# Exponential lower bounds for the Tree-like Hajós Calculus *

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**Abstract**

The *Hajós Calculus* is a simple, nondeterministic procedure which generates precisely the class of non-3-colorable graphs. In this note, we prove exponential lower bounds on the size of *tree-like* Hajós constructions.

## 1 Introduction

The Hajós calculus is a simple, nondeterministic procedure for generating the class of graphs that are not $k$-colorable [Haj]. Mansfield and Welsh [MW] have posed the problem of determining the complexity of this procedure; in particular, it is an open problem whether or not there exists a polynomial-size Hajós construction for every non-3-colorable graph. Because graph 3-colorability is $NP$-complete, if there were polynomial-size Hajós constructions of all non-3-colorable graphs, then $NP = coNP$, so we expect that the Hajós calculus is not polynomially bounded. However, there has been very little progress toward a proof of this conjecture, despite considerable effort.

Pitassi and Urquhart [PU] have recently shown that the Hajós calculus is polynomially bounded if and only if Extended Frege proof systems are polynomially bounded. This result shows that the complexity

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†Supported by Scientific Research Grant, Ministry of Education, japan, No. 02302047 and No. 04650318.

‡Research performed at UCSD and supported by an NSF postdoctoral fellowship.

problem for the Hajós calculus is very difficult, since Extended Frege systems are a very powerful class of proof systems for the propositional calculus, and no techniques currently exist which appear adequate to prove super-polynomial lower bounds for them.

In this paper, we introduce the tree-like Hajós calculus, analogous to tree-like propositional proofs, and prove exponential lower bounds for tree-like Hajós constructions. This is somewhat surprising because in the setting of propositional Frege proofs, it is known that restricting oneself to tree-like proofs does not lead to significantly larger proofs. More precisely, Krajíček [K] has shown that any Frege proof can be efficiently simulated by a tree-like Frege proof. In contrast to this result, our lower bound shows that the Hajós calculus cannot be polynomially simulated by the tree-like Hajós calculus. This loss of efficiency when we restrict to tree-like Hajós derivations can be compared to similar losses in efficiency in weak propositional proof systems, such as Resolution and cut-free Gentzen systems [Urq].

# 2 The Hajós calculus

We define a graph $G$ to be a pair $(V, E)$, where $V$ is a finite set of positive integers, and $E$ a set of unordered pairs of elements of $V$. The *size* of a graph $G$, $|G|$, is the number of its edges. A 3-coloring of a graph is an assignment of one of 3 distinct colors to each of the vertices of the graph. A graph is 3-colorable if and only if there exists a 3-coloring of the graph such that no two adjacent vertices receive the same color.

The Hajós calculus is a collection of initial graphs, together with a finite collection of rules which allows us to derive precisely the class of non-3-colorable graphs. The set of initial graphs in the Hajós calculus contains all graphs isomorphic to $K_4$. There are three rules for generating new graphs:

(1) (Vertex/Edge Introduction) Add (any number of) vertices and edges.

(2) (Join rule) Let $G_1$ and $G_2$ be disjoint graphs, $a_1$ and $b_1$ adjacent vertices in $G_1$, and $a_2$ and $b_2$ adjacent vertices in $G_2$. Construct the graph $G_3$ from $G_1 \cup G_2$ as follows. First, remove edges $[a_1, b_1]$ and $[a_2, b_2]$; then add an edge $[b_1, b_2]$; lastly, contract vertices $a_1$ and $a_2$ into a single vertex, named $a_1$.

(3) (Contraction rule) Contract two nonadjacent vertices into a single vertex, and remove any resulting duplicated edges; the single vertex can be either of the two original vertices.

A construction of a graph $G$ in the Hajós calculus is a sequence of graphs, ending with $G$, such that every graph in the sequence is either an initial graph or follows from previous graphs by one of the above rules. The *size* of a construction is the sum of the sizes of all graphs in the construction. The Hajós calculus is *polynomially bounded* if there exists a polynomial $p$ such that every non-3-colorable graph $G$ has a Hajós construction of size at most $p(|G|)$. In [PU], it was established that the Hajós calculus is polynomially bounded if and only if Extended Frege systems are polynomially bounded.

