A Reconciliation of Logical Representations of Space: from Multidimensional Mereotopology to Geometry

PhD Thesis Defence

Torsten Hahmann

Dept. of Computer Science, University of Toronto

December 07, 2012

Research Problem and Objective I

- 1^{st} Problem: **The expressiveness of current theories of qualitative space is a main hindrance for their practical use, while work on multidimensional theories weaker than classical geometries is limited.**

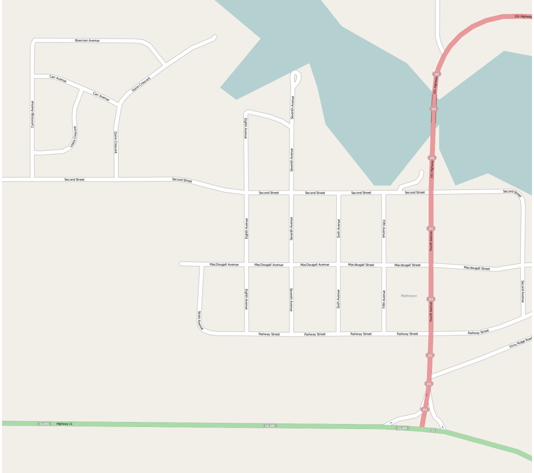
- Objective: Develop a qualitative theory of space that:
  Multidimensional: allows models with entities of multiple dimensions;
  Commonsensical: defines an intuitive set of spatial relations,
  Dimension-independent: not dependent on specific combinations of absolute (numeric) dimensions,
  Atomicity-neutral: admits discrete and continuous models,
  Geometry-consistent: generalizes classical geometries.

→ More expressive, but still intuitive logical theory

→ Basis for 'next-generation qualitative spatial reasoning'

- Driven by capturing street maps, buildings, etc.

Example 1: Simplified Maps

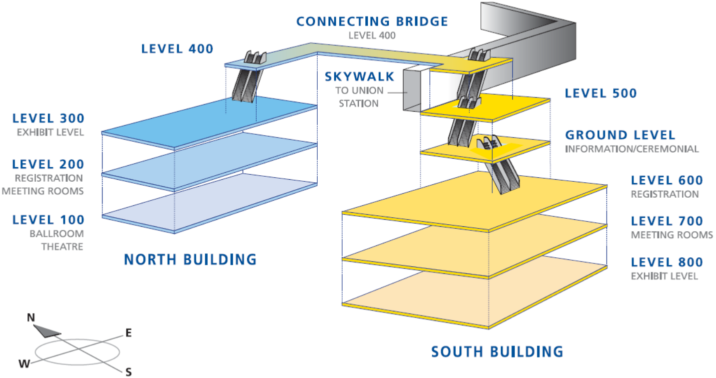


2D: cities, municipalities, lakes, parks;

1D: streets, rail lines;

0D: intersections, bridges, rail crossings.

Example 2: Building maps



3D: entire building;

2D: each floor, stairs, escalators, rooms;

1D: walls, windows, doors;

0D: water fountains, telephones, internet outlets, etc.

Research Problem and Objective II

- 2^{nd} Problem: **How are the various available first-order spatial ontologies, including mereotopologies and geometries, related to the newly developed ontologies and to one another?**

- Objective: Semantically integrate them according to

  Expressivity of their non-logical language: definability
  Which relations and functions are primitive?;
  Restrictiveness of their axioms: non-conservative extensions.

→ Construct (1) hierarchies of ontologies of equal expressivity that are partially ordered by their axioms' restrictiveness; and
(2) partially-order the hierarchies themselves by their non-logical languages' expressivity

  - Often cannot establish full mappings
  - *Comparative* (relative) integration of spatial ontologies to understand shared models and shared inferences

Thesis Outline

2. Methodology: definability and interpretability
3. Literature review: Equidimensional mereotopologies
4. Equidimensional mereotopologies with mereological closures [2]
5. The intended structures

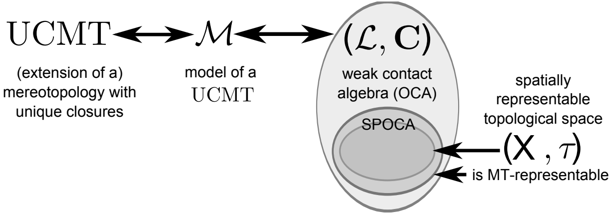
Theory of multidimensional mereotopological space

6. The basic theory [1]
7. Mereological closures operations [1]
8. Relationship to other mereotopologies [2]

Extensions of multidimensional mereotopological space

9. Boundaries and interiors [1,2]
10. Extension with betweenness: Geometries [1,2]
11. Extension with convex hulls: Modelling voids [1]

④ Equidimensional mereotopologies with closures



- Systematic study of equidimensional theories of space through there algebraic counterparts' spatial representability
- Class of mereotopologies of spatial interest: UCMT
  - Uniquely defined closure operations (sum, intersection, complement, universal) for spatial representability

T 2 Every model of UCMT is homomorphic to some orthocompl. CA

  - spatially representable CAs are MT-representable (defined notion)

C 2 An MT-representable complete OCA is a complete SPOCA

④ Equidim. mereotopologies with closures (contd.)

What do the MT-representable CAs that have all closure operations defined mereologically or topologically look like?

