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THE PSPACE-COMPLETENESS OF BLACK-WHITE PEBBLING*

PHILIPP HERTEL† AND TONIANN PITASSI†

Abstract. The complexity of the black-white pebbling game has remained an open problem for 30 years. In this paper we show that the black-white pebbling game is PSPACE-complete.

Key words. complexity, pebbling, PSPACE-completeness, black pebbling, black-white pebbling

AMS subject classification. 68-XX

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1. Introduction. The black-white pebbling game was introduced by Cook and Sethi in 1976 [CS76] in an attempt to separate P from NL. The black-white pebbling game received considerable attention throughout the next decade due to its numerous applications including VLSI design, compilers, and algebraic complexity. In 1983 determining its complexity was rated as "An Open Problem of the Month" in Johnson's *The NP-Completeness Column* [Joh83]. An excellent survey of pebbling results from this period can be found in Pippenger [Pip80]. Recently, there has been a resurgence of interest in pebbling games due to their links with propositional proof complexity [BS02, ET01, Nor06, HU07]. In this paper we prove that the black-white pebbling game is PSPACE-complete.

The black-white pebbling game was preceded by the black pebbling game, which has also been widely studied [Pip80]. Let $\mathcal{G} = (V, E)$ be a directed acyclic graph (DAG) with one distinguished output node, s. In the black pebbling game, a player tries to place a pebble on s while minimizing the number of pebbles placed simultaneously on $\mathcal{G}$. The game is split into distinct steps, each of which takes the player from one pebbling configuration to the next. Initially, the graph contains no pebbles, and each subsequent configuration follows from the previous by one of the following rules:

- At any point the player may place a black pebble on any source node v.
- At any point the player may remove a black pebble from any node v.
- For any node v, if all of v's predecessors have pebbles on them, then the player may place a black pebble on v, or may slide a black pebble from a predecessor u to v.

The black pebbling game models deterministic space-bounded computation. Each node models a result, and the placement of a black pebble on a node represents the deterministic computation of the result from previously computed results. A sequence of moves made by the player is called a *pebbling strategy*. If a strategy manages to pebble s using no more than k pebbles, then that strategy is called a k-pebbling strategy.

The black-white pebbling game is a more powerful extension of the black pebbling game that includes white pebbles, which behave in a dual manner to the black pebbles. As before, the player attempts to place a black pebble on s while minimizing the

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†Department of Computer Science, University of Toronto, Toronto M5S 3G4, ON, Canada (philipp@cs.toronto.edu, toni@cs.toronto.edu).

number of pebbles placed simultaneously on $\mathcal{G}$ at any time. However, unlike the black pebbling game, the black-white pebbling game does not end until every node other than s is empty. So the player must remove any outstanding pebbles once s has been reached. The black-white pebbling game extends the black pebbling game with the addition of the following rules:

- At any point the player may remove a white pebble from any source node v.
- At any point the player may place a white pebble on any node v.
- For any node v with a white pebble on it, the player may slide the pebble to an empty predecessor u if all of v's other predecessors are pebbled, or the player may remove the white pebble if all of v's predecessors are pebbled.
- The game ends when s contains a black pebble and every other node is empty.

As before, the placement of each black pebble is meant to model the derivation of a deterministically computed result, while the placement of each white pebble is meant to model a nondeterministic guess, whose verification requires all of its antecedents to be derived.

In 1978, Lingas showed that a generalization of the black pebbling game, played on monotone circuits instead of DAGs, is PSPACE-complete [Lin78] via a reduction from the quantified boolean satisfiability problem. This was a surprising result since the PSPACE-complete games of the time involved two players and it was clear how the alternation between them led to each game's high complexity. In 1980, Gilbert, Lengauer, and Tarjan elaborated on the basic structure of Lingas's construction to prove the PSPACE-completeness of the black pebbling game on DAGs [GLT80]. The main difficulty in moving from monotone circuits to the more restricted class of DAGs is the creation of an OR widget using only the global bound on the number of permissible pebbles and nodes which act like AND gates.

While the above results settle the complexity of black pebbling, determining the complexity of black-white pebbling has resisted numerous attempts. In contrast to black pebbling, white pebbles allow a much richer choice of strategies since they can be placed anywhere on the graph regardless of previous pebble placements, thereby breaking up the straight inductive pattern obvious in all pure black strategies. Although the black pebbling number of a graph is never more than a square of the black-white pebbling number [Hei81], the addition of white pebbles lowers the pebbling number of many graphs [Wil85, KS88]. Unfortunately, the constructions used for the previous PSPACE-completeness results are both examples of such graphs. As a result, neither can be used to differentiate between true and false quantified boolean formulas (QBFs) in the presence of white pebbles.

We settle Johnson's open problem by building on the construction of [GLT80] to prove the PSPACE-completeness of the black-white pebbling game. The same reduction also provides an infinite family of graphs which require exponential time to minimally black-white pebble, but can be pebbled in linear time if we use just two pebbles more than the minimum. This results in a time/space trade-off similar to that proved in [GLT80] for pure black pebbling.

The rest of this paper is organized as follows. Section 2 contains definitions. Section 3 contains an overview of our proof and a detailed description of our reduction. In section 4 we prove the upper bound, and the lower bound is proved in section 5. In section 6 we present exponential time/space trade-offs for black-white pebbling that follow from our proof, as well as PSPACE-completeness of another version of the black-white pebbling game. Finally, we conclude with a few open problems in section 7.

2. Terminology.

2.1. General definitions.

This section contains definitions which are used continuously throughout the paper. Our main result is a reduction from the quantified boolean satisfiability (QSAT) problem. QSAT is the archetypal PSPACE-complete problem [SM73]. It can be viewed as a generalization of SAT, where conjunctive normal form (CNF) formulas are replaced by QBF formulas and satisfying assignments are replaced by more complicated QSAT models, which are essentially sets of satisfying assignments which respect the quantifiers of the QBF. For example, consider the QBF formula $f = \exists x_1 \forall x_2 \exists x_3 (x_1 \vee x_2) \wedge (\neg x_3 \vee \neg x_2)$. A QSAT model for this formula is a set of truth assignments corresponding to a tree. The root of the tree is labelled with x_1. Since it is an existentially quantified variable, there is only one edge out of this root node, and it is labelled by $x_1 = 1$. The next node is labelled by x_2. Since x_2 is universally quantified, there are two edges out of x_2; the left edge is labelled by $x_2 = 0$, and the right edge is labelled by $x_2 = 1$. The left edge ($x_2 = 0$) goes to a node labelled by x_3, and here we choose $x_3 = 1$; the right edge ($x_2 = 1$) goes to a node also labelled by x_3, and here we choose $x_3 = 0$. Thus the whole tree defines two truth assignments, each of which satisfies the formula, and together these assignments form a QSAT model for f. For a formal definition of quantified boolean satisfiability, please see any standard complexity text such as [Pap94], [Sip96], or [DK00]. Our reduction transforms an instance of the QSAT problem into a DAG $\mathcal{G}$.

Definition 2.1 (in-degree/out-degree). *The* in-degree *of a node v is the number of directed edges directed from other nodes to v. The* out-degree *of a node v is the number of directed edges directed from v to other nodes.*

Definition 2.2 (predecessor/successor). *A node v is a* predecessor *of node u if there is a directed edge from v to u. A node u is a* successor *of node v if v is a predecessor of u.*

Definition 2.3 (ancestor/descendant). *We say that a node v is a* descendant *of another node u if there is a path from u to v. We say that a node u is an* ancestor *of another node v if v is a descendant of u. We say that v is a descendant of u along path ρ if there is a subpath of ρ from u to v. We say that u is an ancestor of v along path ρ if v is a descendant of u along ρ.*

2.2. Black-white pebbling definitions.

We now define terms which we use throughout the paper to discuss black-white pebbling. For each of the following definitions, let $\mathcal{G} = (V, E)$ be a DAG with distinguished output node s. The distinguished output node will also be referred to as the target node, or sink node.

Definition 2.4 (black-white pebbling moves). *The black-white pebbling game has six rules which control how pebbles can be moved on $\mathcal{G}$.*

1. *For any node v, if each of v's predecessors contains a pebble of either color, then a black pebble can be placed on v. We call this move a* black pebble placement onto v.
2. *For any node v, if some predecessor u of v contains a black pebble and each of v's other predecessors contains a pebble of either color, then the black pebble can be slid from u to v. We call this move a* black pebble slide from u to v.
3. *For any node v which contains a black pebble, the black pebble can be removed from v. We call this move a* black pebble removal from v.
4. *For any node v which contains a white pebble, if each of v's predecessors contains a pebble of either color, then the white pebble can be removed from v. We call this move a* white pebble removal from v.

5. *For any node v which contains a white pebble, if some predecessor u of v is empty and each of v's other predecessors contains a pebble of either color, then the white pebble can be slid from v to u. We call this move a* black pebble slide from v to u.
6. *For any node v, a white pebble can be placed on v. We call this move a* white pebble placement onto v.

The state of the game at any moment in time is captured by the notion of a configuration. A configuration encodes which nodes contain black pebbles and which contain white pebbles at that time.

DEFINITION 2.5 (black-white pebbling configuration). *A* black-white pebbling configuration $\mathcal{M}[i]$ *for $\mathcal{G}$ is a pair of sets $(B[i], W[i])$, such that $B[i] \subseteq V$, $W[i] \subseteq V$, and $B[i] \cap W[i] = \emptyset$.*

A black white pebbling strategy is a sequence of configurations, each one following the last by a legal pebbling move. The strategy contains all of the information about how the pebbles move on $\mathcal{G}$ during the entire play of the game.

DEFINITION 2.6 (black-white pebbling strategy). *A* black-white pebbling strategy $\mathcal{S}$ *for $\mathcal{G}$ is a sequence of black-white pebbling configurations $\mathcal{M}[t^{start}], \dots, \mathcal{M}[t^{end}]$, such that $\mathcal{M}[t^{start}] = (\emptyset, \emptyset)$, $\mathcal{M}[t^{end}] = (\{s\}, \emptyset)$, and for all t, $t^{start} \leq t < t^{end}$, $\mathcal{M}[t+1]$ follows from $\mathcal{M}[t]$ by a legal black-white pebbling move.*

The black-white pebbling number measures the space efficiency of a pebbling strategy.

DEFINITION 2.7 (black-white pebbling number). *The* black-white pebbling number of a black-white pebbling configuration $\mathcal{M}[i]$ *is $|B[i]| + |W[i]|$. The* black-white pebbling number of a black-white pebbling strategy $\mathcal{S}$ *is the maximum black-white pebbling number of any configuration of $\mathcal{S}$. If a strategy has pebbling number k, then it is referred to as a k-pebbling strategy. The* black-white pebbling number of a DAG $\mathcal{G}$ *is the minimum k such that $\mathcal{G}$ has a k-pebbling strategy. For any given k, if $\mathcal{G}$ has a k-pebbling strategy, then we say that $\mathcal{G}$ is k-pebblable.*

We can now formally define the black-white pebbling game.

DEFINITION 2.8 (black-white pebbling game). *Given a DAG $\mathcal{G} = (V, E)$ with distinguished output node s and a nonnegative integer k, $k \leq |V|$, the* black-white pebbling game *asks whether $\mathcal{G}$ is k-pebblable.*

As the definition of a black-white pebbling strategy suggests, pebbling moves, such as placements, removals, and slides, occur between configurations. Since configurations are indexed by discrete time units, moves actually occur during the transition from one time unit to the next. For convenience we would like to talk about moves as though they occur at time units, since it is much easier to state that a pebble is placed on a node at a time t, rather than always stating that a pebble is placed on a node between time $t-1$ and t. When we speak about pebbling moves we will therefore associate the action with the configuration to which the action transitions. This gives rise to the following definitions which we use throughout the proof.

DEFINITION 2.9. *We say that v* contains a black pebble *at time t^α if $B[t^\alpha]$ contains v. We say that v* contains a white pebble *at time t^α if $W[t^\alpha]$ contains v. We say that v* contains a pebble *at time t^α if it either contains a black pebble at t^α or contains a white pebble at t^α.*

DEFINITION 2.10. *We say that a node v of $\mathcal{G}$ is* black pebbled *at time t^α of a pebbling strategy $\mathcal{S}$ if $B[t^\alpha - 1]$ does not contain v and $B[t^\alpha]$ does. We say that v is* white pebbled *at time t^α if $W[t^\alpha - 1]$ does not contain v and $W[t^\alpha]$ does. We say that v is* pebbled *at time t^α if it is either black pebbled or white pebbled at time t^α.*

If a node is black pebbled at some time, it is always the result of a black pebble placement or a black pebble slide to the node, but it is sometime unimportant exactly which type of move resulted in the node containing the pebble, so we sometimes employ this more generic language. The same is true for white pebbling and pebbling in general. Similarly, we employ a generic term to capture all pebble removals, as it is sometime unimportant whether a pebble was removed by a removal move or by a slide from the node.

DEFINITION 2.11. *We say that a node v's pebble is* removed *at time t^α if v is contained either in $B[t^\alpha - 1]$ or $W[t^\alpha - 1]$ but is contained in neither $B[t^\alpha]$ nor $W[t^\alpha]$.*

DEFINITION 2.12. *We say that a black pebble is* slid *from v to its successor u at time t^α if $B[t^\alpha - 1]$ contains v and does not contain u, and $B[t^\alpha]$ contains u but not v. Similarly, we say that a white pebble is* slid *from v to its predecessor u at time t^α if $W[t^\alpha - 1]$ contains v and does not contain u, and $W[t^\alpha]$ contains u but not v.*

We use the follow term when discussing sets of nodes which simultaneously contain pebbles.

DEFINITION 2.13. *We say a set of nodes V is* simultaneously black pebbled *at time t^α if some member of V is black pebbled at t^α and all other members of V already contain black pebbles at t^α.*

A black-white pebbling strategy induces a set of pebble assignments, each of which records the interval between the time a pebble is initially placed on a node and the time it is subsequently removed.

DEFINITION 2.14. *A* pebble assignment *$P(v, t^1, t^2)$ records the interval during which a node contains a pebble. The node v contains a pebble continuously from time t^1 to $t^2 - 1$ inclusively, but does not contain a pebble at $t^1 - 1$ or t^2. If the pebble is never removed from v, as in the last assignment of a black pebble to the target, t^2 is infinity. We say that $P(v, t^1, t^2)$* starts *at t^1 and* ends *at t^2.*

Intuitively, if $P(v, t^1, t^2)$ is in the set of pebble assignments induced by some pebbling strategy $\mathcal{S}$, then in $\mathcal{S}$, the node v is pebbled at time t^1 and its pebble is removed subsequently at time t^2. Since two distinct pebbling moves are required to pebble a node and then to remove that node's pebble, for all pebble assignments $P(v, t^1, t^2)$, $t^1 < t^2$. We also note that the following inequalities ($t^3 \leq t^2$, $t^1 \leq t^4$) are used rather than equalities because of the sliding rule.

DEFINITION 2.15. *If v is a predecessor of u, then*

1. *if $P(u, t^3, t^4)$ is a black pebble assignment, we say that $P(v, t^1, t^2)$* supports *$P(u, t^3, t^4)$ if $t^1 < t^3 \leq t^2$ with* support time *t^3,*
2. *if $P(u, t^3, t^4)$ is a white pebble assignment, we say that $P(v, t^1, t^2)$* supports *$P(u, t^3, t^4)$ if $t^1 \leq t^4 < t^2$ with* support time *t^4.*

Intuitively, a pebble assignment supports a black pebble assignment if it allows the pebble to be placed, and a pebble assignment supports a white pebble assignment if it allows the pebble to be removed.

DEFINITION 2.16. *Let $\mathcal{S}$ be a k-pebbling strategy of $\mathcal{G}$. The* support DAG *$\mathcal{D}$ of $\mathcal{S}$ is formed by adding a node for each pebble assignment in $\mathcal{S}$ and then adding edges from each pebble assignment to the pebble assignments which it supports.*

A pebble assignment can only support pebble assignments to one of its successors. Therefore, since $\mathcal{G}$ is acyclic, so too is the support DAG associated with $\mathcal{S}$. The node associated with the pebble assignment to s is a sink in the support DAG for $\mathcal{S}$.

3. Overview of our reduction.

3.1. Proof overview. Formally, the black-white pebbling game takes as input a DAG $\mathcal{G}$ with a special target node s and an integer k and asks whether there is a

k-pebbling strategy for s in $\mathcal{G}$. We prove the following theorem.