A Hajós construction of $G$ can be visualized as a directed acyclic graph, with nodes labelled by graphs, and where the directed edges describe which graphs are used to derive subsequent graphs. A construction is *tree-like* if the underlying directed acyclic graph is a tree; in other words, if each intermediate graph in the construction is only used once. Clearly, any construction can be transformed into a tree-like construction, but the size of the tree-like construction may be substantially larger than the original one.

Pitassi and Urquhart defined the following reformulation of the Hajós calculus, and proved that the set of graphs having polynomial-size Hajós constructions is equal to the set of graphs having polynomial-size $\mathcal{HC}$ constructions. The system $\mathcal{HC}$ has the same set of initial graphs, as well as rules (1) and (3) of the Hajós calculus, but now rule (2) of the Hajós calculus is replaced by the following rule:

(2) (Edge Elimination rule) Let $G_1$ and $G_2$ be two graphs with common vertex set $\{v_1, ..., v_n\}$ which are identical except that $G_1$ contains edges $[v_1, v_2]$ and $[v_2, v_3]$ and not $[v_1, v_3]$, whereas $G_2$ contains edges $[v_1, v_2]$ and $[v_1, v_3]$ and not $[v_2, v_3]$. Then from $G_1$ and $G_2$, we can construct the graph $G_3$ which is graph $G_1$ with edge $[v_2, v_3]$ removed.

# 3 Propositional Proof Systems

A *propositional proof system* is a collection of initial propositional formulas (axioms), together with a finite set of rules which allow us to derive new propositional formulas. A propositional proof system is a

*Frege system* if its axioms consist of all instances of a finite number of tautologies, and it has a finite number of inference rules of a certain form which are sound (i.e. preserve truth). For a more in-depth treatment of propositional proof systems and their relative efficiency, see [CR]. A *proof* of a formula $f$ in a particular proof system is a sequence of formulas ending with $f$ such that each formula in the sequence is either an axiom instance, or follows from previous formulas by one of the inference rules. The *size* of a proof is defined to be the number of occurrences of symbols in it.

## 3.1 Frege systems, Bounded-depth Frege, Renaming

For concreteness, we will work with the following Frege system, $\mathcal{F}$, described in [PU]. However, we could work in any Frege system because it was shown in [CR] that every two Frege systems are $p$-equivalent. The underlying formulas in $\mathcal{F}$ are boolean formulas over the basis $\wedge$, $\vee$ and $\neg$, where $\wedge$ and $\vee$ are unbounded-fanin boolean functions, and $\neg$ is defined as usual. $\mathcal{F}$ will be slightly nonstandard in that it will be a *refutation* system for unsatisfiable formulas in conjunctive normal form. A refutation in $\mathcal{F}$ of an unsatisfiable formula, $A$, is a sequence of formulas, ending with $A$ such that each formula is either an axiom instance or follows from previous formulas by a rule. It is easy to interpret our system $\mathcal{F}$ as a proof system for tautological formulas by simply replacing all $\wedge$'s by $\vee$'s and all $\vee$'s by $\wedge$'s in the following axiom and rules. $\mathcal{F}$ has a single axiom schema, $p \wedge \overline{p}$, and the following rules. In addition, the associative and commutative laws for disjunction and conjunction will be applied tacitly. The symbol $\Leftrightarrow$ indicates that the rule may be applied either from left to right, or from right to left.

$(R0)$ $A \wedge A \wedge B \Rightarrow A \wedge B$, and $A \wedge A \Rightarrow A$,

$(R1)$ $A \Rightarrow A \wedge B$,

$(R2)$ $(A \vee B) \wedge C \Rightarrow A \wedge C$,

$(R3)$ $(A \wedge C), (B \wedge C) \Rightarrow (A \vee B) \wedge C$,

$(R4)$ $A \wedge l, B \wedge \overline{l} \Rightarrow A \wedge B$,

$(R5)$ $A \vee (B_1 \wedge ... \wedge B_k) \Leftrightarrow (A \vee B_1) \wedge ... \wedge (A \vee B_k)$,

$(R6)$ $A \wedge (B_1 \vee ... \vee B_k) \Leftrightarrow (A \wedge B_1) \vee ... \vee (A \wedge B_k)$,

$(R7)$ $\neg(A_1 \wedge ... \wedge A_k) \Leftrightarrow \neg A_1 \vee ... \vee \neg A_k$,

$(R8)$ $\neg(A_1 \vee ... \vee A_k) \Leftrightarrow \neg A_1 \wedge ... \wedge \neg A_k$,

$(R9)$ $\neg\neg A \Leftrightarrow A$.