- Three classes of "minimal" MT-representable OCAs:
  - T 5 An M-closed MT-representable UCMT has an algebraic structure whose lattice is Boolean and whose contact relation satisfies (C0)–(C3)
  - T 6,7 A T- or T′-closed MT-representable UCMT has an algebraic structure whose lattice is a Stonian p-ortholattice and C satisfies (C0)–(C5)

T 8 Every MT-closed MT-representable UCMT has an algebraic structure that is an atomless BCA.

⇒ no new interesting equidimensional mereotopology possible; we are restricted to the expressiveness of the current ones

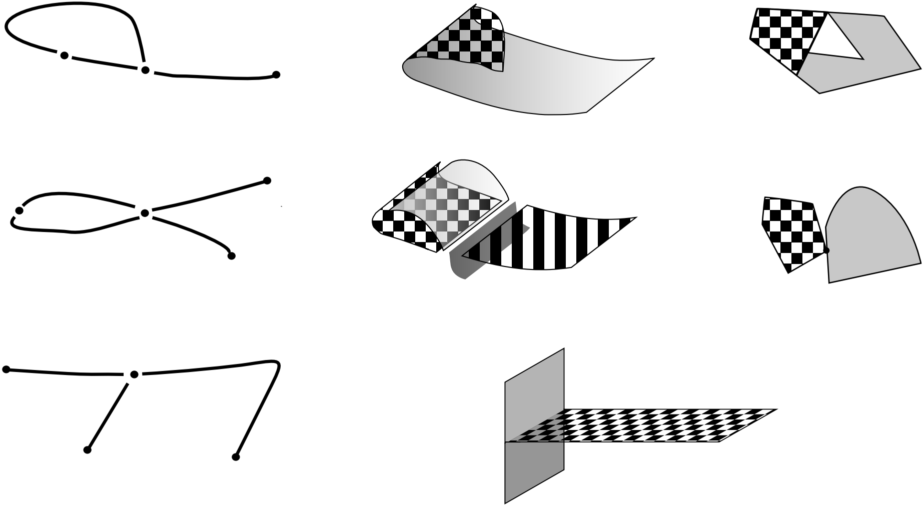
⑤ The intended structures

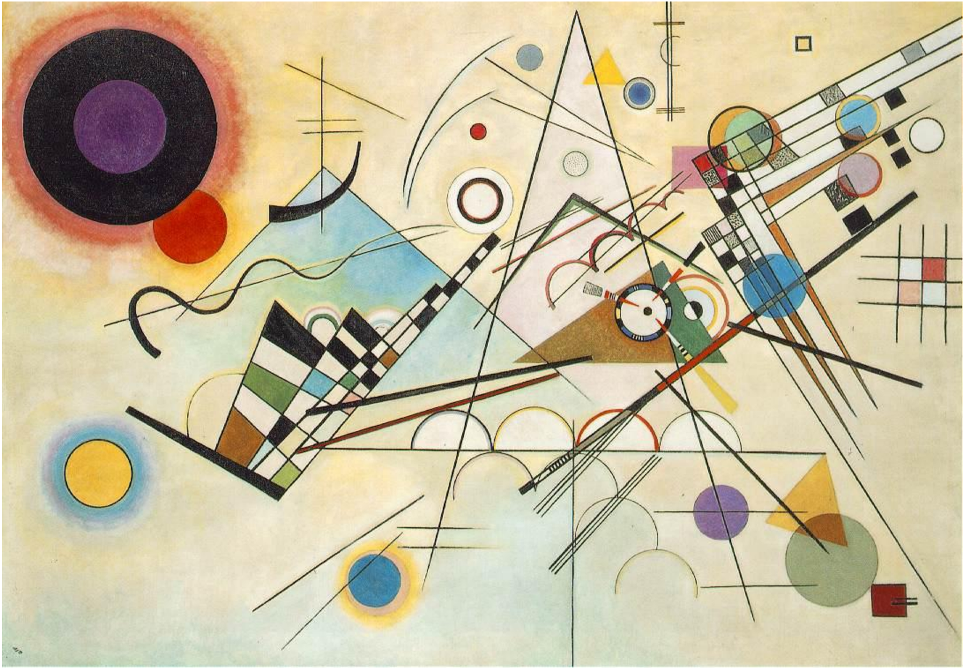
Idea **intended structures in $\mathfrak{M}$ are topologically and dimensionally invariant transformations of simplicial complexes**

- Allows any kind of stretching, bending, rotating, curving, folding, etc.

- Specification of the class of intended structures similar to the definition of simplicial complexes from simplexes
  - Use m-manifolds with boundaries as primitive entities
  - Composite m-manifolds = sets of m-manifolds with boundaries of uniform dimension that do not meet in the interior
  - Class of intended multidimensional structures:
    complex m-manifolds = sets of composite m-manifolds
    (with closure under intersection and complementation)
- Reference for evaluation of our ontologies in Ch. 6–9

Examples of non-atomic composite manifolds





Wassily Kandinsky: Komposition VIII (1923).