THEOREM 1. *The black-white pebbling game is PSPACE-complete.*

We first show that the black-white pebbling game is in PSPACE.

LEMMA 2. *The glack-white pebbling game is in PSPACE.*

Proof. Given $(\mathcal{G}, k)$, we can easily guess a sequence of pebbling configurations that pebbles $\mathcal{G}$ with at most k pebbles, keeping at most two configurations in memory at any given time. Since each configuration requires at most polynomial space, this is an NPSPACE algorithm. Next, we appeal to Savitch's theorem to conclude that the black-white pebbling game is in (deterministic) PSPACE. □

To prove that the black-white pebbling game is PSPACE-hard, we will reduce from QSAT [SM73]. Given a QBF ψ over n variables, we will create a graph $\mathcal{G}$ with the property that ψ is in QSAT iff $\mathcal{G}$ has a $4n + 3$ black-white pebbling strategy.

Our construction is similar at a high-level to [GLT80], in which Gilbert, Lengauer, and Tarjan create a graph from a QBF with the property that the formula is in QSAT iff the graph has a pure k-black pebbling strategy for a specific k. The general idea behind their reduction is to have the black pebbling correspond to the exponential-time procedure that verifies that ψ is in QSAT. The graph is composed of two main parts: a linear chain of clause widgets followed by a linear chain of quantifier widgets. In all strategies which achieve the graph's minimum pebbling number, pebbles must be placed on certain special nodes in a way which corresponds to the lexicographically first truth assignment in the QSAT model for ψ. Since this assignment satisfies ψ's 3CNF, the player is able to successfully pebble through the clause widgets without exceeding the minimum pebbling number. The player can then begin to make progress through the quantifier widgets up to the first universal widget, say widget i. In order to pebble through this widget without exceeding the pebbling number, the player must leave a pebble on a "progress node" in widget i and then repebble the special nodes for the innermost i variables, thereby placing pebbles in a way which corresponds to the lexicographically second truth assignment in the QSAT model. The player can then pebble up through the clause widgets again, and this time use the pebble which was previously placed on the progress node to pebble through widget i, only to have his or her progress arrested at the next universal widget, at which point the process must repeat. Minimally black pebbling the graph corresponding to a true QBF with k universal quantifier widgets therefore requires 2^k time.

In this paper, we want to use ideas similar to those in [GLT80] to construct a graph from a QBF with the property that the formula is in QSAT iff the graph has a k black-white pebbling strategy. Unfortunately, the graphs used in all earlier constructions are easy to pebble once white pebbles are allowed, regardless of whether or not the QBF is in QSAT. Thus the main obstacle in proving hardness of black-white pebbling is to determine how to modify the construction so that white pebbles will be rendered useless. We accomplish this by building a graph which requires the player to use the maximum number of pebbles in *every* configuration of every optimal pure black strategy. The player must maintain a set of black pebbles which are temporarily fixed on the graph along with a wave of black pebbles which moves up the graph, picking up the fixed pebbles along the way. Together, the temporarily fixed set and the wave always contain the maximum number of pebbles, even though pebbles are constantly moving from the fixed set to the wave. The use of any white pebbles, which move down the graph in the opposite direction from the wave, will therefore necessarily lead to a suboptimal pebbling.

However, we run into trouble in the case of existentially quantified variables. The problem stems from the fact that for an existential quantifier widget, we want to be able to pebble up to that widget in either of two different ways—one corresponding to the variable being set to true, and the other way corresponding to the variable being set to false. Thus, there is an implicit OR in this argument. This difficulty was also overcome in [GLT80], in the more limited context of black pebbling. In the context of black-white pebbling, we have to simulate this implicit OR using only AND gates. Any way of doing this will necessarily involve two different pebblings, and it requires subtlety to accomplish this while still prohibiting white pebbles.

3.2. The reduction. To show that the black-white pebbling game is PSPACE-hard, we reduce from QSAT. In our presentation, a QBF $\psi = Q_n x_n Q_{n-1} x_{n-1} \cdots Q_1 x_1 F$, where F is a 3CNF containing m clauses over the n quantified variables $x_n, \dots, x_1$. We have inverted the numbering of the variables simply as a convenience in the proof. Given a QBF ψ, we produce a graph $\mathcal{G}$ whose target node s can be black-white pebbled using at most $4n+3$ pebbles iff ψ is in QSAT. Our construction is designed to penalize any use of white pebbles so that the optimal strategy is all black.

The graph which we construct is composed of $n+m$ widgets, one for each quantified variable and one for each clause in F. As in [GLT80], the quantifier widget for $Q_i x_i$ contains four vertices which represent the variable x_i; we call these nodes x_i, x_i', $\bar{x}_i$, $\bar{x}_i'$. They are divided into two *literal subwidgets*, with x_i' and x_i composing one such subwidget in which x_i' is a predecessor of x_i, and $\bar{x}_i'$ and $\bar{x}_i$ composing the other. We call x_i and $\bar{x}_i$ *top nodes*, and x_i' and $\bar{x}_i'$ *bottom nodes*. The location of pebbles on these four nodes corresponds to the truth value assigned to x_i by the current truth assignment which is being tested by the pebbling. If pebbles are on x_i and $\bar{x}_i'$, then the variable x_i is set to true. If pebbles are on x_i' and $\bar{x}_i$ or if pebbles are on x_i' and $\bar{x}_i'$, then the variable x_i is set to false. Our construction will never allow an assignment to place pebbles on both x_i and $\bar{x}_i$ in an optimal strategy. We will prove that the player can use only black pebbles to set these truth assignments, so if pebbles are on both bottom nodes, x_i' and $\bar{x}_i'$, the player can easily slide one up to put the graph into either the true or false configuration.

The construction of the quantifier widgets relies on a subwidget we call an i-slide. An i-slide is designed to severely restrict the player's pebbling strategies. An example of a 4-slide is shown in Figure 1. Once the bottom nodes of an i-slide are black pebbled, the i-slide strategy, where the bottom pebbles are slid up to the top nodes in the appropriate order, is the only way to pebble the top nodes without exceeding i pebbles.

DEFINITION 3.1. *An i-*slide *is a pair of sets (V, U) together with a set of edges that satisfy the following properties. V is a set of i nodes $v^1, v^2, \dots, v^i$, and U is a set of i nodes $u^1, u^2, \dots, u^i$ such that* (1) *v^j is the predecessor of all nodes v^k such that $k > j$;* (2) *u^j is the predecessor of all nodes u^k such that $k > j$;* (3) *u^j is the predecessor of all nodes v^k such that $k \le j$;* (4) *u^j has at least $i - j + 1$ predecessors from outside of V or U.*

Globally the construction is very much like that in [GLT80]. There are a number of nodes used to encode a truth assignment, which are predecessors to nodes in both clause widgets and quantifier widgets. The clause widgets are connected linearly and can only be pebbled within the space bound of $4n+3$ if the truth assignment encoded by the current pebbling configuration satisfies ψ's CNF part, F. The quantifier widgets are also connected to each other linearly and follow the last clause widget. They slow the advance of the pebbling toward s. In order to advance through them, it will be necessary to repebble the clause widgets numerous times, once for each truth assignment required to show that ψ is in QSAT. Only once the final quantifier widget is pebbled is it possible to pebble the target node s. We now

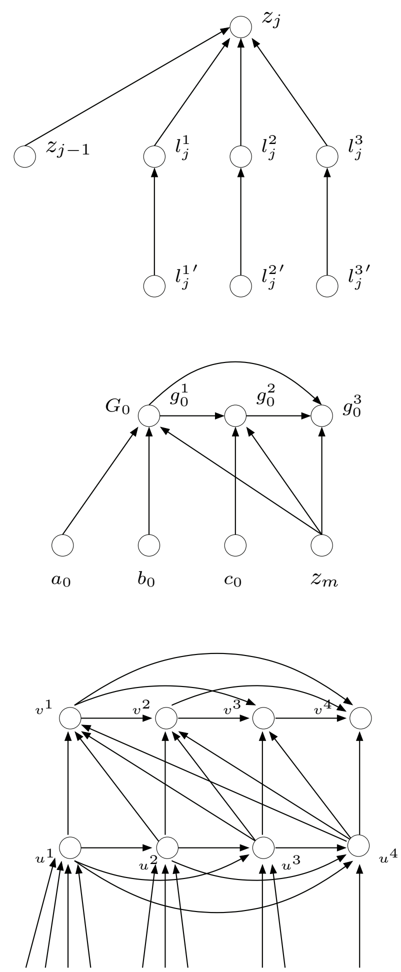

FIG. 1. *A clause widget for clause* $z_j = (l_j^1 \vee l_j^2 \vee l_j^3)$ *(top), the connection of* z_m *to* G_0 *(center), and a 4-slide* $(\{v^1, v^2, v^3, v^4\}, \{u^1, u^2, u^3, u^4\})$ *(bottom).*

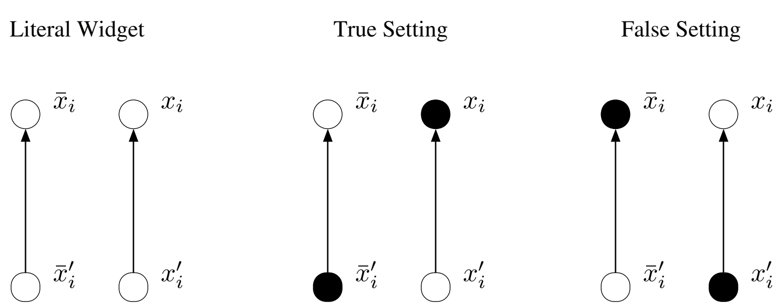

FIG. 2. *A literal widget for variable* x_i *(left). A literal widget for variable* x_i *in the true state (center). A literal widget for variable* x_i *in the false state (right).*

describe the individual widgets and how they are connected. These descriptions are somewhat terse and are meant to be read in conjunction with Figures 1, 2, 3, 4,

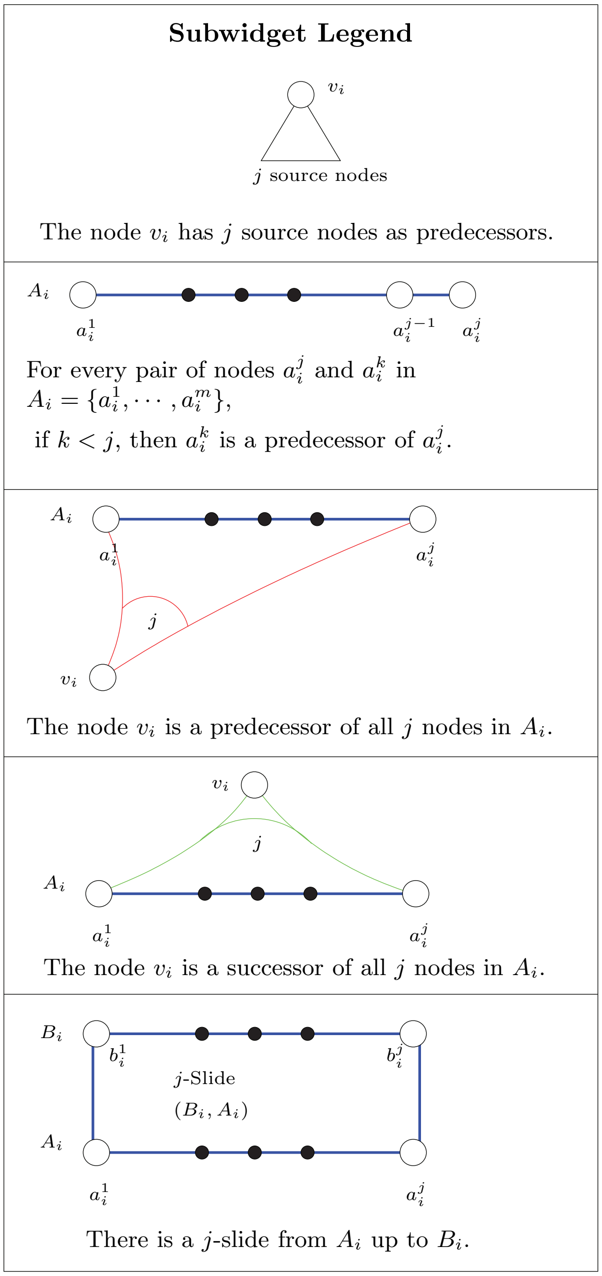

FIG. 3. *Legend explaining the components of Figures* 4 *and* 5.

and 5. Figure 6 shows a small example of a DAG produced from the QBF formula $\psi = \forall x_3 \exists x_2 \forall x_1 (x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$.

The universal widget is depicted in Figure 4. For every i, $1 \leq i \leq n$, if widget i is a universal widget, it is composed of source nodes plus the following five groups of nodes:

- $\{\bar{x}_i, \bar{x}_i', d_i, x_i, x_i', y_i\}$,

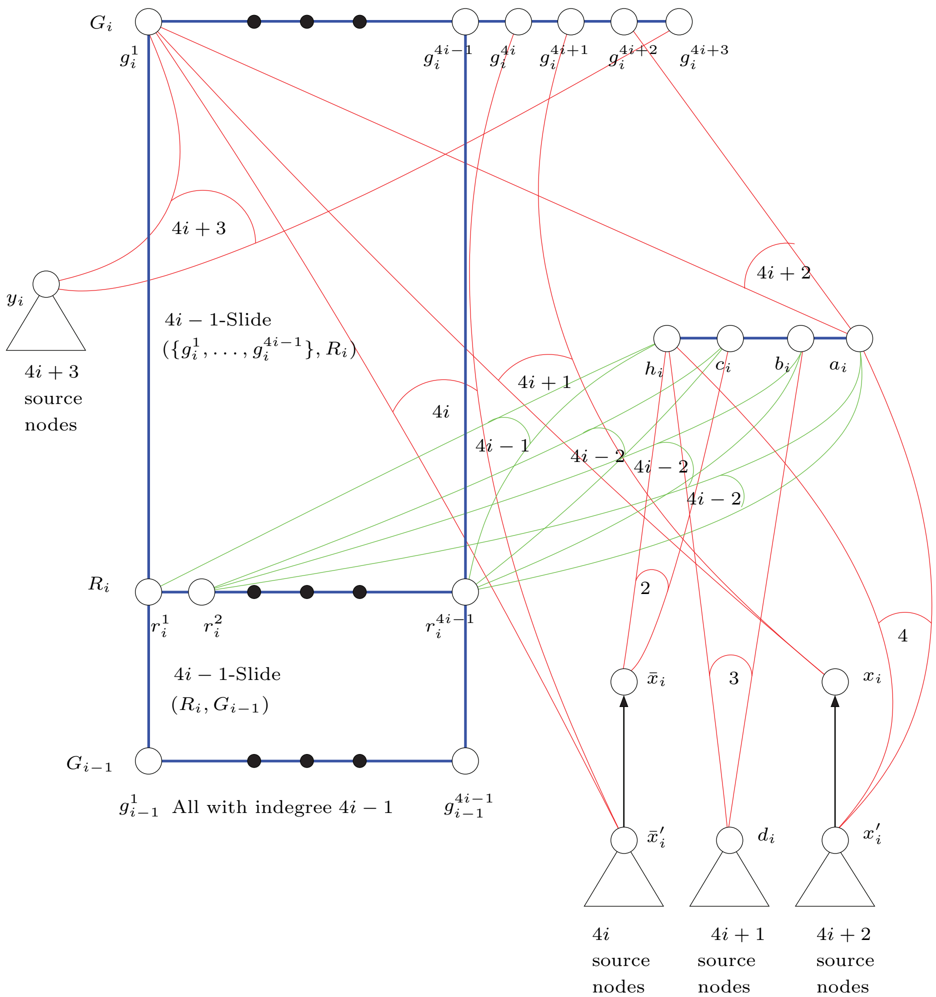

Fig. 4. *A universal widget for the black-white pebbling result.*

- $\{a_i, b_i, c_i, h_i\}$,
- $R_i = \{r_i^1, \ldots, r_i^{4i-1}\}$,
- $G_{i-1} = \{g_{i-1}^1, \ldots, g_{i-1}^{4i-1}\}$, and
- $G_i = \{g_i^1, \ldots, g_i^{4i+3}\}$.