$(R10)$ $0 \wedge A \Leftrightarrow 0$ and $1 \vee A \Leftrightarrow 1$.

The *depth* of a formula is the depth of the boolean tree which represents the formula. We define a depth-$d$ proof in $\mathcal{F}$ of a formula, $A$, to be a proof in $\mathcal{F}$ where each formula occuring in the proof has depth at most $d$. Alternatively, the *depth-$d$ Frege system*, $\mathcal{F}_d$, is the system $\mathcal{F}$, where each formula is restricted to have depth at most $d$.

Another rule used in propositional proofs is the *renaming* that allows us to derive $A[p/q]$ from $A$, where $A[p/q]$ is the formula obtained from $A$ by uniformly substituting the new variable $p$ for the variable $q$ in $A$. If $\mathcal{G}$ is a Frege system, the system which results from $\mathcal{G}$ by adding the renaming rule will be denoted by $\mathcal{RG}$. It is a surprising result due to Buss [Buss] that if we add the Renaming rule to a Frege system, the resulting system is as powerful as Extended Frege. On the other hand, the following lemma shows that if we restrict attention to *tree-like* Frege proofs, then adding the Renaming rule is completely powerless.

**Lemma 1** *Any size $S$, depth $d$ tree-like proof in $\mathcal{RF}$ can be converted into a size $O(S)$, depth $d$ tree-like proof in $\mathcal{F}$ (without Renaming).*

**Proof.** We will prove that any tree-like proof with $r > 0$ renaming instances can be converted into a tree-like proof of the same size with $r-1$ renaming instances. Let $P$ be a tree-like proof of $f$ and let $A(x) \Rightarrow A(y)$ be the first renaming instance in $P$. To eliminate this renaming rule, we uniformly replace the variable $x$ by the variable $y$ in the subproof of $A(x)$, yielding a proof of $A(y)$ of the same size as the original proof of $A(x)$. Let $P'$ be the sequence of formulas in $P$, but with the subproof of $A(y)$ replaced by the above sequence of formulas. Because the original proof was tree-like, $A(x)$ must only be used once (to derive $A(y)$), and therefore, $P'$ is still a tree-like proof of $f$, but with $r-1$ renaming instances. We can now proceed inductively to convert any tree-like proof with $r$ renaming instances into a tree-like proof with zero renaming instances. $\Box$

# 4 Simulation Results

Using reductions between $CNF$ unsatisfiability and graph non-3-colorability, we can translate the rules and axioms of the Hajós calculus to obtain a propositional refutation system, and similarly, we can translate the rules and axioms of a propositional refutation system to obtain a graph calculus. Below we review the translations described in [PU] between propositional formulas and graphs.

## 4.1 Translation from Propositional Formulas to Graphs

Garey, Johnson and Stockmeyer [GJS] showed that $3SAT$ is reducible to 3-Colorability, by obtaining a polynomial-time, many-one transformation from $3CNF$ formulas to graphs. Here, we slightly modify their construction so that it is a transformation from arbitrary $CNF$ formulas to graphs. Let $f$ be a $CNF$ formula, $f = C_1 \wedge C_2 \wedge ... \wedge C_p$, with underlying variables $x_1, ..., x_n$. For notational convenience, let $C_i = (l_1^i \vee l_2^i \vee ... \vee l_q^i)$. Then $Graph(f)$ consists of the following nodes, $N$:

$$\begin{aligned} N &= \{v1, v2, v3\} \cup \{x_i, \overline{x_i} \mid 1 \leq i \leq n\} \cup \\ &\quad \{a_j^i, b_j^i, c_j^i \mid 1 \leq i \leq p, 1 \leq j \leq q-1\}. \end{aligned}$$

