A Theory of Multidimensional Space: Semantic Integration of Spatial Theories that Distinguish Interior from Boundary Contact

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1. Objective: Hierarchy of Logical Theories of Multidimensional Qualitative & Geometric Space

- Semantically integrate logical theories of space
  Here: Theories with interior-boundary distinction
- Show definability or undefinability of relations
- Verify theories with automated theorem provers

Example 1. A map with features of various dimension depicting a part of the University campus in Toronto.

2. Weak Multidimensional Mereotopology (MMT) (CommonSense'11)

SPATIAL CONTAINMENT $Cont(x,y)$
…x is contained in y (dimension-independent)

RELATIVE DIMENSION $x \leq_{dim} y$
…x is of a lower or the same dimension as y

ZERO ENTITY $ZEX(x)$

Relationship between containment and dimension:
(CD-A1) $Cont(x,y) \to x \leq_{dim} y$

⇒ Weak MMT with linear dimension: $CODI_{linear}$

Definable relations:

PARTHOOD $P(x,y) \leftrightarrow Cont(x,y) \wedge x =_{dim} y$

CONTACT $C(x,y) \leftrightarrow \exists z(Cont(z,x) \wedge Cont(z,y))$

⇒ Classification of contact into three types:

(PARTIAL) OVERLAP $PO \Rightarrow x =_{dim} x \cdot y =_{dim} y$
… share a common part

INCIDENCE $Inc \Rightarrow x =_{dim} x \cdot y <_{dim} y$ (or vice versa)
… only a common entity that is part of one

SUPERFICIAL CONTACT $SC \Rightarrow x >_{dim} x \cdot y <_{dim} y$
… contact without common part

Theorem 1 *$\forall x, y \in dom(M)$ in a model M of $CODI_{linear}$, $C(x,y) \leftrightarrow$ exactly one of $PO(x,y)$, $Inc(x,y)$ or $SC(x,y)$.*

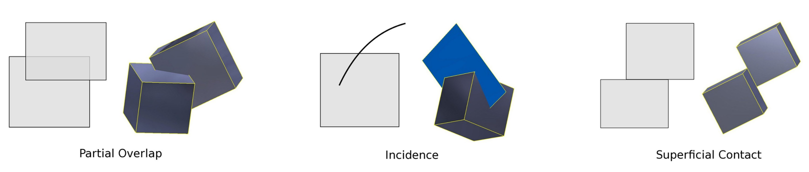


Figure 1: *2D & 3D examples of the three contact relations.*

3. Relationship to Mereotopologies, Incidence Structures and Geometries (IJCAI'11)

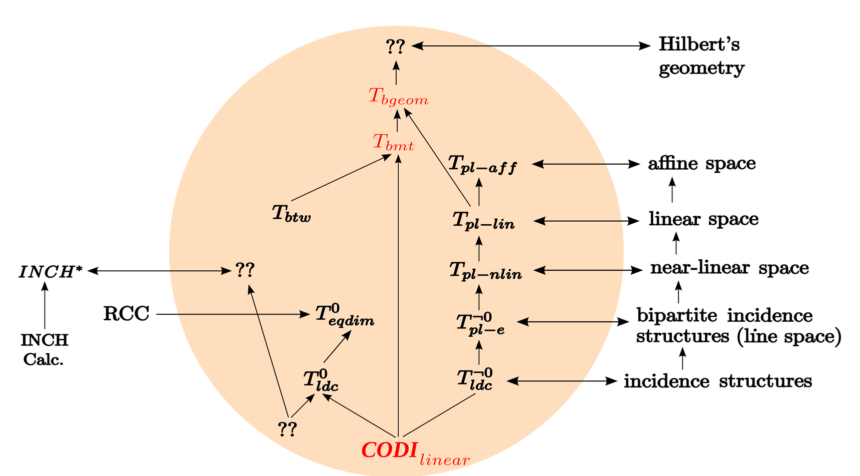


Figure 2: *The hierarchy of spatial theories and their interpretability by external theories (arrows denote extension).*

Non-conservative extensions of T_{ldc} are interpreted by
- … the INCH Calculus (Gotts 1996)
- … the Region Connection Calculus (Randell et al. 1992)
- … k-partite incidence structures
- … (bipartite) incidence geometries

Additional primitive necessary for interpretations by ordered incidence geometries (incl. Hilbert's geometry):

RELATIVIZED BETWEENNESS $Btw(r,a,b,c)$
… in r, b is strictly in between a and c

⇒ MMT with betweenness: T_{bmt}

4. Simple Entities as Objects of the Domain

Restriction to well-behaved n-dimensional spaces that clearly distinguish boundary and interior of each entities.