These are connected as follows. y_i has $4i+3$ source nodes $p_{y_i}^1$ through $p_{y_i}^{4i+3}$ as predecessors, x_i' has $4i+2$ source nodes $p_{x_i}^1$ through $p_{x_i}^{4i+2}$ as predecessors, d_i has $4i+1$ source nodes $p_{d_i}^1$ through $p_{d_i}^{4i+1}$ as predecessors, and $\bar{x}_i'$ has $4i$ source nodes $p_{\bar{x}_i}^1$ through $p_{\bar{x}_i}^{4i}$ as predecessors. The sole predecessor of x_i is x_i', and the sole predecessor of $\bar{x}_i$ is $\bar{x}_i'$. For every pair of nodes g_i^j and g_i^k of G_i, if $j < k$, then g_i^j is a predecessor of g_i^k. Similarly, for every pair of nodes g_{i-1}^j and g_{i-1}^k of G_{i-1}, if $j < k$, then g_{i-1}^j is a predecessor of g_{i-1}^k. The same is true for R_i. The subgraph $(\{g_i^1, \ldots, g_i^{4i-1}\}, R_i)$ forms a $4i-1$ slide. (R_i, G_{i-1}) also forms a $4i-1$ slide. The node h_i is a successor of every node in R_i, the node c_i is a successor of every node in $\{h_i, r_i^2, \ldots, r_i^{4i-1}\}$, the node b_i is a successor of every node in $\{h_i, c_i, r_i^2, \ldots, r_i^{4i-1}\}$, and the node a_i is a

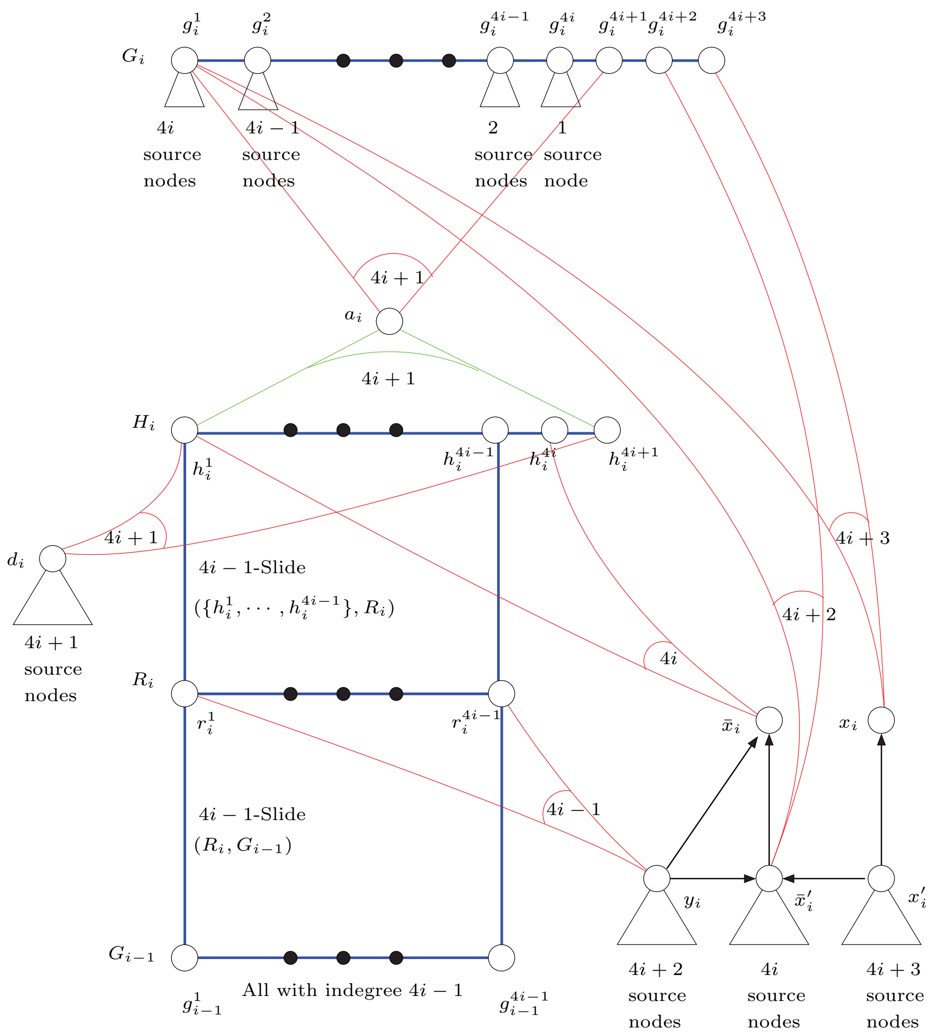

FIG. 5. *An existential widget for the black-white pebbling result.*

successor of every node in $\{h_i, c_i, b_i, r_i^2, \ldots, r_i^{4i-1}\}$. Finally, $\bar{x}_i'$ is a predecessor of every node in $\{g_i^1, \ldots, g_i^{4i}\}$, $\bar{x}_i$ is a predecessor of every node in $\{h_i, c_i\}$, d_i is a predecessor of every node in $\{h_i, c_i, b_i\}$, x_i' is a predecessor of every node in $\{h_i, c_i, b_i, a_i\}$, x_i is a predecessor of every node in $\{g_i^1, \ldots, g_i^{4i+1}\}$, a_i is a predecessor of every node in $\{g_i^1, \ldots, g_i^{4i+2}\}$, and y_i is a predecessor of every node in $\{g_i^1, \ldots, g_i^{4i+3}\}$.

The existential widget is depicted in Figure 5. For every i, $1 \le i \le n$, if widget i is an existential widget, it is composed of some source nodes plus the following four groups of nodes:

- $\{\bar{x}_i, \bar{x}_i', d_i, x_i, x_i', y_i\}$,
- $G_{i-1} = \{g_{i-1}^1, \ldots, g_{i-1}^{4i-1}\}$,
- $R_i \cup H_i \cup \{a_i\}$, where $R_i = \{r_i^1, \ldots, r_i^{4i-1}\}$, and
- $H_i = \{h_i^1, \ldots, h_i^{4i+1}\}$ and $G_i = \{g_i^1, \ldots, g_i^{4i+3}\}$.

x_i' has $4i+3$ source nodes $p_{x_i}^1$ through $p_{x_i}^{4i+3}$ as predecessors, y_i has $4i+2$ source nodes $p_{y_i}^1$ through $p_{y_i}^{4i+2}$ as predecessors, d_i has $4i+1$ source nodes $p_{d_i}^1$ through $p_{d_i}^{4i+1}$ as

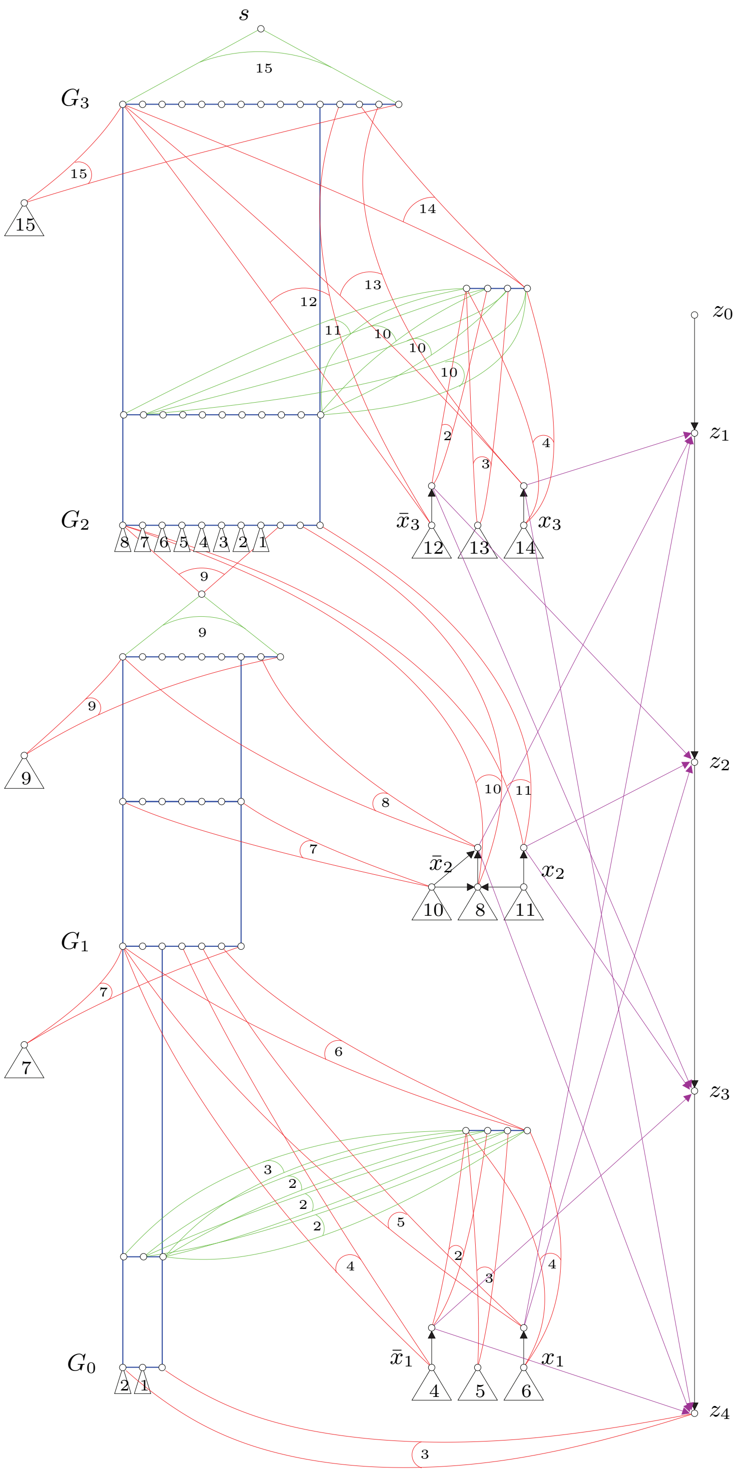

FIG. 6. *An example of* $\mathcal{G}$ *for* $\psi = \forall x_3 \exists x_2 \forall x_1 (x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$.

predecessors, and $\bar{x}'_i$ has $4i$ source nodes $p^1_{\bar{x}_i}$ through $p^{4i}_{\bar{x}_i}$ as predecessors. $\bar{x}'_i$ also has y_i and x'_i as predecessors. The sole predecessor of x_i is x'_i, and the only two predecessors of $\bar{x}_i$ are $\bar{x}'_i$ and y_i. For every pair of nodes g^j_i and g^k_i of G_i, if $j < k$, then g^j_i is a predecessor of g^k_i. The same is true for every pair of nodes in H_i, R_i, and G_{i-1}. Every node $g^j_i \in \{g^1_i, \dots, g^{4i+1}_i\}$ has $4i + 1 - j$ source nodes as predecessors. Also, a_i is a predecessor of every node in $\{g^1_i, \dots, g^{4i+1}_i\}$, $\bar{x}'_i$ is a predecessor of every node in $\{g^1_i, \dots, g^{4i+2}_i\}$, and x_i is a predecessor of every node in $\{g^1_i, \dots, g^{4i+3}_i\}$. Also, a_i is the successor of every node in H_i, d_i is a predecessor of every node in $\{h^1_i, \dots, h^{4i+1}_i\}$, $\bar{x}_i$ is a predecessor of every node in $\{h^1_i, \dots, h^{4i}_i\}$, and $(\{h^1_i, \dots, h^{4i-1}_i\}, R_i)$ forms a $4i - 1$ slide. Finally, y_i is a predecessor of every node in R_i, and (R_i, G_{i-1}) forms a $4i - 1$ slide.

For all i, $1 < i < n$, G_i is part of both widget i and widget $i + 1$. G_0 is special in that it connects the string of quantifier widgets to the string of clause widgets. It is described below. G_n is special because every node in G_n is a predecessor of the target node s. We now describe the m clause widgets.

For each clause C_i from $C_1, \dots C_m$, there is a corresponding node z_i. This node always has four predecessors, one of which is the previous clause node z_{i-1}. The other three, l^1_i, l^2_i, and l^3_i, correspond to the literals which occur in C_i. For example, if the first literal in the ith clause is $\bar{x}_j$, then the node $\bar{x}_j$ from quantifier widget j is one of the predecessors of z_i. z_1 has a special source node z_0 as a predecessor, since it has no previous clause. Finally, we add edges from z_m to all three nodes of G_0. There are also three source nodes a_0, b_0, and c_0 which are connected to G_0. a_0 and b_0 are predecessors of g^1_0, and c_0 is a predecessor of g^2_0. Figure 1 shows both an example of a clause widget and the connection between z_m and G_0. This completes the construction.

In the remainder of the paper, we will show that the following theorem holds with respect to this reduction.

THEOREM 3. *The QBF $\psi = Q_n x_n Q_{n-1} x_{n-1} \dots Q_1 x_1 F$ is in QSAT iff the target node s of $\mathcal{G}$ can be pebbled with $4n + 3$ pebbles.*

Since QSAT is PSPACE-complete [SM73], this shows that the black-white pebbling game is PSPACE-complete. In the upper bound proof, we begin by putting $\mathcal{G}$ into a configuration which corresponds to the first truth assignment in ψ's QSAT model. Then we pebble up through the clause widgets and some of the quantifier widgets before we have to put the graph into a configuration corresponding to the next truth assignment in the QSAT model, etc. In this way we will cycle through and verify all of the assignments in the QSAT model. In the lower bound proof, we will show that every minimal pebbling strategy for $\mathcal{G}$ must follow this. We now define how a pebbling configuration corresponds to a truth assignment.

DEFINITION 3.2 (pebbling configuration to assignment correspondence). *Let the set of all truth assignments over variables $x_{i+1}, \dots, x_n$ be denoted by A_i. Thus each β_i in A_i is a partial assignment that sets the outermost $n-i$ variables of $Q_n x_n \dots Q_1 x_1 F$. For any assignment $\beta_i \in A_i$, define $B[\beta_i]$ to be the pebbling configuration of $\mathcal{G}$ consisting of black pebbles on the following nodes:*

- *For each universally quantified variable x_j of ψ, $j \geq i+1$, if $\beta_i(x_j) = 0$, then $y_j \in B[\beta_i]$, $x'_j \in B[\beta_i]$, $d_j \in B[\beta_i]$, and $\bar{x}_j \in B[\beta_i]$. Otherwise, if $\beta_i(x_j) = 1$, then $y_j \in B[\beta_i]$, $\bar{x}'_j \in B[\beta_i]$, $a_j \in B[\beta_i]$, and $x_j \in B[\beta_i]$.*
- *For each existentially quantified variable x_j of ψ, $j \geq i+1$, if $\beta_i(x_j) = 0$, then $y_j \in B[\beta_i]$, $x'_j \in B[\beta_i]$, $d_j \in B[\beta_i]$, and $\bar{x}_j \in B[\beta_i]$. Otherwise, if $\beta_i(x_j) = 1$, then $y_j \in B[\beta_i]$, $\bar{x}'_j \in B[\beta_i]$, $d_j \in B[\beta_i]$, and $x_j \in B[\beta_i]$.*

Note that the negative assignment correspondences are identical in the existential and universal widgets, while the positive correspondences differ only in that the universal contains the node a_j while the existential contains d_j. We also highlight the fact that any assignment β_i corresponds to $4(n-i)$ pebbles on the graph.

DEFINITION 3.3 (time interval). *We refer to a sequence of configurations from time index* t^α *through time index* t^ω *as the* time interval *from* t^α *through* t^ω*, which we denote as* $[t^\alpha, t^\omega]$.

DEFINITION 3.4 (clamped interval). *For any node* v *and any time units* t^α*,* t^ω *such that* $t^\alpha \leq t^\omega$*, we say that* v *is* clamped *in the interval* $[t^\alpha, t^\omega]$*, denoted as* $v \in \bullet[[t^\alpha, t^\omega]]\bullet$*, iff* v *contains a black pebble during every configuration from* $\mathcal{M}[t^\alpha]$ *through* $\mathcal{M}[t^\omega]$*, i.e., for all* t^**,* $t^\alpha \leq t^* \leq t^\omega$*,* $v \in B[t^*]$.