The set $E$ of edges for $Graph(f)$ is given by:

$$\begin{aligned} E = &\{[v1, v2], [v2, v3], [v1, v3]\} \;\cup\; \{[x_i, \overline{x_i}] | 1 \leq i \leq n\} \\ &\cup \{[v3, x_i], [v3, \overline{x_i}] | 1 \leq i \leq n\} \\ &\cup \{G_{C_i} \mid 1 \leq i \leq q\}, \end{aligned}$$

where $G_{C_i}$ consists of the following edges:

$$\begin{aligned} G_{C_i} = &\{[a_j^i, b_j^i], [b_j^i, c_j^i], [a_j^i, c_j^i] \mid 1 \leq j \leq q-1\} \\ &\cup \{[c_j^i, a_{j+1}^i] \mid 1 \leq j \leq q-2\} \\ &\cup \{[l_1^i, a_1^i], [c_{q-1}^i, v2], [c_{q-1}^i, v3]\} \\ &\cup \{[l_j^i, b_{j-1}^i] \mid 2 \leq j \leq q\} \end{aligned}$$

When $C_i$ is just a single literal, $l$, then the construction for $G_{C_i}$ simply puts $l$ in place of $c_{q-1}$. A key property of the graph $G$ is:

for each $C_i = (l_1^i \vee l_2^i \vee ... \vee l_q^i)$, a proper 3-coloring to $l_1^i, ..., l_q^i$ can be extended to a proper 3-coloring of the underlying vertices of $G_{C_i}$ if and only if at least one of $l_1^i, ..., l_q^i$ has the same color as $v1$. (The color $v1$ thus represents "true".) This observation leads to the following lemma, which was proven in [GJS].

**Theorem 2 (GJS)** *$f$ is satisfiable if and only if $Graph(f)$ is 3-colorable.*

### 4.2 Translation from Graphs to Propositional Formulas

Conversely, if $G$ is any graph, then there exists a propositional 3CNF formula, $Form(G)$ with the property that $G$ is 3-colorable if and only if $Form(G)$ is satisfiable. Following [PU], we define the propositional statement, $Form(G)$, with underlying variables $\{R_x, B_x, G_x \mid x \in V\}$, which expresses "G is 3-colorable" as follows:

$$\bigwedge_{x \in V} (\overline{R}_x \vee \overline{B}_x \vee \overline{G}_x)(R_x \vee B_x \vee G_x)$$
$$(R_x \vee \overline{B}_x \vee \overline{G}_x)(\overline{R}_x \vee \overline{B}_x \vee G_x)(\overline{R}_x \vee B_x \vee \overline{G}_x)$$
$$\bigwedge_{[x,y] \in E} (\overline{R}_x \vee \overline{R}_y)(\overline{G}_x \vee \overline{G}_y)(\overline{B}_x \vee \overline{B}_y).$$

**Lemma 3** *$G$ is 3-colorable if and only if $Form(G)$ is satisfiable.*

## 5 The Main Theorem

**Theorem 4** *(Main Theorem) There exists a family of graphs that requires exponential-size tree-like $\mathcal{HC}$ constructions.*

The general idea of the proof is to show that any tree-like $\mathcal{HC}$ construction of a graph $G$, when translated into a propositional refutation of $Form(G)$, can be simulated efficiently by a small-depth Frege proof. We then obtain the lower bound by using known exponential lower bounds on the size of small-depth Frege proofs of the propositional pigeonhole principle.

By using the translations between propositional formulas and graphs described above, we can translate the Hajós calculus to obtain a new propositional refutation system. Define $Form(\mathcal{HC})$ to be the propositional refutation system obtained by translating the rules and axioms of $\mathcal{HC}$. Specifically, $Form(\mathcal{HC})$ has one axiom: $Form(K_4)$ and three rules, obtained by translating the three rules of $\mathcal{HC}$. Tree-like $Form(\mathcal{HC})$ is the refutation system $Form(\mathcal{HC})$ but where the underlying directed acyclic graph is required to be tree-like.