⑥ Basic multidimensional mereotopological space

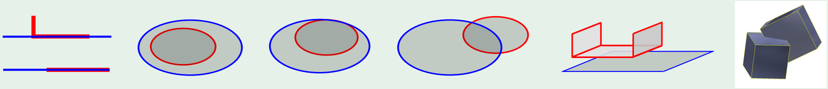
- Start building up qualitative theories by successively increasing the expressivity
- Axiomatization of linear relative dimension: *DI* hierarchy
- Axiomatization of spatial containment: *CO* hierarchy
- Combination to $\mathit{CODI}_{\mathrm{linear}}$

T 1 Partial characterization of the models of $\mathit{CODI}_{\mathrm{linear}}$:
In a model $\mathcal{M}$ of $\mathit{CODI}_{\mathrm{linear}} \cup \{\text{EP-D}\}$, $\mathbf{P}_{\mathcal{M}}$ and $(<_{\mathbf{dim}})_{\mathcal{M}}$ form a partition of $\mathbf{Cont}_{\mathcal{M}}$.

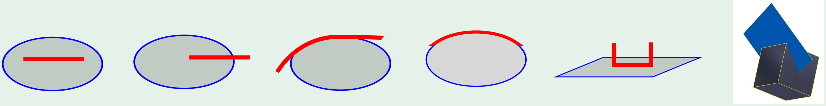
T 2 Three jointly exhaustive and pairwise disjoint (JEPD) types of contact: Partial Overlap, Incidence, Superficial contact definable in *CODI*

Three types of contact definable in $\mathit{CODI}_{\text{linear}}$

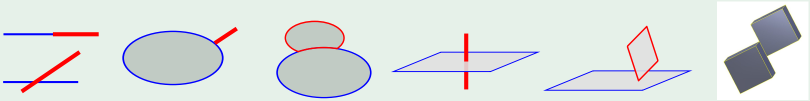
Strong Contact: **(Partial) Overlap** $x \Leftrightarrow x =_{dim} x \cdot y =_{dim} y$



Strong Contact: **Incidence** $\Leftrightarrow x =_{dim} x \cdot y <_{dim} y$ or vice versa



Weak Contact: **Superficial Contact** $\Leftrightarrow x >_{dim} x \cdot y <_{dim} y$



⑦ Mereological closure operations in multidimensional meoreotopological space

- Extension of $\mathit{CODI}_{\text{linear}}$ with mereological closure operations intersection $\cdot$, difference $-$, sum $+$, and universal u

T 1,2,5,7 Closure operations are defined total functions

- Prove mathematical properties of these operations: verifies the axiomatization against our intuitions ("competency questions")
- Strong supplementation for parthood and containment provable in $\mathit{CODI}_{\downarrow}$ (closed under intersection and differences): EP-E1 – EP-E3

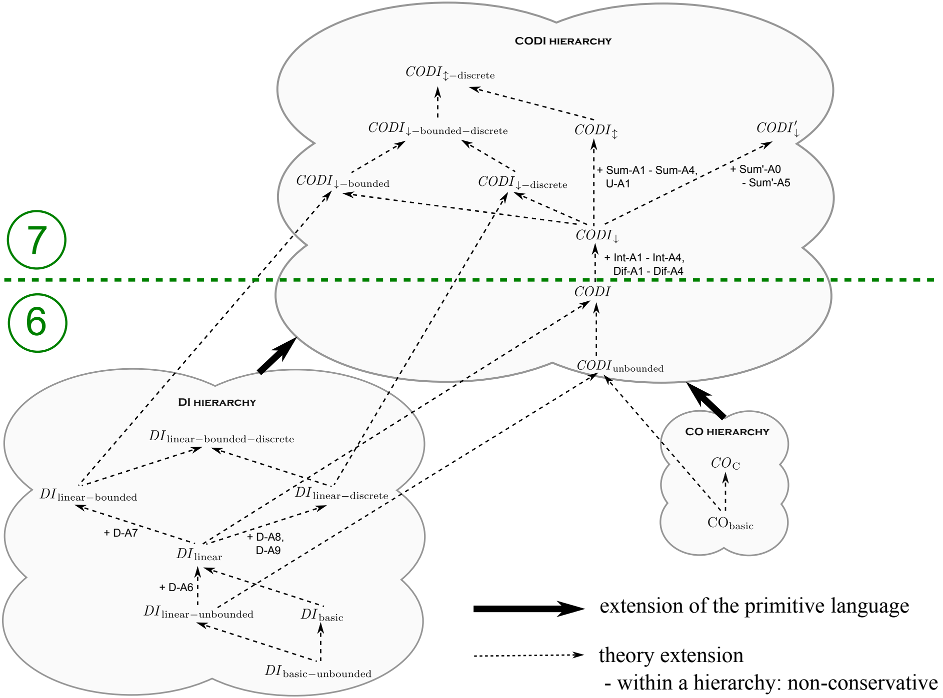
T 4 Satisfiability of $\mathit{CODI}_{\downarrow}$ w.r.t. the intended structures

  - ★ Every intended structure in $\mathfrak{M}$ is a model of $\mathit{CODI}_{\downarrow}$

- Axiomatizability of $\mathit{CODI}_{\downarrow}$ not provable w.r.t. the intended models
  - ★ Open challenge: Is every finite model of $\mathit{CODI}_{\downarrow}$ in $\mathfrak{M}$?
- Distinct structures in $\mathfrak{M}$ may have equivalent models of $\mathit{CODI}_{\downarrow}$

T 6 Characterization of the models of $\mathit{CODI}_{\updownarrow}$ (closed under all closure operations) as "stacks" of Boolean algebras