DEFINITION 3.5 (empty interval). *For any node* v *and any time units* t^α*,* t^ω *such that* $t^\alpha \leq t^\omega$*, we say that* v *is* empty *in the interval* $[t^\alpha, t^\omega]$*, denoted as* $v \in \oslash[[t^\alpha, t^\omega]]\oslash$*, iff* v *contains no pebble during every configuration from* $\mathcal{M}[t^\alpha]$ *through* $\mathcal{M}[t^\omega]$*, i.e., for all* t^**,* $t^\alpha \leq t^* \leq t^\omega$*,* $v \notin B[t^*]$ *and* $v \notin W[t^*]$.

4. Upper bound. Here, we show that if the QBF $\psi = Q_n x_n Q_{n-1} x_{n-1} \dots Q_1 x_1 F$ is in QSAT, then the target node s of $\mathcal{G}$ can be pebbled with $4n+3$ pebbles. We explicitly give a black-white pebbling strategy (which only ever uses black pebbles), which achieves this space bound. In section 5, we prove that essentially this exact strategy is the only way to pebble $\mathcal{G}$ with $4n+3$ pebbles. A thorough understanding of this section is therefore essential in order to understand the lower bound.

We begin by describing a local strategy which will be used repeatedly within the $4n+3$-pebbling strategy used to black pebble s. This local strategy explains how we can simultaneously black pebble U using no more than i pebbles when given an i-slide (V, U) in which every member of V contains a black pebble.

DEFINITION 4.1 (i-slide strategy). *Let* (V, U) *be an* i*-slide. The* i*-slide strategy is very simple. For each* j *from* 1 *to* i*, simply slide the black pebble from* v^j *to* u^j.

LEMMA 4. *If* ψ *is in QSAT, then the target node* s *of* $\mathcal{G}$ *can be pebbled with* $4n+3$ *pebbles.*

Lemma 4 follows from the following more general lemma by setting $i = n$.

LEMMA 5. *For all* i*, and all* $\beta_i \in A_i$ *such that* $\psi\lceil_{\beta_i} \in$ *QSAT, if* $B[\beta_i] \subseteq \bullet[[t^\alpha, t^\omega]]\bullet$*, then we can black pebble* G_i *at some time in* $[t^\alpha, t^\omega]$ *using* $4n+3$ *pebbles.*

Proof. The proof is by induction on i from 0 to n. The base case is when $i = 0$. Let β_0 be any assignment in A_0. Recall that this corresponds to having $4n$ pebbles locked up during the whole pebbling sequence, so we have only three pebbles left to use. Suppose that $Q_n x_n \dots Q_1 x_1 F\lceil_{\beta_0}$ is in QSAT. Then some literal in every clause must be set to true. This implies that for each z_j, $1 \leq j \leq m$, at least one of l_j^1, l_j^2, or l_j^3 is black pebbled in $B[\beta_0]$.

We can black pebble G_0 as follows.

1. Start by putting a black pebble on z_0. Then since at most two of z_1's other predecessors are unpebbled, we have enough free pebbles to black pebble the rest of z_1's predecessors. We know we can black pebble them because if some l_1^k is unpebbled, then $l_1^{k'}$ must contain a black pebble in $B[\beta_i]$.
2. We therefore black pebble all of z_1's empty predecessors.
3. We then slide the pebble from z_0 to z_1 and lift the other (at most two) pebbles which we just put down.
4. Once z_1 is black pebbled, we black pebble z_2 the same way, all the way to z_m.
5. Once z_m is black pebbled, we place the remaining two black pebbles onto a_0 and b_0.

6. We then slide the pebble from a_0 to g_0^1.
7. We then remove the black pebble from b_0 and place it on c_0 and slide it from c_0 to g_0^2.
8. Finally, we slide the pebble from z_m to g_0^3.

Note that this strategy uses only black pebbles. For the inductive step there are two cases depending on whether Q_i is a universal or an existential quantifier.

Case 1. Q_i is a universal quantifier. In this case, both $\psi\lceil_{\beta_i \cup \{x_i\}}$ and $\psi\lceil_{\beta_i \cup \{\bar{x}_i\}}$ are in QSAT. We begin in configuration $B[\beta_i]$ with $4i+3$ free pebbles.

1. Black pebble y_i, followed by x_i', then d_i, and then $\bar{x}_i{}'$.
2. Then slide the pebble from $\bar{x}_i{}'$ to $\bar{x}_i$.
3. At this point we have $4i-1$ pebbles free and can apply the induction hypothesis to black pebble every member of G_{i-1}.
4. Use the i-slide strategy to slide all of G_{i-1}'s pebbles to R_i.
5. Then slide the black pebble from r_i^1 to h_i.
6. Then slide the black pebble from $\bar{x}_i$ to c_i.
7. Then slide the black pebble from d_i to b_i.
8. Then slide the black pebble from b_i to a_i.
9. Remove all pebbles from widget i except for the ones on a_i, x_i', and y_i.
10. Then slide the black pebble from x_i' to x_i and black pebble $\bar{x}_i'$ again.
11. Now apply the induction hypothesis to black pebble every member of G_{i-1} again.
12. Next, use the i-slide strategy to slide all of G_{i-1}'s pebbles to $\{g_i^1, \ldots, g_i^{4i-1}\}$.
13. Then slide $\bar{x}_i'$'s black pebble to g_i^{4i}.
14. Then slide x_i's black pebble to g_i^{4i+1}.
15. Next slide the black pebble from a_i to g_i^{4i+2}.
16. Finally, slide the black pebble from y_i to g_i^{4i+3}.

Case 2. Q_i is an existential quantifier. In this case, either $\psi\lceil_{\beta_i \cup \{x_i\}}$ or $\psi\lceil_{\beta_i \cup \{\bar{x}_i\}}$ is in QSAT. As in the universal case, we begin in $B[\beta_i]$ with $4i+3$ free pebbles.

1. Black pebble x_i', followed by y_i, d_i, and then $\bar{x}_i{}'$.
   The argument now proceeds differently if $\psi\lceil_{\beta_i \cup \{x_i\}}$ is in QSAT or if $\psi\lceil_{\beta_i \cup \{\bar{x}_i\}}$ is in QSAT.

- If $\psi\lceil_{\beta_i \cup \{x_i\}}$ is in QSAT:
   2. Slide the black pebble from x_i' to x_i.
   3. Then apply the induction hypothesis to black pebble G_{i-1}.
   4. Then use the i-slide strategy to slide all of the pebbles from G_{i-1} to R_i.
   5. Then slide the black pebble from y_i to $\bar{x}_i$.
   6. Then use the i-slide strategy to slide all of the pebbles from R_i to $\{h_i^1, \ldots, h_i^{4i-1}\}$.
   7. After that, slide the pebble from $\bar{x}_i$ to h_i^{4i}.
   8. Then slide the pebble from d_i to h_i^{4i+1}.
   9. Then slide the pebble from h_i^{4i+1} to a_i.
   10. At this point remove all the pebbles from the widget so that only $\bar{x}_i'$, x_i, and a_i remain.
   11. Use the $4i$ free pebbles to pebble the source node predecessors of g_i^1 and then slide one to g_i^1 itself.
   12. Remove the pebbles left over on the source nodes and use them to subsequently pebble each g_i^j until g_i^{4i} is pebbled.
   13. At this point slide the pebble from a_i to g_i^{4i+1}, slide the pebble from $\bar{x}_i'$ to g_i^{4i+2}, and finish by sliding the pebble from x_i to g_i^{4i+1}.

- If $\psi\lceil_{\beta_i \cup \{\bar{x}_i\}}$ is in QSAT:
  2. Slide the black pebble from $\bar{x}_i'$ to $\bar{x}_i$.
  3. Then apply the induction hypothesis to black pebble G_{i-1}.
  4. Then use the i-slide strategy to slide all of the pebbles from G_{i-1} to R_i.
  5. Then use the i-slide strategy to slide all of the pebbles from R_i to $\{h_i^1, \ldots, h_i^{4i-1}\}$.
  6. Slide the pebble from $\bar{x}_i$ to h_i^{4i}.
  7. Slide the pebble from d_i to h_i^{4i+1}.
  8. Then slide the pebble from h_i^{4i+1} to a_i.
  9. At this point remove all the pebbles from the widget so that only y_i, x_i', and a_i remain.
  10. Use the $4i$ pebbles that are free to repebble $\bar{x}_i'$.
  11. Then pick the pebble up from y_i and pick up the $4i-1$ pebbles that remain on the source node predecessors of $\bar{x}_i'$.
  12. Slide the pebble from x_i' to x_i.
  13. At this point $\bar{x}_i'$, x_i, and a_i are all pebbled, and we can finish by black pebbling G_i as we did in steps 11, 12, and 13 of the previous case.

We have therefore shown that regardless of whether Q_i is universal or existential, if $\psi\lceil_{\beta_i}$ is in QSAT, then we can pebble through quantifier widget i using no more than $4n+3$ pebbles. And in both cases the strategies use only black pebbles. ∎

5. Lower bound. In this section, we show that if the target node s of $\mathcal{G}$ can be pebbled with $4n+3$ pebbles, then the QBF $\psi = Q_n x_n Q_{n-1} x_{n-1} \ldots Q_1 x_1 F$ is in QSAT.

In our lower bound we will repeatedly show that certain pebble assignments must precede others in any minimal pebbling strategy. If the player can just make unnecessary moves, for example, by using some free pebbles to pebble some node and then immediately remove its pebble, then it is hard to make such an argument. We therefore follow the conventions of [GLT80] and will first define a set of frugal pebbling strategies. These are pebbling strategies which make no unnecessary moves.

Definition 5.1 (necessary pebble assignment). *We classify pebble assignments as* necessary *or* unnecessary*. A pebble assignment $P(v, t^1, t^2)$ is* necessary *if*
1. *it is the last black pebble assignment to the target node, or*
2. *a necessary black pebble assignment $P(u, t^3, t^4)$ exists in the strategy such that $P(v, t^1, t^2)$ supports $P(u, t^3, t^4)$, or*
3. *a necessary white pebble assignment $P(z, t^5, t^6)$ exists in the strategy such that $P(v, t^1, t^2)$ supports $P(z, t^5, t^6)$.*

A pebble placement is unnecessary *if it is not necessary.*

Definition 5.2 (black-white pebbling frugality). *We call a pebbling strategy which contains no unnecessary assignments* frugal.

By the definition above, every necessary pebble assignment except the one to s supports at least one black assignment or at least one white assignment.

Definition 5.3. *Let $P(v, t^1, t^2)$ be a necessary pebble assignment such that $v \neq s$ and let $P(u, t^3, t^4)$ be a necessary pebble assignment supported by $P(v, t^1, t^2)$ such that its support time is minimal among all the necessary pebble assignments that are supported by $P(v, t^1, t^2)$. Then we say that $P(u, t^3, t^4)$* necessitates *$P(v, t^1, t^2)$.*

The support time of the pebble assignment that necessitates $P(v, t^1, t^2)$ can be thought of as the time at which $P(v, t^1, t^2)$ is certified as necessary.

Definition 5.4. *The* chain of necessities *originating at $P(v, t^1, t^2)$ is an ordered set of pebble assignments, starting with the assignment that necessitates $P(v, t^1, t^2)$,* *which is produced by adding a pebble assignment to the set and then adding the pebble assignment which necessitates it, and so on.*

Every pebble assignment $P(u, t^3, t^4)$ in the chain of necessities originating at $P(v, t^1, t^2)$ is necessitated by exactly one other pebble assignment. Therefore, the chain of necessities originating at $P(v, t^1, t^2)$ will always contain exactly one pebble assignment to each of v's descendants along a single path from v to s. Therefore, as long as $v \neq s$, the chain of necessities originating at $P(v, t^1, t^2)$ is nonempty. Note that the chain of necessities originating at $P(v, t^1, t^2)$ is a path from $P(v, t^1, t^2)$ to the last pebble assignment to s in the support DAG $\mathcal{D}$ of $\mathcal{S}$.

DEFINITION 5.5. *Let $P(u, t^3, t^4)$ be a pebble assignment in the chain of necessities originating at $P(v, t^1, t^2)$. The* necessity distance *from $P(v, t^1, t^2)$ to $P(u, t^3, t^4)$ in the chain of necessities originating at $P(v, t^1, t^2)$ is the ordinal of $P(u, t^3, t^4)$ in the chain of necessities originating at $P(v, t^1, t^2)$.*

DEFINITION 5.6. *$J(P(v, t^1, t^2), t^5)$ is the subset of the chain of necessities originating at $P(v, t^1, t^2)$ that is composed of pebble assignments $P(u, t^3, t^4)$, such that $t^3 \leq t^5$.*

It is clear that since every member of the chain of necessities originating at $P(v, t^1, t^2)$ has a unique necessity distance, for each time t^3 some member of the subset $J(P(v, t^1, t^2), t^3)$ must have the maximum necessity distance among all members of $J(P(v, t^1, t^2), t^3)$. Note that $P(v, t^1, t^2)$ is not a member of $J(P(v, t^1, t^2), t^3)$ for any t^3.

We now prove that removing all unnecessary assignments from a k-pebbling strategy for a graph $\mathcal{G}$ results in another k-pebbling strategy for $\mathcal{G}$, and removing all of the configurations between two repeated configurations also results in another k-pebbling strategy. Once we have proved this, we can then limit ourselves to the consideration of frugal pebblings that have no repeated configurations.

LEMMA 6. *Removing all unnecessary pebble assignments from a k-pebbling strategy $\mathcal{S}$ results in another k-pebbling strategy $\mathcal{S}'$ which has no more configurations than $\mathcal{S}$.*

Proof. We first explain what it means for us to remove a pebble assignment from $\mathcal{S}$. The strategy $\mathcal{S}$ is a set of configurations and therefore does not contain a set of pebble assignments at all. Rather it induces a set of pebble assignments. So when we say that we remove a pebble assignment from $\mathcal{S}$ we mean that we modify $\mathcal{S}$ so that the set of pebble assignments which it induces is the same as before except without the one assignment we wish to remove.

We now prove that removing all unnecessary pebble assignments will not affect the correctness of the strategy and will not increase its pebbling number or its length in configurations.

Consider the support DAG $\mathcal{D}$ of $\mathcal{S}$. If there is an unnecessary pebble assignment $P(z, t^\alpha, t^\omega)$ in $\mathcal{D}$, then every one of its descendants must also be unnecessary. Otherwise, the necessary descendant, $P(u, t^3, t^4)$, would necessitate its predecessor in $\mathcal{D}$ which would then cascade to cause all of the pebble assignments on the path from $P(z, t^\alpha, t^\omega)$ to $P(u, t^3, t^4)$, including $P(z, t^\alpha, t^\omega)$, to become necessary.

Since every descendant of an unnecessary pebble assignment is unnecessary, the existence of an unnecessary assignment in $\mathcal{D}$ implies the existence of an unnecessary pebble assignment $P(v, t^1, t^2)$ that is a sink in $\mathcal{D}$. Since $P(v, t^1, t^2)$ does not support any other pebble assignments, we can clearly remove it from $\mathcal{S}$ without affecting the successful pebbling of s. We can therefore remove every unnecessary pebble assignment by repeatedly finding an unnecessary pebble assignment and then removing all the sinks which are descendants of that assignment until we can no longer find any unnecessary pebble assignments.

We now explain how we modify $\mathcal{S}$ so that the set of pebble assignments induced by $\mathcal{S}'$ is the same as the one induced by $\mathcal{S}$ except that it does not include $P(v, t^1, t^2)$. As we have just shown, it is sufficient to consider removing only unnecessary pebble assignments $P(v, t^1, t^2)$ which support no other pebble assignment. We therefore do not have to consider the removal of any white pebble assignments which begin with a slide, as such assignments support the assignment from which the white pebble is slid down, or the removal of any black pebble assignments which end with a slide, as such assignments support the assignments to which the black pebble is slid. This means that there are four kinds of pebble assignments which we might remove from a strategy.