All entities in the domain are *Simple entities*:
- Uniform dimension $m \leq n$
- No self-intersection: No point in interior and boundary
- No singularities or missing lower-dimensional parts
- Composed of atomic (self-connected) simple entities which are only connected in their boundaries (if at all)
- ***Simple atomic entities***: m-manifolds (locally Euclidean in $\mathbb{R}^m$) with boundaries (possibly empty)
- ***Complex entities*** are sets of simple entities.

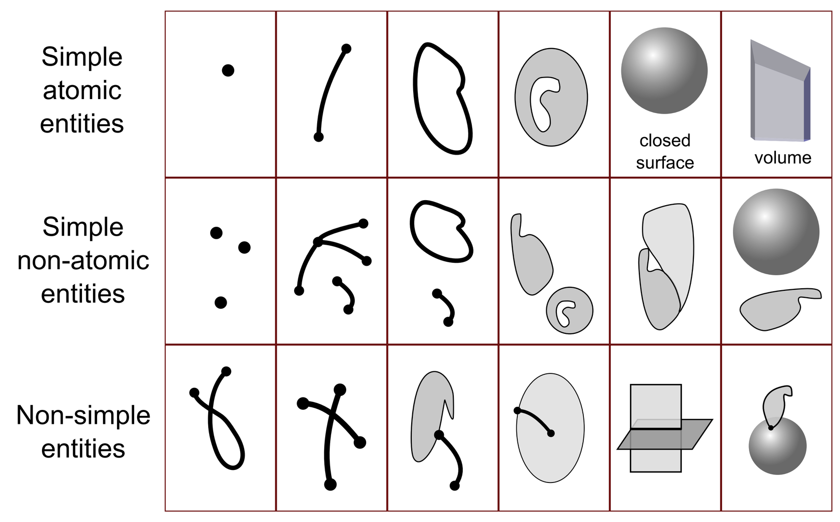
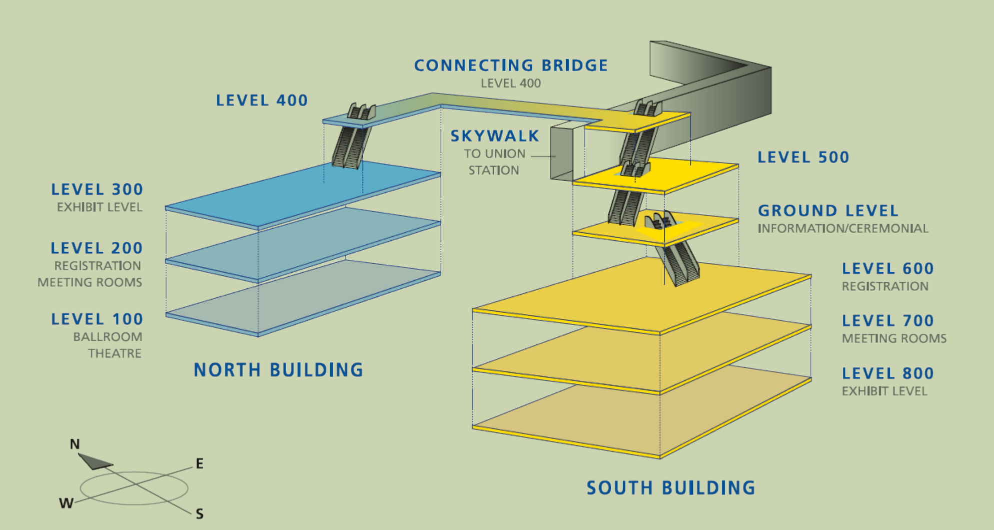


Figure 3: *Examples of simple and non-simple entities.*



Example 2. A 3D building map (simplified CAD drawing) of the Metro Convention Centre, Toronto.

5. Closure Operations in Multidimensional Space

Ensure decomposability of models by closing them under intersections and differences.