- Result: Extended *CODI* hierarchy as basis for the remaining chapters



CODI HIERARCHY
$CODI_{\updownarrow-\text{discrete}}$
$CODI_{\downarrow-\text{bounded}-\text{discrete}}$
$CODI_{\updownarrow}$
$CODI'_{\downarrow}$
+ Sum-A1 - Sum-A4, U-A1
+ Sum'-A0 - Sum'-A5
$CODI_{\downarrow-\text{bounded}}$
$CODI_{\downarrow-\text{discrete}}$
$CODI_{\downarrow}$
+ Int-A1 - Int-A4, Dif-A1 - Dif-A4
7
6
$CODI$
$CODI_{\text{unbounded}}$
DI HIERARCHY
$DI_{\text{linear}-\text{bounded}-\text{discrete}}$
$DI_{\text{linear}-\text{bounded}}$
$DI_{\text{linear}-\text{discrete}}$
+ D-A7
+ D-A8, D-A9
DI_{linear}
+ D-A6
$DI_{\text{linear}-\text{unbounded}}$
DI_{basic}
$DI_{\text{basic}-\text{unbounded}}$
CO HIERARCHY
CO_{C}
CO_{basic}
extension of the primitive language
theory extension
- within a hierarchy: non-conservative

⑧ Relationship to other mereotopologies

- Semantically integrate other theories with the *CODI* hierarchy
  - show how to extend theories from the *CODI* hierarchy to obtain existing mereotopologies
- Equidimensional mereotopology: Region Connection Calculus

T 2 Every model of $CODI_{\updownarrow} \cup$ C-E3 has a substructure that is a BCA (which are known to correspond to RCC models)

(C-E3) $MaxDim(x) \wedge MaxDim(y) \rightarrow$
$[x = y \leftrightarrow \forall z[MaxDim(z) \rightarrow (C(z,x) \leftrightarrow C(z,y))]]$
(extensionality of C amongst regions of maximal dimension)

  - Discussion: Can every model of the RCC be extended to a model of $CODI_{\updownarrow} \cup$ C-E3? No proof, because the extension is somewhat arbitrary.
    - Contact in the *CODI* model is different from contact in the RCC model

⑧ Relationship to other mereotopologies (contd.)

- Multidimensional mereotopology: INCH Calculus

(PA7') Correction of the original INCH Calculus

T 3 $CODI_{\updownarrow} \cup$ C-E4 and $INCH_{\text{calculus}} \cup \{$I-E1 – I-E3$\}$ are definably equivalent

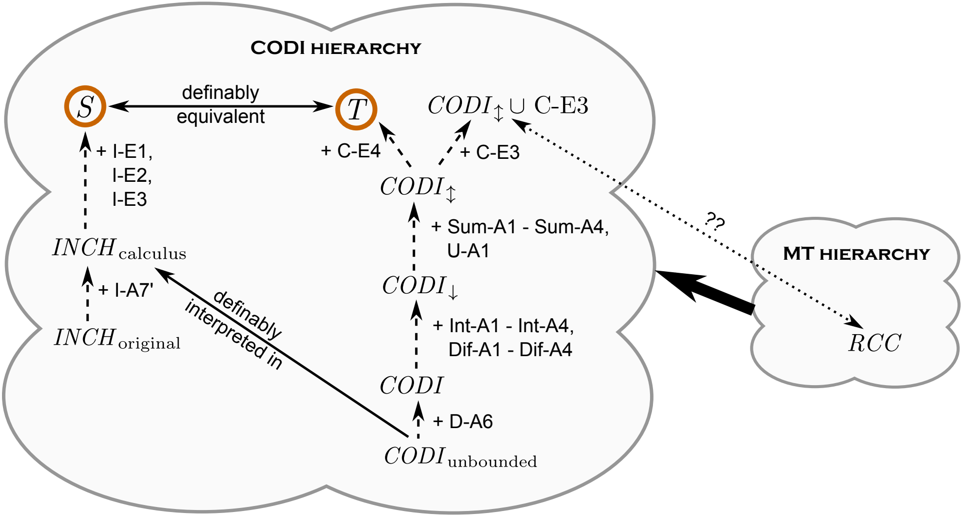
- Established with the mapping axioms I-M1 – I-M10 and I-M1′ – I-M6′

(C-E4) $x \leq_{\text{dim}} y \rightarrow \big[ZEX(x) \vee \exists z, v, w[P(v,x) \wedge Cont(v,z) \wedge P(w,z) \wedge Cont(w,y)]\big]$
(manifestation of relative dimension through a common entity z)

(I-E1) $\exists x[\neg ZEX(x) \wedge \forall y(\neg ZEX(y) \rightarrow GED(y,x))]$
(a non-zero entity of minimal dimension must exist)