- A black pebble assignment $P(v, t^1, t^2)$ that starts with a black pebble slide from a predecessor u of v to v at t^1 and ends with a black pebble removal removal from v at t^2.
  We remove $P(v, t^1, t^2)$ as follows. We begin by removing v from every configuration from $\mathcal{M}[t^1]$ through $\mathcal{M}[t^2 - 1]$. The transition from $\mathcal{M}[t^1 - 1]$ to $\mathcal{M}[t^1]$ therefore becomes a black pebble removal from u, rather than a slide from u to v. Originally, the only difference between $\mathcal{M}[t^2 - 1]$ and $\mathcal{M}[t^2]$ is the fact that $\mathcal{M}[t^2 - 1]$ contained v. Since it no longer does so, $\mathcal{M}[t^2 - 1]$ is identical to $\mathcal{M}[t^2]$. We therefore remove the entire configuration $\mathcal{M}[t^2]$.
- A black pebble assignment $P(v, t^1, t^2)$ that starts with a black pebble placement onto v at t^1 and ends with a black pebble removal from v at t^2.
  As before, we begin to remove $P(v, t^1, t^2)$ by removing v from every configuration from $\mathcal{M}[t^1]$ through $\mathcal{M}[t^2 - 1]$. As in the previous case, we must remove the entire configuration $\mathcal{M}[t^2]$. This time we will also have to remove $\mathcal{M}[t^1]$. Originally, the only difference between $\mathcal{M}[t^1 - 1]$ and $\mathcal{M}[t^1]$ was that $\mathcal{M}[t^1]$ contained v while $\mathcal{M}[t^1 - 1]$ did not, so once v is removed from $\mathcal{M}[t^1]$, they become identical and, as before, we remove one of them.
- A white pebble assignment $P(v, t^1, t^2)$ that starts with a white pebble placement onto v at t^1 and ends with a white pebble slide from v to a predecessor u of v at t^2.
  In this case, we remove v from every configuration from $\mathcal{M}[t^1]$ through $\mathcal{M}[t^2 - 1]$. For the same reasons as are discussed in the previous cases we must also remove $\mathcal{M}[t^1]$.
- A white pebble assignment $P(v, t^1, t^2)$ that starts with a white pebble placement onto v at t^1 and ends with a white pebble removal from v at t^2.
  As before, we remove v from every configuration from $\mathcal{M}[t^1]$ through $\mathcal{M}[t^2 - 1]$, and for the same reasons as are discussed in the previous cases we must also remove $\mathcal{M}[t^1]$ and $\mathcal{M}[t^2]$.

Since all the changes which are made to $\mathcal{S}$ involve removing nodes from configurations and none involve adding any, removing all unnecessary pebble assignments cannot increase the strategy's pebbling number. Similarly, we can see that the removal of each unnecessary pebble assignment requires the removal of at least one configuration and the addition of none, so the removal of any unnecessary pebble assignments also leads to a decrease in the strategy's length. ∎

We also state without proof the following obvious lemma.

LEMMA 7. *Removing all of the configurations between a repeated configuration and its duplicate from a k-pebbling strategy $\mathcal{S}$ results in another k-pebbling strategy $\mathcal{S}'$ which has fewer configurations than $\mathcal{S}$.*

The act of removing all unnecessary pebble assignments from $\mathcal{S}$ may create a repeated configuration, and removing all repeated configurations may create new unnecessary pebble assignments. We can therefore alternate removing all existing unnecessary pebble assignments and removing all repeated configurations until there is a round in which neither is removed. This process ends at some point because there are originally a finite number of configurations in $\mathcal{S}$ and each back-to-back round except the last must remove at least a single configuration. When the process ends, the k-pebbling strategy $\mathcal{S}$ has been transformed into a new k-pebbling strategy $\mathcal{S}'$ which, by definition, contains no unnecessary pebble assignments or repeat configurations.

We will assume from now on that every pebbling strategy is frugal and contains no repeat configurations. We may at times remind the reader that we are dealing with such strategies, but we would like to point out that in general *all* strategies are assumed to be frugal.

We now prove a series of technical lemmas, culminating with Lemma 11. Lemma 11 is used repeatedly throughout the induction which forms the bulk of the lower bound proof. In particular, it implies that the i-slide strategy described in the upper bound proof is the only strategy which can successfully pebble the top level of the i-slide using no more than i pebbles.

LEMMA 8. *Let $P(v, t^1, t^2)$ be a necessary pebble assignment and let t^ω be a time unit such that $t^1 \leq t^\omega$. If a black pebble assignment $P(u, t^3, t^4) \in J(P(v, t^1, t^2), t^\omega)$, where $t^3 < t^1$, then there must be a white pebble assignment in $J(P(v, t^1, t^2), t^\omega)$ that is on the graph at $t^1 - 1$.*

Proof. Consider a black pebble assignment $P(u, t^3, t^4) \in J(P(v, t^1, t^2), t^\omega)$, where $t^3 < t^1$, that is closest to $P(v, t^1, t^2)$ in necessity distance. Consider the chain of necessities between $P(v, t^1, t^2)$ and $P(u, t^3, t^4)$.

By the definition of necessary, $P(u, t^3, t^4)$ cannot necessitate $P(v, t^1, t^2)$ because $t^3 < t^1$. $P(u, t^3, t^4)$ must therefore necessitate another pebble assignment $P(z, t^5, t^6)$ in the chain of necessities originating at $P(v, t^1, t^2)$. The necessity distance of $P(v, t^1, t^2)$ to $P(z, t^5, t^6)$ is clearly 1 less than the necessity distance from $P(v, t^1, t^2)$ to $P(u, t^3, t^4)$. Since $P(u, t^3, t^4)$ is a black assignment that necessitates $P(z, t^5, t^6)$, it follows that $t^5 < t^3$, and therefore $t^5 < t^1$. Since $P(u, t^3, t^4)$ is the closest black assignment to $P(v, t^1, t^2)$ that is placed before t^1, $P(z, t^5, t^6)$ must be a white assignment and there can be no black pebble assignments placed before t^1 with lower necessity distance from $P(v, t^1, t^2)$ in $J(P(v, t^1, t^2), t^\omega)$.

We will prove that since there is no black pebble assignment that starts before t^1 with lower necessity distance from $P(v, t^1, t^2)$ than $P(z, t^5, t^6)$ in $J(P(v, t^1, t^2), t^\omega)$, there must be a white pebble assignment in $J(P(v, t^1, t^2), t^\omega)$ which remains on the graph at the time step immediately preceding t^1. The proof is by induction on the necessity distance of $P(v, t^1, t^2)$ to $P(z, t^5, t^6)$.

Basis. In the basis, the necessity distance from $P(v, t^1, t^2)$ to $P(z, t^5, t^6)$ is 1, so $P(z, t^5, t^6)$ is the white pebble assignment which necessitates $P(v, t^1, t^2)$. By the definition of necessary, this means that $t^1 \leq t^6$. So $t^5 < t^1 \leq t^6$, as required.

Induction step. If $t^6 \geq t^1$, then the lemma holds. On the other hand, suppose that $t^6 < t^1$. By the definition of necessitates, the assignment which $P(z, t^5, t^6)$ necessitates in the chain of necessities originating at $P(v, t^1, t^2)$ must have been started before t^6 and therefore before t^1. It is also 1 unit of necessity distance closer to $P(v, t^1, t^2)$ than $P(z, t^5, t^6)$ is. So this previous assignment also cannot be black by the assumptions of the lemma and must therefore be white. We therefore apply the induction hypothesis to find the white pebble assignment that is on the graph at $t^1 - 1$. $\square$

LEMMA 9. *Let $P(v, t^1, t^2)$ be a necessary pebble assignment. If $t^\alpha \le t^1 \le t^\omega$ and at time $t^1 - 1$ no descendant of v contains a white pebble, then no pebble assignment whose node is in $\bullet[[t^\alpha, t^\omega]]\bullet$ is in $J(P(v, t^1, t^2), t^\omega)$.*

Proof. If a node that is not a descendant of v is in $\bullet[[t^\alpha, t^\omega]]\bullet$, then it is by definition not in $J(P(v, t^1, t^2), t^\omega)$ since the chain of necessities originating at $P(v, t^1, t^2)$ (and any subset of it) contains only pebble assignments to v's descendants. We can therefore restrict our attention to the descendants of v that are in $\bullet[[t^\alpha, t^\omega]]\bullet$.

We first note that there are no white pebbles on any descendant of v at time $t^1 - 1$. By Lemma 8 if there is a black pebble assignment $P(u, t^3, t^4)$ in $J(P(v, t^1, t^2), t^\omega)$ such that $t^3 < t^1$, then there must be a white pebble assignment in $J(P(v, t^1, t^2), t^\omega)$ that is on the graph at $t^1 - 1$, a contradiction. ∎

LEMMA 10. *Let $P(v, t^1, t^2)$ be a necessary pebble assignment such that $v \ne s$. If $t^2 \le t^\omega$, then regardless of whether $P(v, t^1, t^2)$ is a black or a white pebble assignment, the member of $J(P(v, t^1, t^2), t^\omega)$ which has the maximum necessity distance among all members of $J(P(v, t^1, t^2), t^\omega)$ must be on the graph at t^ω.*

Proof. Since v is not the target node, there exists some member of the subset $J(P(v, t^1, t^2), t^\omega)$, $P(u, t^3, t^4)$, which has the maximum necessity distance among all members of $J(P(v, t^1, t^2), t^\omega)$. If $P(u, t^3, t^4)$ has no successor in the chain of necessities justifying $P(v, t^1, t^2)$, then u must be the target node and t^4 must be infinity, in which case $P(u, t^3, t^4)$ clearly is on the graph at t^ω.

Suppose, on the other hand, that $P(u, t^3, t^4)$ has a successor, $P(z, t^5, t^6)$, in the chain of necessities justifying $P(v, t^1, t^2)$. Since $P(u, t^3, t^4)$ has the greatest necessity distance from $P(v, t^1, t^2)$ of any member of $J(P(v, t^1, t^2), t^\omega)$, and the necessity distance from $P(v, t^1, t^2)$ to $P(z, t^5, t^6)$ is greater by 1, $P(z, t^5, t^6)$ is not in $J(P(v, t^1, t^2), t^\omega)$. Therefore, $t^\omega < t^5 < t^6$. So if $P(z, t^5, t^6)$ is a white pebble assignment, then $t^4 > t^6 > t^\omega$, and if $P(z, t^5, t^6)$ is a black pebble assignment, then $t^4 \ge t^5 > t^\omega$. In either case, $t^3 < t^\omega + 1 \le t^4$, so $P(u, t^3, t^4)$ is on the graph at t^ω. ∎

DEFINITION 5.7. *We say that a node is* uniquely black pebblable *during the interval $[t^1, t^2]$ if it can be black pebbled only once within the interval and can never be white pebbled at all within the interval.*

LEMMA 11. *Consider a (frugal) k-pebble strategy, S, of a graph G. Let v be a node of $\mathcal{G}$, $v \ne s$, such that*

1. *no descendant of v contains a white pebble at any time in $[t^\alpha, t^\omega]$,*
2. *there is a set E of $k - c$ clamped nodes in $\bullet[[t^\alpha, t^\omega]]\bullet$ such that v has c predecessors not in E, and*
3. *v contains no pebble at t^α and no white pebble at t^ω.*

Then v is uniquely black pebblable in $[t^\alpha, t^\omega]$.

Proof.

Case 1. Suppose v is white pebbled at time t^1, $t^\alpha \le t^1 \le t^\omega$. Then its white pebble must be removed at $t^2 \le t^\omega$. By Lemma 10, at every time unit t^3 between t^2 and t^ω inclusive, there must be a pebble on some node z of some member of $J(P(v, t^1, t^2), t^3)$. Since $\mathcal{G}$ is acyclic, z is not one of the predecessors of v. By Lemma 9, z is not in $\bullet[[t^\alpha, t^\omega]]\bullet$, so it is not in E. Thus at $t^2 - 1$, there are $k - c$ black pebbles in E, 1 pebble on z, and 1 white pebble on v. This means that at most $c - 2$ pebbles can be placed on v's predecessors without exceeding the limit of k. Therefore, it is impossible to remove v's white pebble at t^2. It is therefore impossible to white pebble v in the first place between t^α and t^ω.

Case 2. Suppose v is black pebbled at time t^1 and its black pebble is removed at

$t^2 < t^\omega$. By Lemma 10, at every time unit t^3 between t^2 and t^ω inclusive, there must be a pebble on some node z of some member of $J(P(v, t^1, t^2), t^3)$ at t^3. Since $\mathcal{G}$ is acyclic, z is not one of the predecessors of v. By Lemma 9, z is not in $\bullet[[t^\alpha, t^\omega]]\bullet$, so it is not in E. Thus at each t^3, there are $k - c$ black pebbles in E and 1 pebble on z. This means that at most $c - 1$ pebbles can be placed on v's predecessors without exceeding the limit of k. Therefore, it is impossible to put a black pebble on v at any t^3 between t^2 and t^ω. It is therefore impossible to black pebble v for a second time between t^α and t^ω. ∎

LEMMA 12. *Consider a (frugal) k-pebbling strategy $\mathcal{S}$ of a graph $\mathcal{G}$. Let v be a node of $\mathcal{G}$, $v \neq s$, such that*

1. *no descendant of v contains a white pebble at time t^α,*
2. *there is a set E of $k - c$ clamped nodes in $\bullet[[t^\alpha, t^\omega]]\bullet$ such that v has c source node predecessors not in E,*
3. *v contains no pebble at t^α and no white pebble at t^ω, and*
4. *v is black pebbled at some time t^*, $t^\alpha < t^* \leq t^\omega$.*

Then v is uniquely black pebblable during this interval, and the black pebble must remain on v until at least the last time in $[t^\alpha, t^\omega]$ that v supports another pebble assignment. Furthermore, if there are no white pebbles on any nonsource nodes of $\mathcal{G}$ at t^α, then v must be the first node of $\mathcal{G}$ pebbled during this interval other than one of its own source node predecessors.

Proof. First we will argue that neither v nor any descendant of v can be white pebbled during the interval $[t^\alpha, t^*]$. Suppose for the sake of contradiction that v' is the first node such that either $v' = v$ or v' is a descendant of v, and v' is white pebbled at time t^1, $t^\alpha < t^1 \leq t^*$. Since v is black pebbled at t^*, and v contains c predecessors, and only c pebbles are free, the white pebble placed on v' at time t^1 must be removed at some time $t^2 < t^*$. By Lemma 10, at every time unit t^3 between t^2 and t^ω, there must be a pebble on some node z of some member of $J(P(v, t^1, t^2), t^3)$. Since $\mathcal{G}$ is acyclic, z is not one of the predecessors of v. By Lemma 9, z is not in E. So as in Lemma 11, it is therefore impossible to white pebble v' during the interval $[t^\alpha, t^*]$ because there will not be enough pebbles left to black pebble v at t^*.

We will now prove that if there are no nonsource nodes containing white pebbles at t^α, then v is the first node of G pebbled during this interval other than its predecessors. Since v is black pebbled at time t^*, at the preceding step, all c of the source nodes of v contain pebbles. Any other pebble placed between t^α and t^* must be removed by $t^* - 1$, but this is impossible, using Lemmas 9 and 10 and the fact that the graph has no white pebbles on nonsource nodes at time t^α. Therefore v is the first node of G pebbled during $[t^\alpha, t^\omega]$ other than one of its own source node predecessors.

As in Lemma 11, we can now argue that it is impossible to black pebble v for a second time during this interval. Assume for the sake of contradiction that v contains a black pebble during the interval t^* to t^1, $t^1 < t^\omega$, and then is black pebbled for a second time at time t^2, $t^1 < t^2 \leq t^\omega$. The first black pebble placement at t^* means, by Lemmas 10 and 9 and the fact that there are no white pebbles on any descendant of v at t^*, that some pebble must remain on some descendant of v during the entire interval $[t^*, t^\omega]$, but then this leaves too few pebbles for black pebbling v for a second time.

Because v does not contain a white pebble at time t^ω, if v were white pebbled after t^* and before t^ω, the white pebble must be removed before t^ω, and now by an argument similar to that of the second black pebbling, there are not enough pebbles to do this. Thus, it is also impossible to white pebble v at any time after t^* and before t^ω. Finally, since v is uniquely black pebblable in $[t^\alpha, t^\omega]$, the black pebble must remain on v until at least the last time in $[t^\alpha, t^\omega]$ that v supports another pebble assignment. ∎

Definition 5.8. *An* unblocked path ρ *from node* z *to node* v *in* $\mathcal{G}$ *at time* t^α *is a path from* z *to* v *in* $\mathcal{G}$ *such that at time* t^α*,* v *is either empty or has a white pebble, and all other nodes in* ρ *contain no pebbles.*

The following lemma is very similar to one that appeared in [GT78].