Definable function: INTERSECTION $x \cdot y$
…intersection of the greatest common dimension

(Int-A1) $\neg C(x,y) \to ZEX(x \cdot y)$ (empty intersection)
(Int-A2) $x \cdot y = y \cdot x$ (intersection commutative)
(Int-A3) $\neg ZEX(x \cdot y) \to Cont(x \cdot y, x)$ ($x \cdot y$ is contained in the intersecting entities)
(Int-A4) $Cont(z,x) \wedge Cont(z,y) \to z \leq_{dim} x \cdot y$ ($x \cdot y$ is of the greatest dimension of the intersection)
(Int-A5) $Cont(z,x) \wedge Cont(z,y) \wedge z =_{dim} x \cdot y \to P(z, x \cdot y)$ (greatest intersection of greatest dimension)

Definable function: DIFFERENCE $x - y$
…difference of the same dimension as x

(Dif-A1) $P(z, x-y) \leftrightarrow P(z,x) \wedge \neg PO(z, x \cdot y)$ (constitution of the difference $x-y$)
(Dif-A2) $PP(y,x) \to PP(x-y, x)$ (non-empty difference $x-y$ if y is a proper part of x)
(Dif-A3) $P(y,x) \wedge Cont(z,x) \wedge Min(z) \to [Cont(z, x-y) \vee Cont(z,y)]$ (minimal entities in x contained in y or $x-y$)

Definable relation: SELF-CONNECTEDNESS $Con(x)$

(Con-D) $Con(x) \leftrightarrow \forall y[PP(y,x) \to C(y, x-y)]$

⇒ MMT with downward mereological closures $CODI_{\downarrow}$

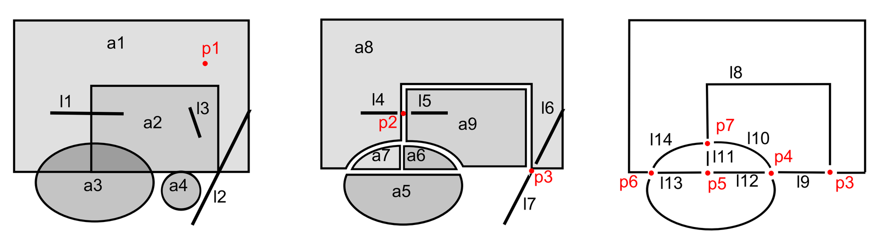


Figure 4: *A model of $CODI_{\downarrow}$ decomposed by intersections and differences into simple atomic entities.*

6. Distinguishing Interiors from Boundaries

BOUNDARY CONTAINMENT $BCont(x,y)$
…x is contained in the boundary of y

$BCont$ **is not definable in** $CODI_{\downarrow}$:

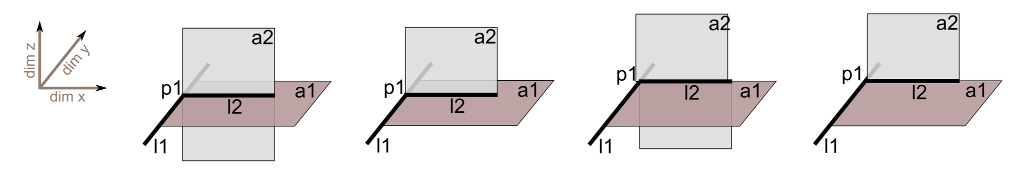


Figure 5: *Four models equivalent in $CODI_{\downarrow}$ but not equivalent in $CODI_{\downarrow}B$. Their extensions of $BCont$ differ: in the left model neither $BCont(l2,a1)$ nor $BCont(l2,a2)$, in the middle models one of them holds, and in the right model both hold.*

