Multidimensional Mereotopology with Betweenness

Torsten Hahmann & Michael Grüninger

Semantic Technologies Laboratory,
Dept. of Computer Science & Dept. of Mechanical and Industrial Engineering,
University of Toronto

torsten@cs.toronto.edu, gruninger@mie.utoronto.ca

1. Objective: A Theory for Qualitative Spatial Maps

QUALITATIVELY ABSTRACT geometry to represent maps qualitatively while preserving essential aspects for route descriptions (without distances or absolute directions)

SEMANTICALLY INTEGRATE theories of space: mereotopologies, incidence structures, and geometries

MODULARLY DESIGN a hierarchy of first-order-theories of qualitative space in a bottom-up fashion

2. Main idea: Abstraction of Geometry

ABSTRACT HILBERT'S ORDERED INCIDENCE GEOMETRY QUALITATIVELY SO THAT ONLY TOPOLOGY & ORDER MATTER (WITHOUT CURVATURE, SHAPE, OR DISTANCES).

We can describe city or building maps as sets of interconnected multidimensional features of uniform dimensions including areal features (street blocks, administrative regions, parks, forests, water bodies), linear features (roads, rivers, rail lines), and point features (intersections, bridges, rail crossings, points of interest) using a standard geometry weakened by two common geometric assumptions:

(a) Two distinct entities A, B of equal dimension determine a unique C of next-highest dimension (*line axiom*);

(b) For two distinct entities A, B of equal dimension in a higher-dimensional space, there exists two more entities C, D of the same dimension so that C is in between A and B, and B is between A and D (*continuity axiom*).

3. Methodology: 'Not Stronger Than Necessary'

COMPILE THE MOST BASIC AXIOMS OF SPACE IN A WEAK THEORY AND AXIOMATIZE SUCCESSIVELY STRONGER EXTENSIONS INTERPRETED BY KNOWN SPATIAL THEORIES.

1. Show that the weak theory is a mereotopology (i.e. the usual contact and parthood are definable): Construct theories stronger than existing mereotopologies without having to add essential mereotopological axioms
2. Show that suitable extensions are interpreted by known incidence structures and geometries (up to affine $\sim$)
3. Extend the theory with multidimensional betweenness: a generalized version of geometric betweenness
4. Show that suitable extensions are interpreted by known ordered incidence geometries (such as ordered affine $\sim$, betweenness $\sim$, and Hilbert's $\sim$)

4. Weak Multidimensional Mereotopology

- Primitive relations:

  SPATIAL CONTAINMENT $Cont(x, y)$
  $\ldots x$ is contained in y (dimension-independent)

  RELATIVE DIMENSION $x \leq_{dim} y$
  $\ldots x$ is of a lower or the same dimension as y

  ZERO ENTITY $ZEX(x)$
- Relationship between containment and dimension:

  $Cont(x, y) \to x \leq_{dim} y$ (CD-A1)

MODELS: Partial order defined by containment with superimposed linear order over equivalence classes of equidimensional entities (cf. Fig. 3)

- Definable relations:

  CONTACT $C(x, y) \leftrightarrow \exists z(Cont(z, x) \wedge Cont(z, y))$ (C-D)

  PARTHOOD $P(x, y) \leftrightarrow Cont(x, y) \wedge x =_{dim} y$ (EP-D)
- Classification of contact into three types:

  (PARTIAL) OVERLAP $PO \Rightarrow x =_{dim} x \cdot y =_{dim} y$
  ... share a common part

  INCIDENCE $Inc \Rightarrow x =_{dim} x \cdot y <_{dim} y$ (or vice versa)
  ... only a common entity that is part of one

  SUPERFICIAL CONTACT $SC \Rightarrow x >_{dim} x \cdot y <_{dim} y$
  ... contact without common part