Lemma 13. *Suppose there is an unblocked path* ρ *from* z *to* v *at time* t^α*. Suppose we either want to black pebble* v *at* t^* *or remove a white pebble from* v *at* t^**,* $t^\alpha \le t^* \le t^\omega$*, and end in a configuration in which all of* v*'s predecessors on* ρ *are empty at* t^ω*. Then each node on* ρ *must be pebbled at some time* t^1 *and have its pebble removed at some time* t^2*, where* $t^\alpha < t^1 < t^2 \le t^\omega$*.*

Proof. The proof is by induction on the length of ρ.

Basis. When $|\rho| = 1$, z is one of v's direct predecessors. Then in order to black pebble v or remove a white pebble from v, z must be pebbled no later than t^*. Since z is empty at t^α and t^ω, z must be pebbled at a time t^1, $t^\alpha < t^1$, and its pebble must be removed at some time $t^2 \le t^\omega$.

Induction step. When $|\rho| = d$, we know that the length of the subpath of ρ, call it ρ', from z's successor y on ρ to v is $d-1$. Since ρ is empty at t^α and again at t^ω and ρ' is a subpath of ρ, ρ' is empty at t^α and t^ω. We can therefore apply the induction hypothesis to conclude that y must be pebbled at some time t^2, $t^\alpha < t^2$, and have its pebble removed at some time $t^3 \le t^\omega$.

If y is black pebbled at time t^2, then z must be pebbled after t^α but before t^2, and its pebble must be removed after t^2 but no later than t^ω. If y is white pebbled at time t^2, then z must be pebbled after t^α but no later than t^3, and its pebble must be removed after t^3 but no later than t^ω. ∎

The next lemma roughly states the following. A pebble, v, can be removed as soon as it is no longer needed. Also, if during some interval, a pebble on v is needed only for u, and u is pebbled multiple times during this interval, then these multiple pebblings can be replaced by just one pebble placement on u.

Lemma 14. *Let* $P(v, t^1, t^2)$ *be a necessary black pebble assignment in a frugal* k*-pebbling strategy,* $\mathcal{S}$*, let the black pebble assignment* $P(u, t^3, t^4)$*,* $t^3 \le t^2$*, be the pebble assignment which has the latest support time,* t^**, of any pebble assignment supported by* $P(v, t^1, t^2)$*, and let* t' *be a time such that*

1. $t' < t^* < t^2$*,*
2. *from* t' *through* $t^2 - 1$*,* $P(v, t^1, t^2)$ *supports only pebble assignments to* u*, and*
3. *all of* u*'s predecessors, other than* v*, are clamped from* $t' - 1$ *through* $t^* - 1$*.*

Then there is another frugal k*-pebbling strategy* $\mathcal{S}'$ *in which* $P(v, t^1, t^2)$ *is replaced with* $P(v, t^1, t')$ *and all of the pebble assignments which* $P(v, t^1, t^2)$ *supported after* t' *are replaced with the single black pebble assignment* $P(u, t', t^4)$*.*

Proof. We will show that the k-pebbling strategy $\mathcal{S}'$ exists by showing how we can modify $\mathcal{S}$ to a new k-pebbling strategy such that the set of pebble assignments induced by $\mathcal{S}'$ is just like that induced by $\mathcal{S}$ except that the pebble is slid from v to u at the earliest possible time, t', and then remains on u until t^4, essentially replacing numerous late assignments of pebbles to u with a single earlier one.

Basically, we will modify $\mathcal{S}$ to produce $\mathcal{S}'$ by removing v from every configuration from $\mathcal{M}[t']$ through $\mathcal{M}[t^2 - 1]$ and adding u to the black set of every configuration from $\mathcal{M}[t']$ through $\mathcal{M}[t^3 - 1]$ which does not already contain it. In this way, we merge all of the pebble assignments to u which $P(v, t^1, t^2)$ supported after t' into a single pebble assignment $P(u, t', t^4)$ and shorten the pebble assignment to v to $P(v, t^1, t')$. Making this wholesale change requires some careful bookkeeping to make sure that no syntactic errors creep into $\mathcal{S}'$. We will therefore consider each configuration in turn from $\mathcal{M}[t'-1]$ to $\mathcal{M}[t^2]$ and describe the changes we must make to each one.

We begin by considering $\mathcal{M}[t'-1]$. There is already a move which transitions $\mathcal{M}[t'-1]$ to $\mathcal{M}[t']$ in $\mathcal{S}$. Now u is in $\mathcal{M}'[t'-1]$ and v is not in $\mathcal{M}'[t']$, so the differences between $\mathcal{M}'[t'-1]$ and $\mathcal{M}'[t']$ are as if two moves, the original one plus the slide from v to u, are made simultaneously in $\mathcal{S}'$. This is not legal since each configuration must follow from the last by exactly one legal move. We therefore add a configuration $\mathcal{M}'[t'-.5]$ between $\mathcal{M}'[t'-1]$ and $\mathcal{M}'[t']$ which allows us to split the changes over two time units so that the slide from v to u happens first, and the other move happens second.

We now consider an arbitrary configuration $\mathcal{M}[i]$ which is between (and including) $\mathcal{M}[t']$ and $\mathcal{M}[t^2]$. When we consider $\mathcal{M}[i]$, we will also consider $\mathcal{M}[i-1]$ so that we can see what kind of pebbling move $\mathcal{M}[i]$ is the result of. Then we will explain how we modify $\mathcal{M}[i]$ to produce $\mathcal{M}'[i]$. Since v contains a pebble from t^1 to t^2, every $\mathcal{M}[i]$ will contain v. But then we have the following cases, depending on what happens with u.

- Neither $\mathcal{M}[i-1]$ nor $\mathcal{M}[i]$ contains u.
  In this case, we produce $\mathcal{M}'[i]$ by removing v from $\mathcal{M}[i]$ and adding u to $B'[i]$. $\mathcal{M}[i-1]$ is transformed into $\mathcal{M}'[i-1]$ in the previous step.
  In this case, a pebble is either placed onto some other node w or removed from another node w at this time. In either case, v did not support the move, so the move is still legal in $\mathcal{S}'$.
- $\mathcal{M}[i-1]$ does not contain u, but $\mathcal{M}[i]$ does.
  In this case u could be in $W[i]$ or in $B[i]$.
  We produce $\mathcal{M}'[i]$ by removing v from $\mathcal{M}[i]$ and, regardless of whether $u \in W[i]$ or $u \in B[i]$, putting $u \in B'[i]$. $\mathcal{M}[i-1]$ is transformed into $\mathcal{M}'[i-1]$ in the previous step.
  Since u is not in $\mathcal{M}[i-1]$, but it is in $\mathcal{M}[i]$, a pebble is placed onto u in the move which transitions from $\mathcal{M}[i-1]$ to $\mathcal{M}[i]$. This may or may not have been part of a slide from some node w to u.
  If the pebble placement was part of a slide, then there are two differences between $\mathcal{M}[i-1]$ and $\mathcal{M}[i]$, namely, that $\mathcal{M}[i-1]$ does contain the node w where the pebble is slid from but does not contain u, and that $\mathcal{M}[i]$ does contain u but does not contain w. Note that $w \neq v$ since v is in every configuration that we are considering. When we add u to $\mathcal{M}'[i-1]$, the slide becomes a removal of a pebble from w rather than a slide, so everything is okay and no other moves are affected.
  If the pebble placement was not part of a slide, then the presence of u is the only difference between $\mathcal{M}[i-1]$ and $\mathcal{M}[i]$. So when u is added to $\mathcal{M}[i-1]$, $\mathcal{M}'[i-1]$ and $\mathcal{M}'[i]$ become identical. In this case we must therefore remove this repetition by removing $\mathcal{M}'[i]$ from $\mathcal{S}'$.
- $\mathcal{M}[i-1]$ does contain u, but $\mathcal{M}[i]$ does not.
  In this case, we produce $\mathcal{M}'[i]$ by removing v from $\mathcal{M}[i]$ and adding u to $B[i]$. $\mathcal{M}[i-1]$ is transformed into $\mathcal{M}'[i-1]$ in the previous step.
  Since u is in $\mathcal{M}[i-1]$, but it is not in $\mathcal{M}[i]$, a pebble is removed from u in the move which transitions from $\mathcal{M}[i-1]$ to $\mathcal{M}[i]$. This may or may not have been part of a slide from u to some node w.

If the pebble removal was part of a slide, then there are two differences between $\mathcal{M}[i-1]$ and $\mathcal{M}[i]$, namely, that $\mathcal{M}[i-1]$ does not contain the node w where the pebble is slid to but does contain u, and that $\mathcal{M}[i]$ does not contain u but does contain w. Note that $w \neq v$ since v is in every configuration that we are considering. When we add u to $\mathcal{M}'[i]$, the slide becomes the addition of a pebble to w rather than a slide, so everything is okay and no other moves are affected.

If the pebble removal was not part of a slide, then the presence of u is the only difference between $\mathcal{M}[i-1]$ and $\mathcal{M}[i]$. So when u is added to $\mathcal{M}[i]$, $\mathcal{M}'[i-1]$ and $\mathcal{M}'[i]$ become identical. In this case we must therefore remove this repetition by removing $\mathcal{M}'[i]$ from $\mathcal{S}'$.

- Both $\mathcal{M}[i-1]$ and $\mathcal{M}[i]$ contain u.

  In this case, u could be contained in either both $W[i-1]$ and $W[i]$ or in both $B[i-1]$ and $B[i]$. It is not possible for a single node to contain two different colored pebbles in two consecutive configurations because every pebble removal move results in a configuration in which the node from which the pebble was removed is empty.

  We produce $\mathcal{M}'[i]$ by removing v from $\mathcal{M}[i]$ and, regardless of whether $u \in W[i]$ or $u \in B[i]$, putting $u \in B'[i]$. $\mathcal{M}[i-1]$ is transformed into $\mathcal{M}'[i-1]$ in a separate step.

  Since u and v are in both original configurations, the move which transitions from $\mathcal{M}[i-1]$ to $\mathcal{M}[i]$ has nothing to do with u or v. A pebble is either placed onto some other node w or removed from another node w at this time. In either case, v did not support the move, so the move is still legal in $\mathcal{S}'$.

In all these cases, we always remove at least as many pebbles from the configuration as we add, so the pebbling number does not increase.

We now consider $\mathcal{M}[t^2]$. $\mathcal{M}[t^2-1]$ contains v, while $\mathcal{M}[t^2]$ does not. In $\mathcal{S}$, v's pebble could be removed with a slide or not. If it is removed with a slide, then the slide has to be to u, since v contains a black pebble and only supports assignments to u after t'. In this case there are two differences between $\mathcal{M}[t^2-1]$ and $\mathcal{M}[t^2]$, namely, that $\mathcal{M}[t^2-1]$ does not contain the node u where the pebble is slid to but does contain v, and that $\mathcal{M}[t^2]$ does not contain v but does contain u. When we remove v to $\mathcal{M}'[t^2-1]$, the slide becomes the addition of a pebble to u rather than a slide, so everything is okay and no other moves are affected.

If the pebble is not removed with a slide, then there is only one difference between $\mathcal{M}[t^2-1]$ and $\mathcal{M}[t^2]$, namely, that v is in $\mathcal{M}[t^2-1]$ and not in $\mathcal{M}[t^2]$. When v is removed to produce $\mathcal{M}'[t^2-1]$, $\mathcal{M}'[t^2-1]$ and $\mathcal{M}'[t^2]$ become the same, and we must remove $\mathcal{M}'[t^2]$ from $\mathcal{S}'$.

Finally, if $t^4 > t^2$, then we must still modify every configuration $\mathcal{M}[i]$ from $\mathcal{M}[t^2]$ to $\mathcal{M}[t^4]$. Since $\mathcal{M}[i]$ is beyond t^2, we modify $\mathcal{M}[i]$ only if $W[i]$ contains u. In this case we produce $\mathcal{M}'[i]$ by removing u from $W'[i]$ and adding it to $B'[i]$.

It remains to argue that the new strategy is frugal. The only pebble placements whose necessity can be affected by the modifications described above are the pebble placements to u's predecessors. But since u's predecessors are clamped during this whole time, we argue that the pebble placement will still be necessitated in the modified strategy. Let u' be a predecessor of u. One case is when u' is necessitated by a placement to a successor other than u, in which case the event still occurs as before. The second case is when u' is necessitated by a placement onto u before t'; again this event still occurs. The final case is when u' is necessitated by the placement onto u on or after t'. In this case, u' will become necessitated by $P(u, t', t^4)$. ∎

LEMMA 15. *Let $\mathcal{G}$ be a DAG and let $\mathcal{S}$ be a k-pebbling strategy for $\mathcal{G}$. Let v be a node of $\mathcal{G}$ and let t' be a time such that*

1. *v is empty at t',*
2. *v is black pebbled at $t'+1$,*
3. *there is a set E of $k-1$ nodes in $\bullet[[t', t'+1]]\bullet$,*
4. *one of v's predecessors, u, is not in E.*

Then u must contain a black pebble at t' and must be empty at $t'+1$.

Proof. Since u is a predecessor of v and v is black pebbled at $t'+1$, u must contain some pebble at t'. Since there are already $k-1$ pebbles in E and 1 pebble on u, adding another would exceed the space bound of k pebbles, so v's pebble cannot be a new one. But since every member of E is clamped in $\bullet[[t', t'+1]]\bullet$, we cannot reuse one of those pebbles to black pebble v at $t'+1$. Therefore, u's pebble must slide from u to v at $t'+1$, since this is the only pebble left and a slide is the only way to remove a pebble from u and place a pebble on v during the same time step. This means that u's pebble must be black at t', since only black pebbles can slide from a node to a successor. Furthermore, u must be empty at $t'+1$ as a result of the slide. ∎

We have now reached the point where (using the lemmas so far) we can prove exactly how an i-slide must be pebbled.

LEMMA 16. *Let $\mathcal{G}$ be a DAG with target node s, let (V, U) be an i-slide in $\mathcal{G}$, and let t^α, t^1, and t^ω be times such that*

1. *$t^\alpha < t^1 \leq t^\omega$,*
2. *there is a set of $k-i$ nodes, none of which is in V or W, or is an ancestor of any node in V or W in $\bullet[[t^\alpha, t^\omega]]\bullet$,*
3. *all nodes of U and V are empty at t^α,*
4. *v^1 is black pebbled at t^1 and $v^1 \in \bullet[[t^1, t^\omega]]\bullet$,*
5. *every node in V contains some pebble at t^ω,*
6. *there are never more than k pebbles used during $[t^\alpha, t^\omega]$.*

Then U must be simultaneously black pebbled at some time t^1-1, at which time every member of V must be empty.

Proof. Since there are $k-i$ nodes in $\bullet[[t^\alpha, t^\omega]]\bullet$ and every member of (V, U) has in-degree i, only v^1's predecessors can contain pebbles at t^1-1.

All i members of U are predecessors of v, and each has in-degree i. Every member of U must be empty by t^ω. If some member of U contained a white pebble at t^1-1, it could not be removed at some point in $[t^1, t^\omega]$ since v^1 and $k-i$ other nodes are in $\bullet[[t^1, t^\omega]]\bullet$. Therefore, at t^1-1, every member of U must contain a black pebble. Since this uses up all of the free pebbles, every member of V must be empty at this time. ∎

With these lemmas in hand, we are now ready to prove our main lower bound lemma.

LEMMA 17. *Let ψ be a QBF and let $\mathcal{G}$ be the corresponding graph. If s has a $4n+3$ black-white pebbling strategy in $\mathcal{G}$, then ψ is in QSAT, and any $4n+3$ black-white pebbling strategy requires $\Omega(2^k)$ steps, where k is the number of universal quantifiers in ψ.*

We first note that s has $4n+3$ predecessors, G_n. Each of these nodes has in-degree $4n+3$. So no node of G_n could ever contain a white pebble while s contains a black pebble, because there would not be enough free pebbles to remove it. Therefore, in order to pebble s, G_n must first be simultaneously black pebbled. Lemma 17 therefore follows from the more general Lemma 18 by setting $i = n$, $t^\alpha = t^{start}$, and $t^\omega = t^{end}$.