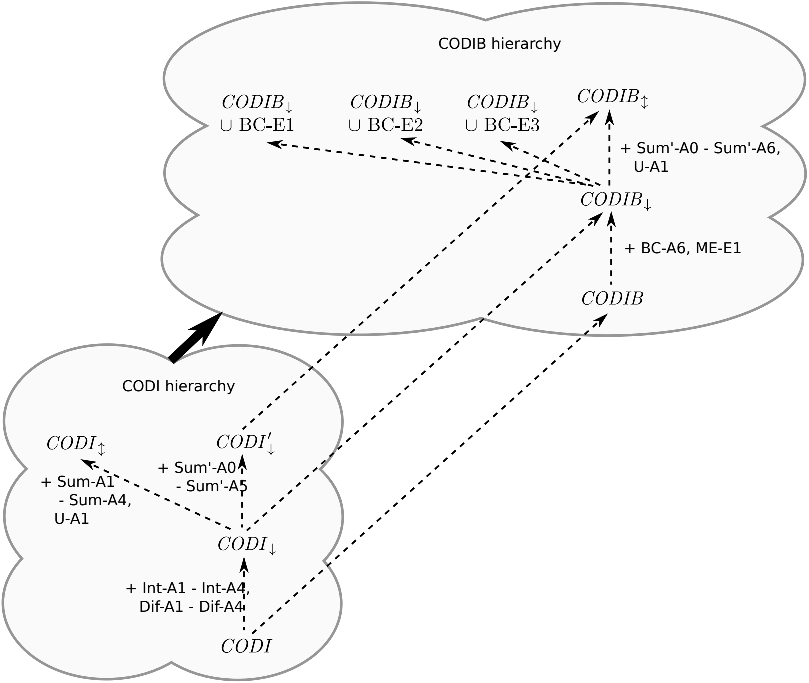
(I-E2) $\exists u \forall x[INCH(u,x)]$
(an entity exists that includes a chunk of any other entity)

(I-E3) $\exists u \forall x[CS(u,x)]$
(an entity exists of which every entity is a constituent)


CODI HIERARCHY
S
T
definably
equivalent
$CODI_{\updownarrow} \cup$ C-E3
+ I-E1,
I-E2,
I-E3
+ C-E4
+ C-E3
$CODI_{\updownarrow}$
+ Sum-A1 - Sum-A4,
U-A1
??
MT HIERARCHY
$INCH_{\text{calculus}}$
+ I-A7'
$INCH_{\text{original}}$
definably
interpreted in
$CODI_{\downarrow}$
+ Int-A1 - Int-A4,
Dif-A1 - Dif-A4
RCC
CODI
+ D-A6
$CODI_{\text{unbounded}}$

9 Boundaries and Interiors

- Some distinct intended structures are not distinguishable by the primitive language of *CODI*
  - Boundary containment vs. interior containment is undefinable
- Extension: new primitive relation of boundary containment
  - T 1 defined relations tangential and interior containment are JEPD
  - T 2 Satisfiability of $CODI_{\downarrow} \cup \{$BC-A1 – BC-A4$\}$ w.r.t. the intended structures in the restricted class $\mathfrak{M}$
  - T 3 defined relations tangential and interior parthood are JEPD
    - Prove interesting properties of the relations
    - Define two notions of boundary parts
- More fine-grained classification of contact relations based on whether interiors, boundaries, or both are in contact
  - T 4 C iff at least one of IO, IBC, IBC^{-1}, and BO holds
  - T 5 set of 9 JEPD binary relations, which generalize the relations from Egenhofer & Herring 1991, Clementini, et al. 1993, McKenney et al. 2005 to the finite-dimensional case with manifolds with boundaries

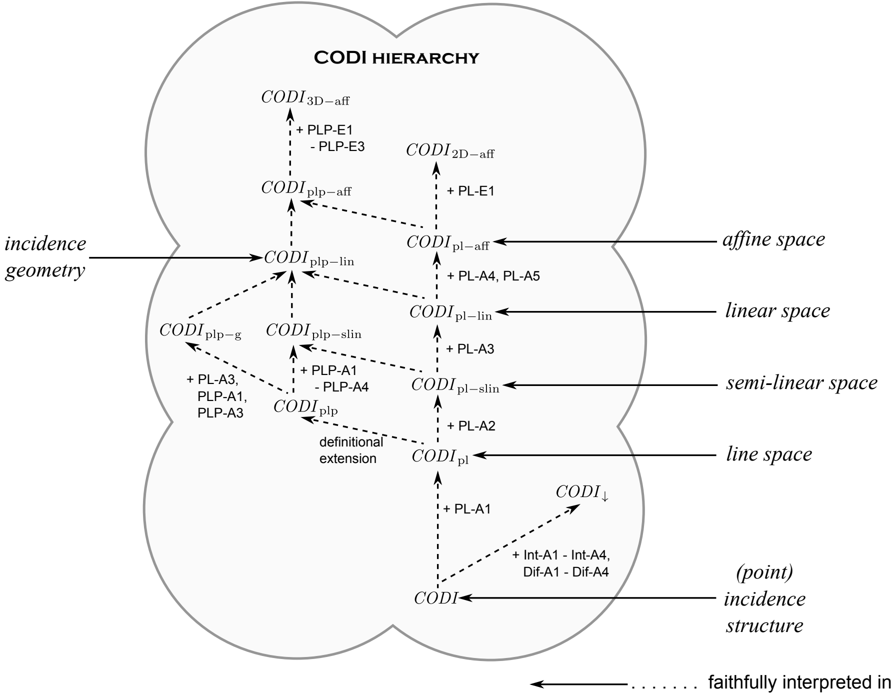

CODIB hierarchy
$CODIB_{\downarrow} \cup$ BC-E1
$CODIB_{\downarrow} \cup$ BC-E2
$CODIB_{\downarrow} \cup$ BC-E3
$CODIB_{\updownarrow}$
+ Sum'-A0 - Sum'-A6, U-A1
$CODIB_{\downarrow}$
+ BC-A6, ME-E1
$CODIB$
CODI hierarchy
$CODI_{\updownarrow}$
$CODI'_{\downarrow}$
+ Sum'-A0 - Sum'-A5
+ Sum-A1 - Sum-A4, U-A1
$CODI_{\downarrow}$
+ Int-A1 - Int-A4, Dif-A1 - Dif-A4
$CODI$