(BC-A1) $BCont(x,y) \to \exists z\,[SC(y,z) \wedge Cont(x,y) \wedge Cont(x,z)]$ (boundaries separate two distinct entities)
(BC-A2) $SC(x,y) \wedge MaxDim(x) \wedge Cont(z,x) \wedge Cont(z,y) \to BCont(z,x)$ (necessarily in boundary of x)
(BC-A3) $SC(x,y) \wedge P(x,v) \wedge Cont(y,v) \wedge Cont(z,x) \wedge Cont(z,y) \to BCont(z,x)$ (necessarily in boundary of x)
(BC-A4) $P(x,v) \wedge P(y,v) \wedge SC(x,y) \wedge Cont(z,x) \wedge Cont(z,y) \wedge z \prec_{dim} v \to \neg BCont(z,v)$ (not in boundary of v)
(BC-A5) $C(x,y) \wedge Con(x) \wedge Con(y) \wedge \neg Cont(x,y) \wedge \neg Cont(y,x) \wedge P(x,v) \wedge Cont(y,v) \to \exists z[BCont(z,x) \wedge Cont(z,y)]$ (generalized Jordan Curve Theorem)
(BC-T1) $BCont(x,y) \to Cont(x,y) \wedge x <_{dim} y$ ('thin' boundary)

⇒ MMT with interior-boundary distinction: $CODI_{\downarrow}B$

Definable relation: INTERIOR CONTAINMENT

$ICont(x,y) \leftrightarrow Cont(x,y) \wedge \forall z[Cont(z,x) \to \neg BCont(z,y)]$
…x is contained in the interior of y

Definable relation: TANGENTIAL CONTAINMENT

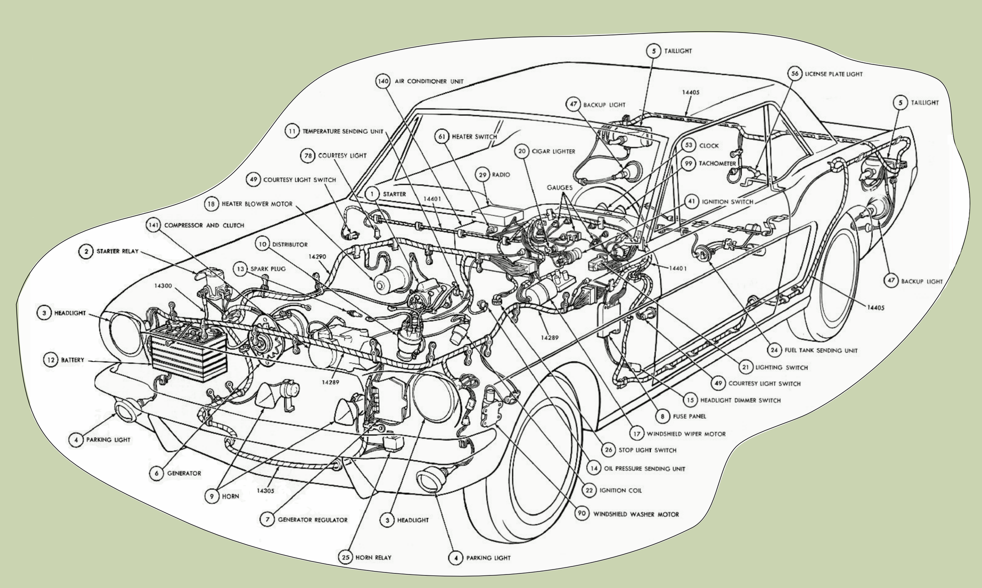
$TCont(x,y) \leftrightarrow Cont(x,y) \wedge \exists z[Cont(z,x) \wedge BCont(z,y)]$
…x is tangentially contained in y

Theorem 2 *$\forall x, y \in dom(M)$ in a model M of $CODI_{\downarrow}B$, $Cont(x,y) \leftrightarrow$ exactly one of $ICont(x,y)$ or $TCont(x,y)$.*

7. Result: 9-intersections definable in $CODI_{\downarrow}B$

- Can semantically integrate logical theories of space with interior-boundary distinction
- Can fit those into the hierarchy of logical theories of space

	y°	∂y	y^{-}
x°	$\exists z[ICont(z,x) \wedge ICont(z,y)]$	$\exists z[ICont(z,x) \wedge BCont(z,y)]$	$\neg Cont(x,y)$
∂x	symm.	$\exists z[(BCont(z,x) \wedge BCont(z,y)]$	$\wedge\exists z[BCont(z,x) \wedge \neg Cont(z,y)]$
x^{-}	symm.	symm.	$\exists z[(\neg Cont(z,x) \wedge \neg Cont(z,y)]$



Example 3. Parts and wiring of a 1964 Ford Mustang as a simplified 3D CAM drawing.

8. References

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