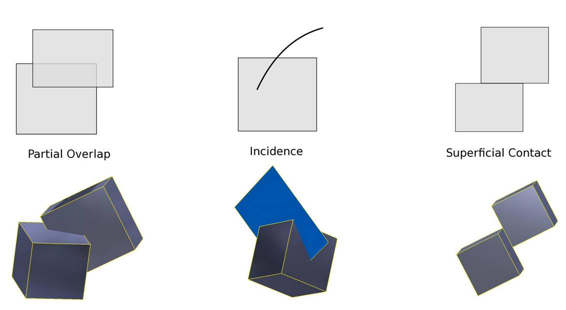

Figure 1: *Partial overlap, incidence, and superficial contact each in 2D and 3D (from left to right).*

5. A "Sketch Map" of an Island and its Model in the Weak Multidimensional Mereotopology

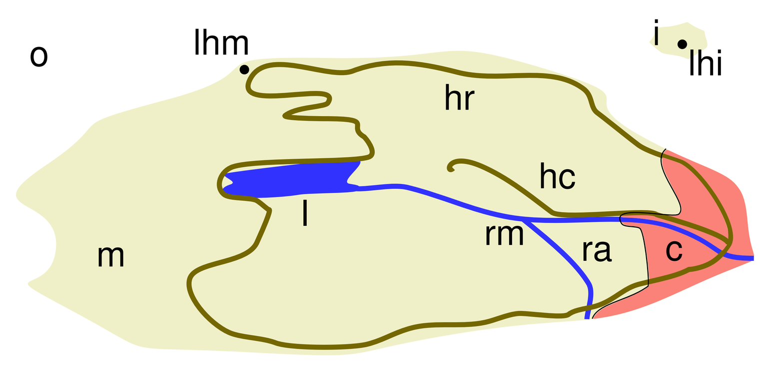

Figure 2: *Map with entities of various dimensions:*
2D: *ocean, **m**ain island, small **i**sland, **c**ity, **l**ake;*
1D: *river **m**ain and **a**rm, **h**ighway **r**ing and **c**entral;*
0D: ***l**ight**h**ouse **m**ain, **l**ight**h**ouse **i**sland.*

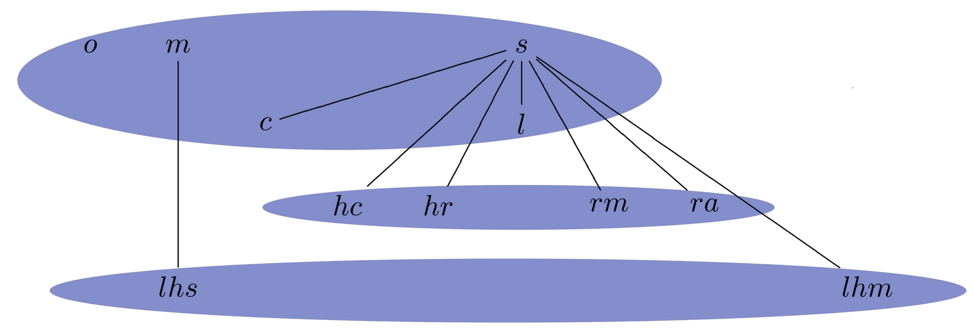

Figure 3: *The model as partial defined by the containment relation amongst the primary spatial objects of Fig. 2. Each bubble contains entities of identical dimension; parthood is the containment relation within bubbles.*