(10) Relationship to Incidence Geometries

- **Relationship to incidence structures**
  - T 1 Every model of *CODI* defines a (point) incidence structure
  - T 2 Point incidence structures can be definably expanded to *CODI* models
    - ⋆ *CODI* faithfully interprets the theory of point incidence structures

- **Relationship to planar (bipartite) incidence geometries**
  - T 3,5 Every model of $\mathit{CODI}_{\mathrm{pl}}$ ($\mathit{CODI}_{\mathrm{pl-slin}}$, $\mathit{CODI}_{\mathrm{pl-lin}}$, $\mathit{CODI}_{\mathrm{pl-aff}}$) defines a line (semi-linear, linear, affine) space.
  - T 4,6 Any line (semi-linear, linear, affine) space can be definably expanded to a model of $\mathit{CODI}_{\mathrm{pl}}$ ($\mathit{CODI}_{\mathrm{pl-slin}}$, $\mathit{CODI}_{\mathrm{pl-lin}}$, $\mathit{CODI}_{\mathrm{pl-aff}}$).
    - ⋆ $\mathit{CODI}_{\mathrm{pl}}$ ($\mathit{CODI}_{\mathrm{pl-slin}}$, $\mathit{CODI}_{\mathrm{pl-lin}}$, $\mathit{CODI}_{\mathrm{pl-aff}}$) faithfully interprets the theory of line (semi-linear, linear, affine) spaces

- **Relationship to (tripartite) incidence geometries**
  - T 7 Any model $\mathcal{M}$ of $\mathit{CODI}_{\mathrm{plp-lin}}$ defines an incidence geometry.
  - T 8 Any incidence geometry can be definably expanded to a model of $\mathit{CODI}_{\mathrm{plp-slin}}$ ($\mathit{CODI}_{\mathrm{pl-lin}}$, $\mathit{CODI}_{\mathrm{pl-aff}}$).
    - Shows in principle how to reconstruct any finite-dimensional geometry
    - Defines a mereotopological generalization of incidence geometry
    - Discussion of when a mereotopology becomes a geometry


CODI HIERARCHY
$CODI_{3D-aff}$
+ PLP-E1
- PLP-E3
$CODI_{2D-aff}$
+ PL-E1
$CODI_{plp-aff}$
$CODI_{pl-aff}$
incidence geometry
$CODI_{plp-lin}$
affine space
+ PL-A4, PL-A5
$CODI_{pl-lin}$
linear space
$CODI_{plp-g}$
$CODI_{plp-slin}$
+ PL-A3
+ PLP-A1
- PLP-A4
$CODI_{pl-slin}$
semi-linear space
+ PL-A3,
PLP-A1,
PLP-A3
$CODI_{plp}$
+ PL-A2
definitional
extension
$CODI_{pl}$
line space
$CODI_{\downarrow}$
+ PL-A1
+ Int-A1 - Int-A4,
Dif-A1 - Dif-A4
$CODI$
(point)
incidence
structure
.... faithfully interpreted in

⑩ Extension with Betweenness: Ordered Geometries

- **Motivation: even when capturing space qualitatively we often want to preserve spatial orderings, for example, for street maps**

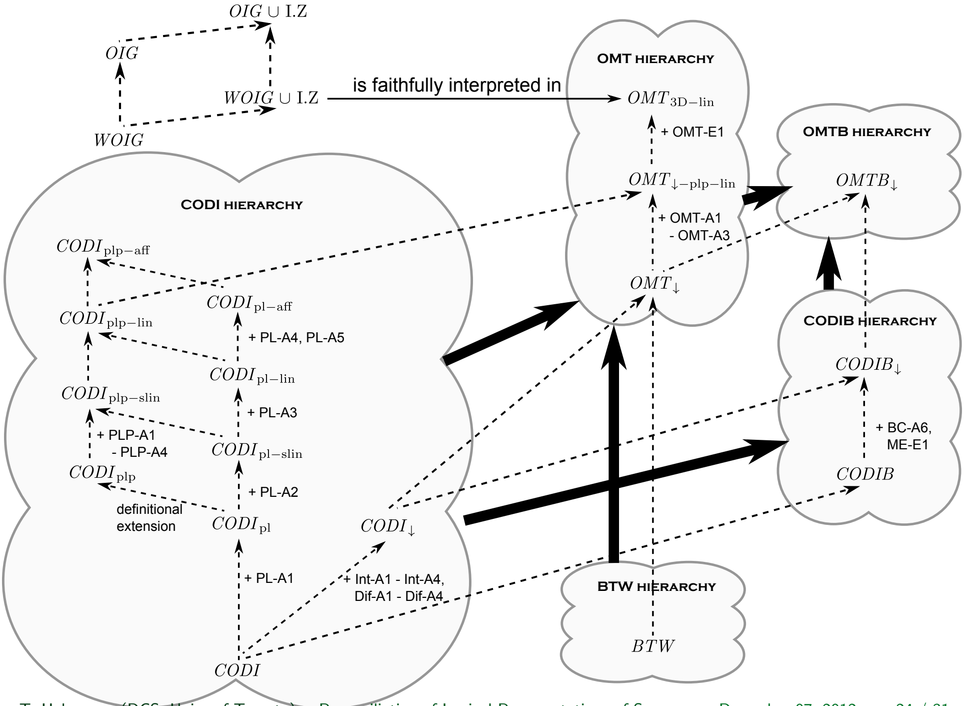
- Extension: quaternary primitive relation of relativized betweenness
  - Not definable in the languages of *CODI* or *CODIB*
  - A multidimensional version of betweenness in a new hierarchy, *BTW*
  - Combining *BTW* and *CODI* results in ordered mereotopologies *OMT*
  - Discussion of the required strength of the geometry to define convexity

- Relationship to ordered incidence geometries

  T 9 Any model of $OMT_{\mathrm{3D-lin}}$ defines a weak ordered incidence geometry.