LEMMA 18. *Let ψ be a QBF and let G be the corresponding graph. Suppose there exists a frugal, optimal pebbling strategy of G such that the following holds. For all*

i, $1 \leq i \leq n$, and for all $\beta_i \in A_i$, there exist times t^α, t^ω such that the following five conditions are satisfied:

1. *the only members of $\mathcal{G}$ which contain pebbles at t^α are members of $B[\beta_i]$,*
2. *every member of G_i contains a black pebble at t^ω,*
3. *$B[\beta_i] \subseteq \bullet[[t^\alpha, t^\omega]]\bullet$,*
4. *there are never more than $4n+3$ pebbles on the graph at any time during $[t^\alpha, t^\omega]$, and*
5. *no pebble is placed on any descendant of g_i^{4i+3} during the interval $[t^\alpha, t^\omega]$.*

Then we can conclude that $\psi\lceil_{\beta_i}$ is in QSAT and requires $\Omega(2^k)$ units of time between t^α and t^ω, where k is the number of universal quantifiers among the i innermost quantifiers.

Proof. The proof is by induction on i from 0 to n.

Basis. The base case is $i = 0$. Let β_0 be any assignment in A_0 and suppose there exist times t^α and t^ω such that $B[\beta_0] \subseteq \bullet[[t^\alpha, t^\omega]]\bullet$. We will show that simultaneously black pebbling G_0 at t^ω without ever exceeding $4n+3$ pebbles or placing a pebble on a descendant of g_0^3 requires that ψ is in QSAT.

In order to black pebble z_j or remove a white pebble from z_j we must either black pebble z_{j-1} or remove a white pebble from z_{j-1}. In order to black pebble any node in G_0, we must pebble z_m. Inductively, this means that at some point, for every single z_j, it is necessary to either black pebble it or remove a white pebble from it. But every z_j (except z_0) has four predecessors, l_j^1, l_j^2, l_j^3, z_{j-1}. Therefore, in order to pebble z_j at least one l_j^k must be black pebbled in $B[\beta_0]$. But in this case, β_0 satisfies clause j of ψ. Since every z_j must either be black pebbled or be removed, β_0 must satisfy every clause of ψ. Therefore $\psi\lceil_{\beta_0}$ is in QSAT. Clearly, for every ψ, this process takes $\Omega(1)$ time to complete.

Induction step. We now prove the induction step in which we will show that if $B[\beta_i] \subseteq [t^\alpha, t^\omega]$ and we can simultaneously black pebble $G_i = \{g_i^1, \ldots, g_i^{4i+3}\}$ at time t^ω without placing a pebble on any descendant of g_i^{4i+3} while using no more than $4i+3$ pebbles during $[t^\alpha, t^\omega]$, then $\psi\lceil_{\beta_i}$ is in QSAT and there must be at least $\Omega(2^k)$ time units between t^α and t^ω, where k is the number of universally quantified variables among the innermost i variables of ψ. We are concerned that we never use more than $4i+3$ pebbles, because β_i, which is clamped during this time, contains $4n-4i$ pebbles, which leaves us at most $4i+3$ to use before we exceed our global bound of $4n+3$ pebbles.

The induction step is split into two main cases depending on whether the ith quantifier of ψ is a universal quantifier or an existential quantifier. The universal case appeals to the induction hypothesis twice, once to prove that $\psi\lceil_{\beta_i \cup \{\bar{x}_i\}}$ is in QSAT and once to prove that $\psi\lceil_{\beta_i \cup \{x_i\}}$ is in QSAT. Together, the cases allow us to conclude that $\psi\lceil_{\beta_i}$ is in QSAT. The existential case is split into two subcases, each of which appeals to the induction hypothesis once. In one case we prove that $\psi\lceil_{\beta_i \cup \{\bar{x}_i\}}$ is in QSAT, and in the other case we prove that $\psi\lceil_{\beta_i \cup \{x_i\}}$ is in QSAT. Again, this allows us to conclude that $\psi\lceil_{\beta_i}$ is in QSAT. Since the universal case appeals to the induction hypothesis twice, each time using $\Omega(2^{k-1})$ units of time, we can conclude that black pebbling all the members of G_i requires $\Omega(2^k)$ time. For the existential case, the number of universals in the innermost i widgets is the same as the number of universals in the innermost $i-1$ widgets, so the single application of the induction hypothesis allows us to conclude that black pebbling all the members of G_i requires $\Omega(2^k)$ time.

Each case of the induction step is very complex. We will therefore present each argument as an itemized sequence of key points, each of which is justified separately.

We will use this sequence of points to build a table of intervals during some of which we will prove that certain nodes must contain pebbles and during some of which we will prove that certain nodes must be empty. These tables are shown in Figures 7, 8, and 9. Figure 7 summarizes the intervals for the induction step when the ith quantifier is a universal quantifier. Figures 8 and 9 summarize the intervals for the two subcases of the induction step when the ith quantifier is an existential quantifier. In each figure, time is shown on the x-axis, while the nodes under consideration label the y-axis. Each interval during which a node must contain a black pebble is shown as a thick blue line next to the name of the node, while each interval during which the node must be empty is shown as a thinner red line. The sequence of intervals for each node is also labeled with the key point from the proof which justifies it. The shaded green regions represent the times during which we appeal to the induction hypothesis. Of critical importance is the sequence of intervals which enter the green region since these encode the configuration which the graph is in at the moment when we appeal to it.

One important point that we make now and use constantly throughout both cases is that for all $\beta_i \in A_i$, $|B[\beta_i]| = 4n - 4i$. Since our space bound is $4n + 3$ and $B[\beta_i] \in \bullet[[t^\alpha, t^\omega]]\bullet$, there are therefore at most $4i + 3$ free pebbles available at any time in $[t^\alpha, t^\omega]$. We will therefore often refer to $4i + 3$ as the space bound during the induction step and just take for granted that $B[\beta_i]$ fills the rest.

Case 1 (see Figure 7). Q_i is a universal quantifier. We will show that in order to black pebble G_i we must necessarily pass through a number of all-black configurations, including black pebbling G_{i-1} twice, once with black pebbles on y_i, x'_i, d_i, and $\bar{x}_i$ (the false configuration) and once with black pebbles on y_i, $\bar{x}'_i$, a_i, and x_i (the true configuration). We will work backwards through this strategy, showing that all minimal black-white pebbling strategies for $\mathcal{G}$ must pass through the set of specified configurations by showing that certain nodes must contain pebbles at certain times and must be empty at other times.

We want to prove that the vertices must be pebbled in the order $y_i, \bar{x}_i, d_i, \bar{x}'_i$, after which we will be in a position to apply the induction hypothesis.

- (1.1) *No descendant of any node in $G_i \cup G_{i-1} \cup R_i \cup \{a_i, b_i, c_i, h_i\}$ that is outside of widget i contains a white pebble at any time during $[t^\alpha, t^\omega]$.*
  The only descendants of any node in $G_i \cup G_{i-1} \cup R_i \cup \{a_i, b_i, c_i, h_i\}$ which are outside of widget i are all descendants of g_i^{4i+3}. The only members of these which can contain a pebble at t^α are those in $B[\beta_i]$, so no descendant of any of those nodes can contain a white pebble at t^α. Furthermore, by the assumptions of the induction step, we know that no descendants of g_i^{4i+3} can be pebbled during $[t^\alpha, t^\omega]$, so none can contain a white pebble at any time in $[t^\alpha, t^\omega]$.
- (1.2) *Every node of widget i other than members of G_i is in $\oslash[[t^\omega, t^\omega]]\oslash$.*
  Between t^α and t^ω there are at most $4i + 3$ free pebbles. G_i has exactly $4i + 3$ members, and every member of G_i must contain a black pebble at t^ω. No node of G_i is in $B[\beta_i]$. Therefore there are not enough free pebbles to keep one on any other node of widget i at t^ω. A similar argument will be used repeatedly to show that certain nodes must be empty at certain times between t^α and t^ω.
- (1.3) *g_i^{4i+3} is uniquely black pebblable in $[t^\alpha, t^\omega]$, $g_i^{4i+3} \in \oslash[[t^\alpha, t^{14}_{g_i^{4i+3}} - 1]]\oslash$, and $g_i^{4i+3} \in \bullet[[t^{14}_{g_i^{4i+3}}, t^\omega]]\bullet$, where $t^{14}_{g_i^{4i+3}} = t^\omega$; and for each j, $1 \le j \le 4i + 2$,*

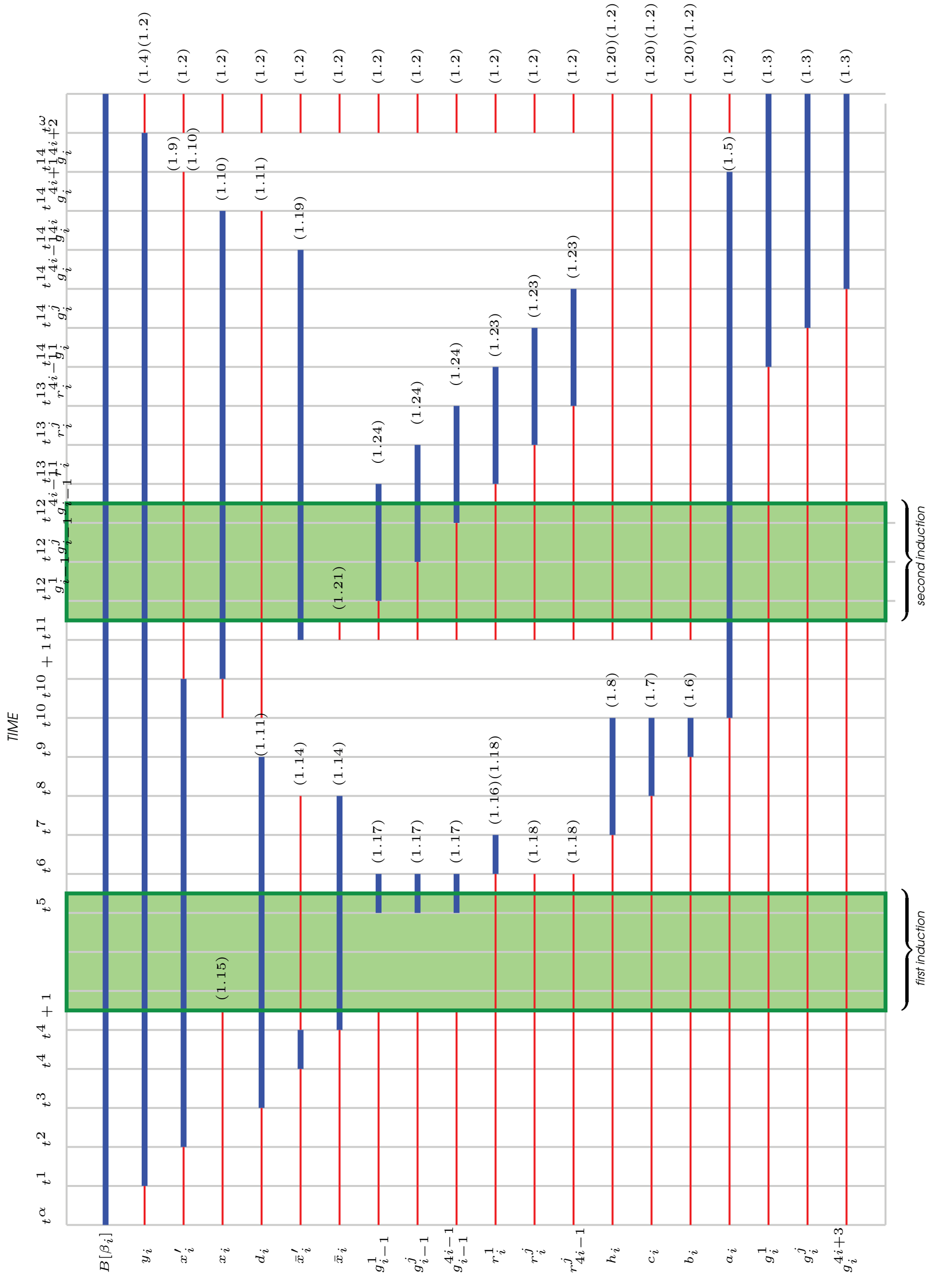

FIG. 7. *Subintervals of* $[t^\alpha, t^\omega]$ *during which nodes of the universal widget must be empty or must contain a black pebble. Intervals during which a node must contain a black pebble are shown as a thick blue line, while intervals during which a node must be empty are shown as a thinner red line.*

g_i^j is uniquely black pebblable in $[t^\alpha, t^\omega]$, $g_i^j \in \oslash[[t^\alpha, t^{14}_{g_i^j} - 1]]\oslash$, and $g_i^j \in \bullet[[t^{14}_{g_i^j}, t^\omega]]\bullet$, where $t^{14}_{g_i^j} < t^{14}_{g_i^{j+1}}$.

By (1.1) none of g_i^{4i+3}'s descendants contains any white pebbles at any time in $[t^\alpha, t^\omega]$. Also, g_i^{4i+3} has $4i+3$ predecessors, none of which is in $\bullet[[t^\alpha, t^\omega]]\bullet$, since they must all be empty at t^α. Furthermore, g_i^{4i+3} cannot contain a white pebble at t^ω, since every member of G_i must contain a black pebble at t^ω. We can therefore apply Lemma 11 to conclude that g_i^{4i+3} is uniquely black pebblable in $[t^\alpha, t^\omega]$ at some time $t^{14}_{g_i^{4i+3}}$, and since it must contain a black pebble until t^ω, $g_i^{4i+3} \in \bullet[[t^{14}_{g_i^{4i+3}}, t^\omega]]\bullet$ and $g_i^{4i+3} \in \oslash[[t^\alpha, t^{14}_{g_i^{4i+3}} - 1]]\oslash$.

We can now prove by induction from $j = 4i+2$ down to $j = 1$ that g_i^j is uniquely black pebblable in $[t^\alpha, t^\omega]$ and is clamped in $\bullet[[t^{14}_{g_i^j}, t^\omega]]\bullet$, where $t^{14}_{g_i^j} < t^{14}_{g_i^{j+1}}$. The argument is exactly the same as for g_i^{4i+3} except that g_i^j has more descendants than g_i^{4i+3}, namely, every g_i^l, where $l > j$. But by the induction hypothesis we know that all of these nodes are uniquely black pebblable in $[t^\alpha, t^\omega]$ and therefore cannot contain a white pebble at any time in $[t^\alpha, t^\omega]$. So we can still apply Lemma 11 to conclude that g_i^j is uniquely black pebblable in $[t^\alpha, t^\omega]$.

Furthermore, since g_i^j is a predecessor of each g_i^l where $l > j$, it must be black pebbled before g_i^l can be. Therefore, $t^{14}_{g_i^j} < t^{14}_{g_i^{j+1}}$. Futhermore, because each member of g_i^j must contain a black pebble at t^ω but cannot be repebbled after $t^{14}_{g_i^j}$, $g_i^j \in \bullet[[t^{14}_{g_i^{4i+3}}, t^\omega]]\bullet$ and $g_i^j \in \oslash[[t^\alpha, t^{14}_{g_i^j} - 1]]\oslash$.

Finally, note that when g_i^{4i+3} is black pebbled at $t^{14}_{g_i^{4i+3}}$, every member of G_i must contain a black pebble and all are clamped through t^ω. Since every other pebbled node is also clamped through t^ω and there are no free pebbles, no moves can be made between $t^{14}_{g_i^{4i+3}}$ and t^ω. Therefore, $t^{14}_{g_i^{4i+3}} = t^\omega$.

- (1.4) *y_i is uniquely black pebblable in $[t^\alpha, t^\omega]$ at time t^1, $y_i \in \oslash[[t^\alpha, t^1 - 1]]\oslash$, and $y_i \in \bullet[[t^1, t^{14}_{g_i^{4i+3}} - 1]]\bullet$, where no nodes other than y_i's predecessors are pebbled between t^α and t^1.*

  In order to black pebble any member of G_i, we must first pebble y_i. Neither y_i nor any of its $4i+3$ source node predecessors is in $\bullet[[t^\alpha, t^\omega]]\bullet$ since all of widget i is empty at t^α.

  By (1.3) g_i^{4i+3} must be black pebbled at time $t^{14}_{g_i^{4i+3}}$. Also, g_i^{4i+3} has $4i+3$ predecessors, including $4i+2$ members of G_i, which by (1.3) are clamped until t^ω. By (1.2) y_i must be empty at t^ω. We can therefore apply Lemma 15 to conclude that y_i must contain a black pebble at $t^{14}_{g_i^{4i+3}} - 1$.