**Lemma 5** *Any size $S$ tree-like proof in $Form(\mathcal{HC})$ can be converted into a size $O(S)$ tree-like proof in $\mathcal{F}_2$.*

**Proof.** Pitassi and Urquhart [PU] showed that all rules with the exception of the contraction rule can be efficiently simulated by a a depth-2 Frege system. After the conversion to propositional formulas, the contraction rule becomes precisely the renaming rule. In the context of dag-like proof systems, this rule is powerful enough to allow an efficient simulation of arbitrary substitution. (This is the key observation used to show that if the Hajos calulus is polynomially bounded, then so are Extended Frege proofs.) However, in the context of tree-like proofs, the renaming rule is powerless. We proceed more formally below.

Let $P$ be a tree-like proof of $f$ in $Form(\mathcal{HC})$; we will show how to modify $P$ to obtain a depth-2 Frege proof, $P'$, of $f$. Tree-like $Form(\mathcal{HC})$ has one axiom, $Form(K_4)$, and three rules. It was shown in [PU] that any instance of the axiom $Form(K_4)$ can be derived by a constant-sized tree-like proof in $\mathcal{F}_2$. They also showed that any application of rule (1) or (2) can also be linearly simulated by tree-like $\mathcal{F}_2$. (These simulations are straightforward.) Given $P$, we first apply this translation to obtain a new proof $P'$ where all axioms and rules (1) and (2) of $Form(\mathcal{HC})$ have been replaced by their $\mathcal{F}_2$ simulations. The final rule (3) is the renaming rule, and thus $P'$ is a polynomial-sized, depth-2 proof of $f$ in $\mathcal{RF}$. Furthermore because $P$ was tree-like, $P'$ will also be tree-like. Now by Lemma 1, all renaming instances can be removed to obtain a polynomial-size, depth-2, tree-like Frege proof of $f$. $\Box$

**Lemma 6** *Let $f$ be an unsatisfiable CNF formula. Then there exists a $c > 0$ such that if there exists a size $S$, depth $k$ proof of $Form(Graph(f))$ in $\mathcal{F}$, then there exists a size $S^c$, depth $k+3$ proof of $f$ in $\mathcal{F}$.*

**Proof.** The above lemma is implicit in [PU], but we prove it here for completeness. Let $f = C_1 \wedge C_2 \wedge .. \wedge C_q$ be a CNF formula, with underlying variables $\{x_1, ..., x_n\}$. The proof follows the basic idea behind the [GJS] proof that if $Graph(f)$ is non-3-colorable, then $f$ is unsatisfiable. We will define a particular coloring for $Graph(f)$, based on the propositional variables $\{x_1, ..., x_n\}$.

Let $C_i = (l_1 \vee l_2 \vee ... \vee l_q)$ be a clause in $f$, where each $l_k$ is a literal over $\{x_1, ..., x_n\}$. We will first define variable substitutions for all $1 \leq j \leq q-1$, and $1 \leq k \leq q$.

$$R_{l_k} \leftrightarrow l_k;\ B_{l_k} \leftrightarrow \overline{l_k};\ G_{l_k} \leftrightarrow 0$$
$$R_{\overline{l_k}} \leftrightarrow \overline{l_k};\ B_{\overline{l_k}} \leftrightarrow l_k;\ G_{\overline{l_k}} \leftrightarrow 0$$
$$R_{a^i_j} \leftrightarrow (\bar{l}_1 \wedge ... \wedge \overline{l_{j+1}});\ B_{a^i_j} \leftrightarrow (l_1 \vee ... \vee l_j) \wedge \overline{l_{j+1}};$$
$$\quad G_{a^i_j} \leftrightarrow l_{j+1}$$
$$R_{b^i_j} \leftrightarrow 0;\ B_{b^i_j} \leftrightarrow l_{j+1};\ G_{b^i_j} \leftrightarrow \overline{l_{j+1}};$$
$$R_{c^i_j} \leftrightarrow (l_1 \vee ... \vee l_{j+1});\ B_{c^i_j} \leftrightarrow (\overline{l_1} \wedge ... \wedge \overline{l_{j+1}});$$
$$\quad G_{c^i_j} \leftrightarrow 0;$$
$$R_{v1} \leftrightarrow 1;\ B_{v1} \leftrightarrow 0;\ G_{v1} \leftrightarrow 0$$
$$R_{v2} \leftrightarrow 0;\ B_{v2} \leftrightarrow 1;\ G_{v2} \leftrightarrow 0$$
$$R_{v3} \leftrightarrow 0;\ B_{v3} \leftrightarrow 0;\ G_{v3} \leftrightarrow 1$$

