholattices
- Explore the larger space of contact algebras (and equi-dimensional mereotopology in general)
  - e.g. topological representability of p-ortholattices by regular regions