  T 10 Any weak ordered incidence geometry defines a model of $OMT_{\mathrm{3D-lin}}$.
  - But $WOIG \cup$ I.Z faithfully interpreted in $OMT_{\mathrm{3D-lin}}$, that is, existence of zero region is the only difference

- Qualitative analogues to ordered incidence geometries

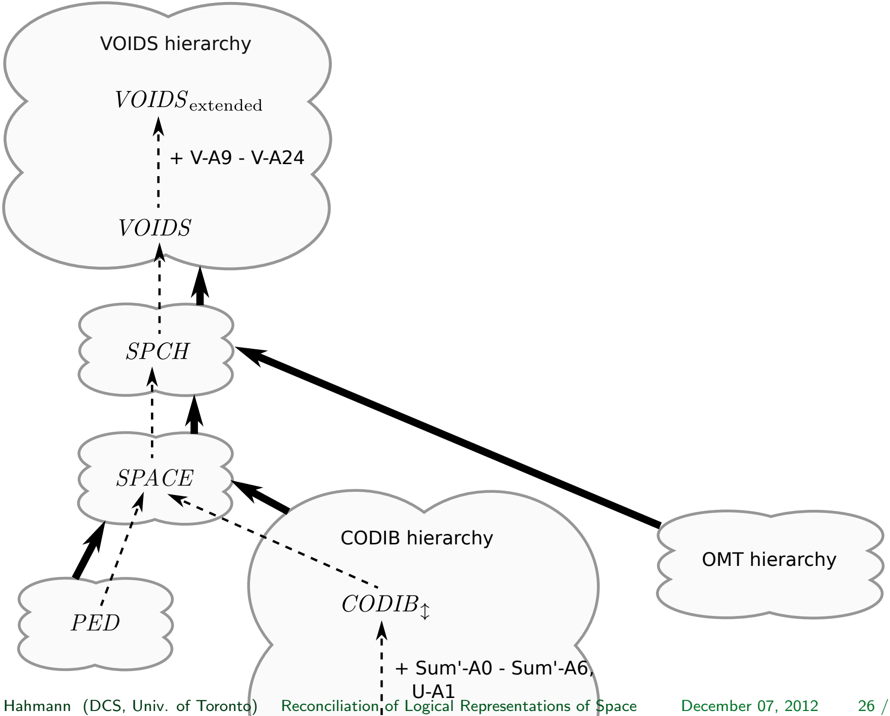


$OIG \cup \mathrm{I.Z}$
OIG
$WOIG \cup \mathrm{I.Z}$
$WOIG$
is faithfully interpreted in
OMT HIERARCHY
$OMT_{\mathrm{3D-lin}}$
+ OMT-E1
$OMT_{\downarrow-\mathrm{plp-lin}}$
+ OMT-A1
- OMT-A3
$OMT_{\downarrow}$
OMTB HIERARCHY
$OMTB_{\downarrow}$
CODIB HIERARCHY
$CODIB_{\downarrow}$
+ BC-A6,
ME-E1
$CODIB$
CODI HIERARCHY
$CODI_{\mathrm{plp-aff}}$
$CODI_{\mathrm{pl-aff}}$
$CODI_{\mathrm{plp-lin}}$
+ PL-A4, PL-A5
$CODI_{\mathrm{pl-lin}}$
$CODI_{\mathrm{plp-slin}}$
+ PL-A3
+ PLP-A1
- PLP-A4
$CODI_{\mathrm{pl-slin}}$
$CODI_{\mathrm{plp}}$
+ PL-A2
definitional
extension
$CODI_{\mathrm{pl}}$
$CODI_{\downarrow}$
+ PL-A1
+ Int-A1 - Int-A4,
Dif-A1 - Dif-A4
$CODI$
BTW HIERARCHY
BTW

⑪ Extension with Convex Hulls: Physical Voids

Utilize the axiomatization of abstract space in a specific setting:
Ontology of Hydrogeology (rock formations and water bodies)

- Extend the axiomatization of abstract space by physical space (objects and matter): Layered Mereotopology (Donnelly, 2003)
  - New: axiomatize distinction between matter and objects
- Fit in convex hulls: not in the defined setting from *ordered mereotopology* but in a more general setting as primitive relation
- Classification of physical voids
  - by the void's self-connectedness (simple vs. complex void)
  - by the host's self-connectedness (gap vs. hole)
  - by the void's external connectedness (cavity vs. hollow vs. tunnel)
  - by granularity distinction (voids in matter vs. voids in objects)
- Can still prove consistency for this sizeable complex ontology (roughly 120 axioms, 60 distinct non-logical symbols, 40 existentials)


VOIDS hierarchy
$VOIDS_{\text{extended}}$
+ V-A9 - V-A24
$VOIDS$
$SPCH$
$SPACE$
PED
CODIB hierarchy
$CODIB_{\updownarrow}$
+ Sum'-A0 - Sum'-A6,
U-A1
OMT hierarchy