6. Relationship to Other Mereotopologies, Incidence Structures, and Incidence Geometries

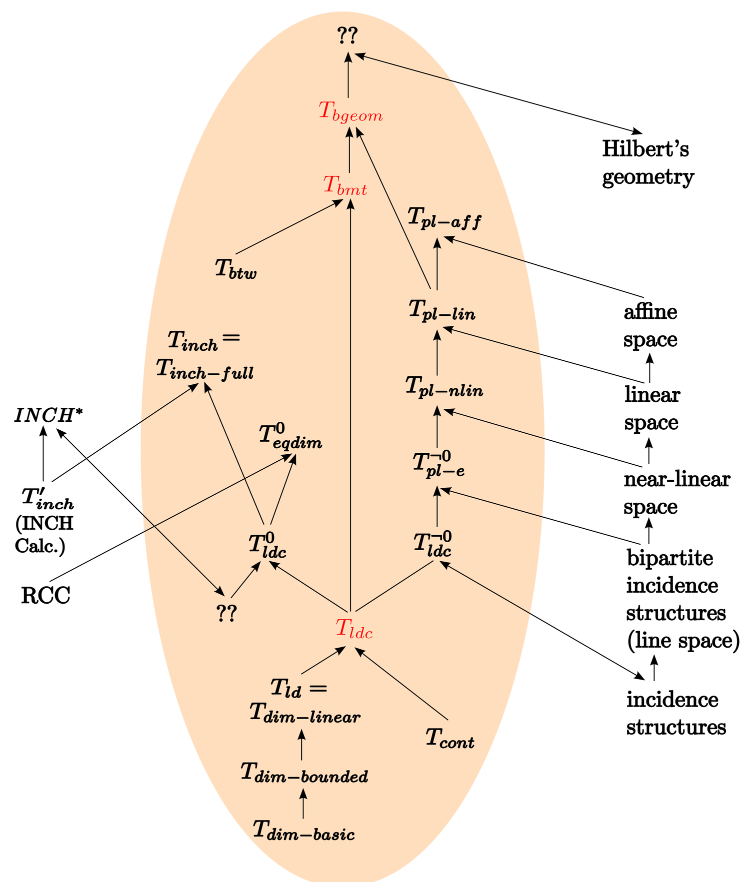

Figure 4: *The hierarchy of spatial theories and their interpretability by external theories (arrows denote extension).*

Interpretation by the INCH Calculus (Gotts 1996)

- Primitive relation $INCH(x, y)$
  $\ldots x$ includes a chunk (equidimensional part of) y

Theorem 2 *The INCH Calculus faithfully interprets the theory T_{inch}, an extension of T^0_{ldc}.*

- Extensionality of INCH
- Equidimensional sums and differences

ONGOING WORK: Interpretations in both directions

- Must weaken T_{inch} and strengthen the INCH Calculus

Interpretation by the Region Connection Calculus (RCC: Randell et al. 1992)

- Primitive relation: $C(x, y) \ldots x$ and y are in contact

Theorem 4 *For a model M of T^0_{eqdim} there exists a model N of RCC such that N is definably interpreted in M.*

- Extensionality of C amongst regions of highest dim.
- Sums, intersections, complements, and universal
- Connectedness & infinite divisibility

An interpretation in the reverse direction cannot exist: Equidimensional mereotopology is not capable of defining a multidimensional mereotopology

Interpretation by k-partite incidence structures

Theorem 5 *The axiomatization of the class of incidence structures faithfully interprets the theory T^{-0}_{ldc}.*

- Inc is really a classical incidence relation

Interpretation by bipartite incidence geometries

Theorem 6 *The structure $\langle \mathbf{Pt}, \mathbf{L} \rangle$ of a model of $T_{pl-nlin}$ (T_{pl-lin}, T_{pl-aff}) with the incidence relation $x * y \Leftrightarrow [(Pt(x) \vee L(x)) \wedge (Pt(y) \vee L(y)) \wedge Inc(x, y) \vee x = y]$ in M is a near-linear (linear, affine) space.*

- Extends to k-partite incidence geometries

7. Multidimensional Betweenness

When is Betweenness Necessary?