Let $P$ be the size $S$, depth $k$ Frege proof of $Form(Graph(f))$. We will show how to obtain a depth $k+3$, polynomial size Frege proof of $f$ from $P$. We first uniformly make the variable substitutions described above throughout the proof $P$. This results in a new sequence of formulas such that the final formula, $f'$, is $Form(Graph(f))$, but with the above variable substitutions. It is not too hard to check that this new sequence of formulas is still a valid Frege proof (of $f'$). Furthermore, this new proof has polynomial size and depth $k+3$ because the substitutions are all constant-size with depth at most 3. It is left to show that from a proof of $f'$, we can obtain a small-depth, polynomial-size proof of $f$. We will argue this informally. Consider the clauses in $f'$. It can be checked that all clauses with the exception of clauses of the form $(\overline{B_{c^i_{q-1}}} \vee \overline{B}_{v2})$ can be simplified to the constant 1 by elementary reasoning. For each $i$, $1 \leq i \leq q$, the only clause remaining is $(\overline{B_{c^i_{q-1}}} \vee \overline{B}_{v2})$, which becomes $\overline{B_{c^i_{q-1}}} = C_i$ after applying the substitution. Combining all of the $C_i$'s, we can then easily derive $f$. This final derivation of $f$ from $f'$ does not increase the depth, and the size is linear in the size of $f'$. □

The propositional pigeonhole principle, $PHP_n$, is a very well-known tautology stating that there is no one-to-one map from $n+1$ pigeons to $n$ holes. We encode $PHP_n$ using variables $P_{ij}$, $1 \leq i \leq n+1$, $1 \leq j \leq n$. Using these variables, $PHP_n$ can be written as the disjunction of the following clauses:

$$\bigwedge\{\neg P_{ij} \mid 1 \leq j \leq n\},\ 1 \leq i \leq n+1;$$

$$\bigwedge\{P_{ik}, P_{jk}\},\ i \neq j,\ 1 \leq i,j \leq n+1,\ 1 \leq k \leq n.$$

Note that the size of $PHP_n$ is $O(n^3)$. The following theorem shows that any small-depth Frege proof of $PHP_n$ requires exponential size.

**Theorem 7** *([BIKPPW]) For $n$ sufficiently large, any depth-$d$ Frege proof of the negation of the propositional pigeonhole principle, $\neg PHP_n$, requires size at least $2^{\Omega(n^{6^{-d}})}$.*

Using the above lemmas and theorem, we will now prove our main theorem.

**Proof of Main Theorem.** We will show that for all $n$ sufficiently large, any tree-like $\mathcal{HC}$ construction of $Graph(\neg PHP_n)$ requires exponential size. Assume for sake of contradiction that there exists a size $S$, $S < 2^{\Omega(n^{6^{-d}})}$, tree-like $\mathcal{HC}$ proof of $Graph(\neg PHP_n)$, where $d = 5$. Then there exists a size $S^{O(1)}$, depth-2 Frege proof of $Form(Graph(\neg PHP))$, by Lemma 5. Now by Lemma 6, there exists a size $S^{0(1)}$, depth-5 Frege proof of $\neg PHP_n$. But this contradicts Theorem 7. □

**Corollary 8** *The Hajós calculus cannot be polynomially simulated by the tree-like Hajós calculus.*

**Proof.** Cook and Reckhow [CR] have shown that there are polynomial-size Extended Frege proofs of the propositional pigeonhole principle. By the simulation results in [PU], it follows that $Graph(\neg PHP_n)$ has a polynomial-size Hajós construction. Thus, $Graph(\neg PHP_n)$ is a family of graphs that have polynomial-size Hajós constructions, but by the proof of the above Theorem, do not have polynomial-size tree-like Hajós constructions. □

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