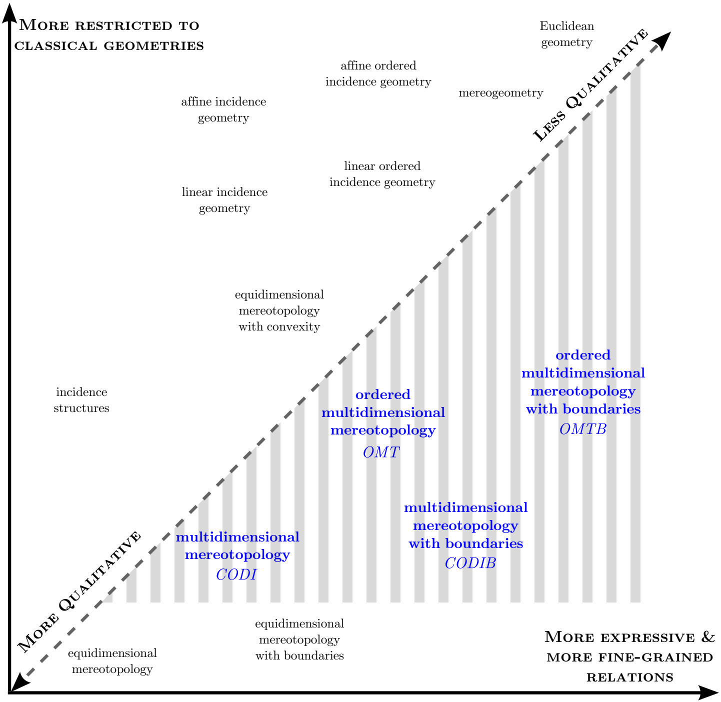
Verification of the Developed Ontologies

- **JEPD relations:** classification of spatial relations; lends itself to spatial calculi (6.2, 9.1, 9.3, 9.4 (not disjoint), Ch. 11)
- **Model Characterization:** understanding and verification of theories w.r.t. the intended structures (satisfiability, T 7.4, 9.2) or w.r.t. well-understood algebraic structures (4.2 − 4.10, 6.1, 7.6, 8.1)
- **Cross-verification:** theory relationships to other ontologies
  - Compare: integration results (next)
- **Competency questions:** proved many expected properties of certain relations; mostly automated proofs
- **Non-trivial consistency:** constructed models to show that any relation can have a non-empty extension

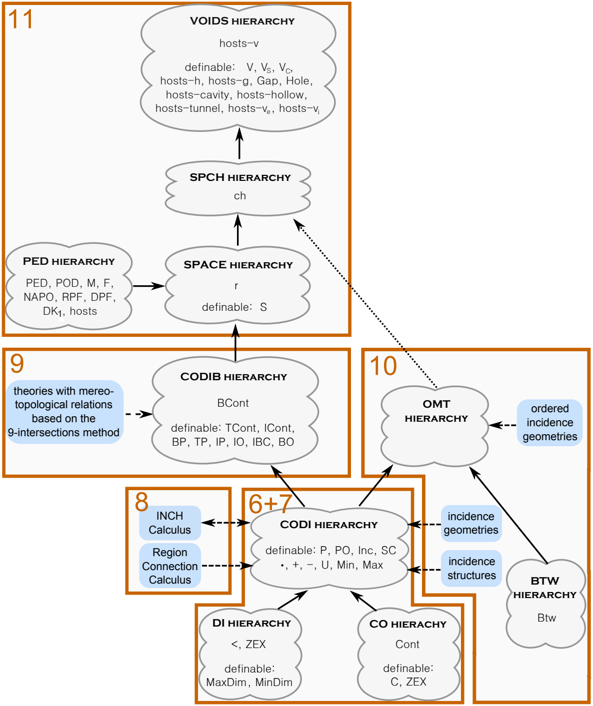
Integration Results

- **Theory Relationships:** mapping between theories that are extensions of *CODI* and external spatial theories
  - Full theory integration (definably equivalence between theories): 8.3
  - Faithful interpretation (conservative extension, possibly language extension) established through model expansions: 10.2, 10.4, 10.6, 10.8
  - Definable interpretation (possibly non-conservative extension) established when all models of the interpreting theory define models of the interpreted theory: 8.2, 10.1, 10.3, 10.5, 10.7, 10.9, 10.10
  - Implicit interpretability via the intended structures: 9.5
- **Definability:** closure operations are defined (7.1, 7.2, 7.5, 7.7)
- **Non-definability:** give two structures that have identical models in one language but distinct models in a more expressive language

SUMMARY OBJECTIVE 1



SUMMARY OBJECTIVE 2



Summary

- Developed new qualitative ontologies of space that are more expressive than previously available mereotopologies and formally studied their expressivity and their logical relationships
  - Proposed a characterization of multidimensional qualitative space
  - First well-understood theory of multidimensional mereotopology
  - Not fixed in number of dimensions, not tied to points or regions
- Established formal relationships (theory interpretations and relationships between classes of models) to understand how various ontologies of space relate to one another

  $\Rightarrow$ first step toward integration of spatial data

Lessons learned

- Manual ontology verification and integration is arduous
- Automated reasoning often successful without much manual tweaking

$\Rightarrow$ Suggests **ontology verification** and **ontology integration** can be largely automated in practise