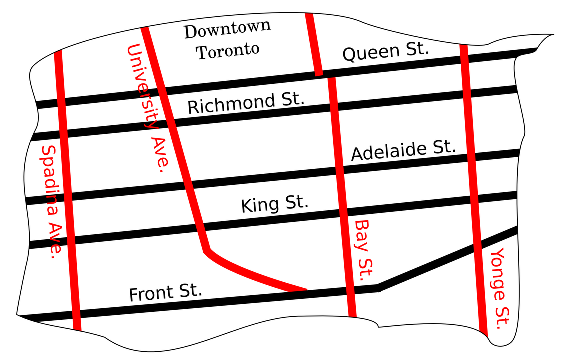

Figure 5: *Permuting Queen, Richmond, Adelaide, King, and Front St. does not change the map's representation in the multidimensional mereotopology without betweenness.*

Multidimensional Axiomatization of Betweeness

- Primitive relation: quaternary betweenness $Btw(r, a, b, c)$
  $\ldots$ in r, b is strictly in between a and c

(B-A1) $Btw(r, a, b, c) \to a \neq b \neq c \neq a$ (strong $\equiv$ irreflexive)

(B-A2) $Btw(r, a, b, c) \to Btw(r, c, b, a)$ (outer symmetry)

(B-A3) $Btw(r, a, b, c) \to \neg Btw(r, a, c, b)$ (strict $\equiv$ acyclic)

(B-A4) $Btw(r, x, a, b) \wedge Btw(r, a, b, y) \to Btw(r, x, a, y)$ (outer transitivity)

(B-A5) $Btw(r, x, a, b) \wedge Btw(r, a, y, b) \to Btw(r, x, a, y)$ (inner transitivity)

(BMT-A1) $Btw(r, x, y, z) \to x =_{dim} y =_{dim} z \prec r \wedge Cont(x, r) \wedge Cont(y, r) \wedge Cont(z, r)$
(betweenness only amongst equidimensional entities contained in a common entity of next highest dimension)

Not an axiom of betweenness (see Fig. 6), but necessary in the extensions to ordered incidence geometries which ,e.g., assumes total orderability of points on a line:

(BMT-E1) $x =_{dim} y =_{dim} z \prec r \wedge \neg C(x, y) \wedge \neg C(x, z) \wedge \neg C(y, z) \wedge Cont(x, r) \wedge Cont(y, r) \wedge Cont(z, r) \to [Btw(r, x, y, z) \vee Btw(r, x, z, y) \vee Btw(r, y, x, z)]$
(three disconnected equidimensional entities contained in an entity of next highest dimension are totally orderable)

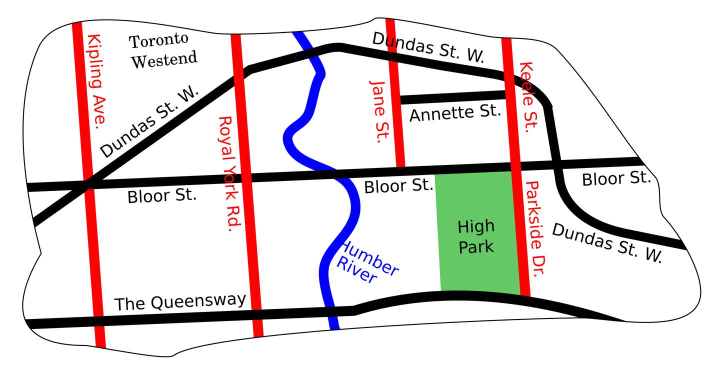

Figure 6: *Intersecting streets, such as Dundas, Bloor, and The Queensway, can usually not be ordered. But Dundas, Annette, and Bloor might still be orderable.*

Future Challenge: Disentangle and Axiomatize the Various Usages and Interpretations of Betweenness

SEPARATION: The Humber separates Royal York from Jane

ENCLOSURE: Dundas and Bloor enclose Annette

VARIOUS STRENGTHS OF PARTIAL BETWEENNESS:

Parkside between Humber River and Dundas? (ambigious)

Jane between Humber River and Keele? (likely)

Jane between Humber River and Parkside? (unlikely)

IJCAI 2011, 22nd International Joint Conference on Artificial Intelligence, July 16-22, 2011, Barcelona, Spain