  We therefore know that y_i is empty at t^α, must be empty at t^ω, and must contain a black pebble at some point in between. By the inductive hypothesis, we know that there are no white pebbles on the graph at time t^α. We can therefore apply Lemma 12 to conclude that y_i is uniquely black pebblable in $[t^\alpha, t^\omega]$, where no nodes other than y_i's predecessors are pebbled between t^α and t^1. Furthermore, $y_i \in \oslash[[t^\alpha, t^1 - 1]]\oslash$, and since y_i is a predecessor of g_i^{4i+3}, $y_i \in \bullet[[t^1, t^{14}_{g_i^{4i+3}} - 1]]\bullet$.

Our argument now divides into two sections. In order to simultaneously black pebble G_i, we must black pebble g_i^{4i+1}, which requires that both a_i and $\{g_i^1, \ldots, g_i^{4i}\}$

must contain pebbles. In the first part of the argument we prove that, in order to pebble a_i, $\psi\lceil_{\beta_i \cup \{\bar{x}_i\}}$ must be QSAT, and that $\Omega(2^{k-1})$ units of time must pass between t^α and the time a_i is pebbled, where k is the number of universally quantified variables among the innermost i variables of ψ. In the second part of the argument, we argue that pebbling each member of $\{g_i^1, \ldots, g_i^{4i}\}$ without exceeding our space bound requires that $\psi\lceil_{\beta_i \cup \{x_i\}}$ is in QSAT, and that $\Omega(2^{k-1})$ units of time pass between the time a_i is pebbled and t^ω. This will allow us to conclude that black pebbling G_i requires that $\psi\lceil_{\beta_i}$ is in QSAT and requires $\Omega(2^k)$ time when widget i is a universal quantifier widget.

- (1.5) *a_i is uniquely black pebblable in $[t^\alpha, t^{14}_{g_i^{4i+3}} - 1]$, $a_i \in \oslash[[t^\alpha, t^{10} - 1]]\oslash$, and $a_i \in \bullet[[t^{10}, t^{14}_{g_i^{4i+2}} - 1]]\bullet$, where $t^{10} < t^{14}_{g_i^1}$.*
  In order to black pebble g_i^1, we must first pebble a_i at some time t^{10}, before $t^{14}_{g_i^1}$. By (1.4), we know that $a_i \in \oslash[[t^\alpha, t^1]]\oslash$.
  By (1.4), $y_i \in \bullet[[t^1, t^{14}_{g_i^{4i+3}} - 1]]\bullet$, so there are at most $4i+2$ free pebbles during $[t^1, t^{14}_{g_i^{4i+3}} - 1]$. By (1.3) and (1.1), we know that no descendants of a_i can contain a white pebble at any time in $[t^\alpha, t^\omega]$. Since a_i has $4i+2$ predecessors, none of which is y_i, and since a_i must be empty at $t^{14}_{g_i^{4i+3}} - 1$, we can appeal to Lemma 11 to conclude that a_i is uniquely black pebblable in the interval $[t^1, t^{14}_{g_i^{4i+3}} - 1]$. Let t^{10} be the time at which a_i is black pebbled. Then since a_i is a predecessor of g_i^1 and g_i^1 must be black pebbled at $t^{14}_{g_i^1}$, $t^{10} < t^{14}_{g_i^1}$. Also, since a_i can be pebbled only once before $t^{14}_{g_i^{4i+3}} - 1$, $a_i \in \oslash[[t^\alpha, t^{10} - 1]]\oslash$ and a_i's pebble must stay in place until the last of its successors, g_i^{4i+2}, is pebbled, so $a_i \in \bullet[[t^{10}, t^{14}_{g_i^{4i+2}} - 1]]\bullet$.
- (1.6) *b_i is uniquely black pebblable in $[t^\alpha, t^{14}_{g_i^{4i+3}} - 1]$, $b_i \in \oslash[[t^\alpha, t^9 - 1]]\oslash$, and $b_i \in \bullet[[t^9, t^{10} - 1]]\bullet$, where $t^9 < t^{10}$.*
  In order to black pebble a_i at time t^{10} we must pebble b_i at some time t^9, before t^{10}. By (1.4), we know that $b_i \in \oslash[[t^\alpha, t^1]]\oslash$.
  By (1.4), $y_i \in \bullet[[t^1, t^{14}_{g_i^{4i+3}} - 1]]\bullet$, so there are at most $4i+2$ free pebbles during $[t^1, t^{14}_{g_i^{4i+3}} - 1]$. By (1.3), (1.1), and (1.5), we know that no descendants of b_i can contain a white pebble at any time in $[t^1, t^{14}_{g_i^{4i+3}} - 1]$. Since b_i has $4i+2$ predecessors, none of which is y_i, and since b_i must be empty at $t^{14}_{g_i^{4i+3}} - 1$, we can appeal to Lemma 11 to conclude that b_i is uniquely black pebblable in the interval $[t^1, t^{14}_{g_i^{4i+3}} - 1]$. Let t^9 be the time at which b_i is black pebbled. Then since b_i is a predecessor of a_i, $t^9 < t^{10}$. Also, since b_i can be pebbled only once before $t^{14}_{g_i^{4i+3}} - 1$, $b_i \in \oslash[[t^\alpha, t^9 - 1]]\oslash$ and b_i's pebble must stay in place until a_i is pebbled, so $b_i \in \bullet[[t^9, t^{10} - 1]]\bullet$.
- (1.7) *c_i is uniquely black pebblable in $[t^\alpha, t^{14}_{g_i^{4i+3}} - 1]$, $c_i \in \oslash[[t^\alpha, t^8 - 1]]\oslash$, and $c_i \in \bullet[[t^8, t^{10} - 1]]\bullet$, where $t^8 < t^9$.*
  In order to black pebble b_i at time t^9 we must pebble c_i at some time t^8, before t^9. By (1.4), we know that $c_i \in \oslash[[t^\alpha, t^1]]\oslash$.
  By (1.4), $y_i \in \bullet[[t^1, t^{14}_{g_i^{4i+3}} - 1]]\bullet$, so there are at most $4i+2$ free pebbles during $[t^1, t^{14}_{g_i^{4i+3}} - 1]$. By (1.3), (1.1), (1.5), and (1.6), we know that no descendants of c_i can contain a white pebble at any time in $[t^1, t^{14}_{g_i^{4i+3}} - 1]$. Since c_i has $4i+2$

predecessors, none of which is y_i, and since c_i must be empty at $t^{14}_{g_i^{4i+3}} - 1$, we can appeal to Lemma 11 to conclude that c_i is uniquely black pebblable in the interval $[t^1, t^{14}_{g_i^{4i+3}} - 1]$. Let t^8 be the time at which c_i is black pebbled. Then since c_i is a predecessor of b_i, $t^8 < t^9$. Also, since c_i can be pebbled only once before $t^{14}_{g_i^{4i+3}} - 1$, $c_i \in \oslash[[t^\alpha, t^8 - 1]]\oslash$ and c_i's pebble must stay in place until a_i is pebbled, so $c_i \in \bullet[[t^8, t^{10} - 1]]\bullet$.

- (1.8) *h_i is uniquely black pebblable in $[t^\alpha, t^{14}_{g_i^{4i+3}} - 1]$, $h_i \in \oslash[[t^\alpha, t^7 - 1]]\oslash$, and $h_i \in \bullet[[t^7, t^{10} - 1]]\bullet$, where $t^7 < t^8$.*
In order to black pebble c_i at time t^8 we must pebble h_i at some time t^7, before t^8. By (1.4), we know that $h_i \in \oslash[[t^\alpha, t^1]]\oslash$.
By (1.4), $y_i \in \bullet[[t^1, t^{14}_{g_i^{4i+3}} - 1]]\bullet$, so there are at most $4i + 2$ free pebbles during $[t^1, t^{14}_{g_i^{4i+3}} - 1]$. By (1.3), (1.1), (1.5), (1.6), and (1.7), we know that no descendants of h_i can contain a white pebble at any time in $[t^1, t^{14}_{g_i^{4i+3}} - 1]$. Since h_i has $4i + 2$ predecessors, none of which is y_i, and since h_i must be empty at $t^{14}_{g_i^{4i+3}} - 1$, we can appeal to Lemma 11 to conclude that h_i is uniquely black pebblable in the interval $[t^1, t^{14}_{g_i^{4i+3}} - 1]$. Let t^7 be the time at which h_i is black pebbled. Then since h_i is a predecessor of c_i, $t^7 < t^8$. Also, since h_i can be pebbled only once before $t^{14}_{g_i^{4i+3}} - 1$, $h_i \in \oslash[[t^\alpha, t^7 - 1]]\oslash$ and h_i's pebble must stay in place until a_i is pebbled, so $h_i \in \bullet[[t^7, t^{10} - 1]]\bullet$.
- (1.9) *x'_i is uniquely black pebblable in $[t^1, t^{14}_{g_i^{4i+3}} - 1]$, $x'_i \in \oslash[[t^\alpha, t^2 - 1]]\oslash$, and $x'_i \in \bullet[[t^2, t^{10} - 1]]\bullet$, where $t^1 < t^2 < t^{14}_{g_i^{4i+3}} - 1$ and no nodes other than x'_i's predecessors are pebbled between t^1 and t^2.*
We first prove that $x'_i \in \oslash[[t^\alpha, t^1]]\oslash$. x'_i is not a member of $B[\beta_i]$, so it is empty at t^α. From (1.4) we know that x'_i cannot be pebbled between t^α and t^1 and therefore must remain empty at t^1.
x'_i must also be empty at $t^{14}_{g_i^{4i+3}} - 1$ since g_i^{4i+3} has $4i + 3$ predecessors, none of which is x'_i. We now show that it must be black pebbled at some time between t^1 and $t^{14}_{g_i^{4i+3}} - 1$.
We know that x'_i must contain some pebble at $t^{10} - 1$ since it is a predecessor of a_i. This pebble must be removed by $t^{14}_{g_i^{4i+3}} - 1$ since x'_i is not one of g_i^{4i+3}'s $4i + 3$ predecessors. By (1.4) $y_i \in \bullet[[t^{10}, t^{14}_{g_i^{4i+3}} - 1]]\bullet$, by (1.5) $a_i \in \bullet[[t^{10}, t^{14}_{g_i^{4i+2}} - 1]]\bullet$, and by (1.3) $g_i^1 \in \bullet[[t^{14}_{g_i^{4i+2}} - 1, t^{14}_{g_i^{4i+3}} - 1]]\bullet$, so at least two of the $4i+3$ free pebbles must be clamped at each time unit in $[t^{10}, t^{14}_{g_i^{4i+3}} - 1]$. This means that there are not enough free pebbles to remove a white pebble from x'_i during $[t^{10}, t^{14}_{g_i^{4i+3}} - 1]$. So x'_i must contain a black pebble at $t^{10} - 1$. We therefore know that x'_i must be empty at t^1 and $t^{14}_{g_i^{4i+3}} - 1$ and must contain a black pebble at some point in $[t^1, t^{14}_{g_i^{4i+3}} - 1]$. By (1.4), we know that $y_i \in \bullet[[t^1, t^{14}_{g_i^{4i+3}} - 1]]\bullet$. By the induction hypothesis and (1.4), we know that no nodes contain white pebbles at t^α, y_i is black pebbled at t^1, and no nodes other than y_i's predecessors are pebbled between t^α and t^1. Thus there are no white pebbles on nonsource nodes of the graph at t^1. Since x'_i has $4i+2$ source nodes as predecessors, we can therefore apply Lemma 12 to conclude

that x'_i is uniquely black pebblable in $[t^1, t^{14}_{g_i^{4i+3}} - 1]$ and x'_i is pebbled at t^2, $t^1 < t^2 < t^{14}_{g_i^{4i+3}} - 1$, where no nodes other than x'_i's predecessors are pebbled between t^1 and t^2, so $x'_i \in \oslash[[t^\alpha, t^2 - 1]]\oslash$. Since x'_i is a predecessor of a_i, $x'_i \in \bullet[[t^2, t^{10} - 1]]\bullet$.

- (1.10) *$x_i \in \oslash[[t^{10} - 1, t^{10}]]\oslash$, $x_i \in \bullet[[t^{10} + 1, t^{14}_{g_i^{4i+1}} - 1]]\bullet$, $x'_i \in \bullet[[t^2, t^{10}]]\bullet$, and $x'_i \in \oslash[[t^{10} + 1, t^{14}_{g_i^{4i+3}} - 1]]\oslash$.*

  x_i must be empty at $t^{10} - 1$ since by (1.4) $y_i \in \bullet[[t^1, t^{14}_{g_i^{4i+3}} - 1]]\bullet$ and a_i has $4i + 2$ predecessors, none of which is x_i or y_i. It must remain empty at t^{10} since a_i is pebbled at t^{10}, not x_i, so $x_i \in \oslash[[t^{10} - 1, t^{10}]]\oslash$.

  By (1.5), (1.6), (1.7), and (1.8) we already know that three of x'_i's successors, a_i, b_i, c_i, and h_i can be pebbled only a single time in $[t^1, t^{14}_{g_i^{4i+3}} - 1]$ and none is pebbled after t^{10}. x'_i only has a single other successor x_i.

  Since x'_i is not one of g_i^1's $4i+3$ predecessors, it must be empty at $t^{14}_{g_i^1} - 1$. But x_i must contain a pebble by $t^{14}_{g_i^1} - 1$ since it is a predecessor of g_i^1. Furthermore, the pebble must be black because g_i^1 has $4i + 3$ predecessors, none of which is x'_i. Therefore x'_i must be empty and, by (1.3), must remain empty until t^ω, so a white pebble cannot be removed from x_i. x_i is therefore the only successor of x'_i that is pebbled after t^{10} and before x'_i is empty at $t^{14}_{g_i^1} - 1$. We can therefore apply Lemma 14 to conclude that x'_i's pebble can be slid to x_i at $t^{10} + 1$, so $x_i \in \bullet[[t^{10} + 1, t^{14}_{g_i^{4i+1}} - 1]]\bullet$.

  We will pause for a moment to make some general comments about our use of Lemma 14. Observe that in applying Lemma 14, we are slightly modifying our strategy. However, the new strategy will still satisfy all five conditions of our lemma, and, moreover, all of the properties that we have proved so far continue to hold for our new strategy. In what follows, we will be applying Lemma 14 several times. Each time Lemma 14 is applied, it will be applied in the same general way, to argue that we can move a pebble up from either x'_i or $\bar{x}'_i$. Thus we will be slightly modifying the strategy as we go along, but again the new strategy will always continue to satisfy the five conditions of our lemma, as well as all of the properties that we have proved up until that point.

  Now continuing on with the argument, by (1.9) x'_i is uniquely black pebblable in $[t^1, t^{14}_{g_i^{4i+3}} - 1]$, $x'_i \in \oslash[[t^{10} + 1, t^{14}_{g_i^{4i+3}} - 1]]\oslash$. We can also extend the result from (1.9) to conclude that $x'_i \in \bullet[[t^2, t^{10}]]\bullet$ since x'_i is uniquely black pebblable in $[t^1, t^{14}_{g_i^{4i+3}} - 1]$ and must therefore contain a pebble until the last time one of its successors is black pebbled during this time.

- (1.11) *d_i is uniquely black pebblable in $[t^2, t^{14}_{g_i^{4i+1}} - 1]$, $d_i \in \oslash[[t^\alpha, t^3 - 1]]\oslash$, $d_i \in \oslash[[t^{10} - 1, t^{14}_{g_i^{4i+1}} - 1]]\oslash$, and $d_i \in \bullet[[t^3, t^9 - 1]]\bullet$, where no nodes other than d_i's predecessors are pebbled between t^2 and t^3, and $t^2 < t^3 < t^8$.*

  In order to pebble c_i at t^8 we must first pebble d_i at some time t^3, $t^3 < t^8$.

  We first prove that $d_i \in \oslash[[t^\alpha, t^2]]\oslash$. d_i is not a member of $B[\beta_i]$, so it is empty at t^α. From (1.4) we know that d_i cannot be pebbled between t^α and t^1, and by (1.9) we know that d_i cannot be pebbled between t^1 and t^2 and therefore must remain empty at t^2.

  d_i must also be empty at $t^{10} - 1$ since, by (1.4), $y_i \in \bullet[[t^1, t^{14}_{g_i^{4i+3}} - 1]]\bullet$ and a_i

has $4i+2$ predecessors, none of which is d_i or y_i. We now show that d_i must be black pebbled at some point in $[t^2, t^{10}-1]$.

As we have already noted, d_i must contain some pebble at t^8-1 and must be empty at $t^{10}-1$. Suppose, for the sake of contradiction, that d_i contains a white pebble at t^8-1. Then its pebble must be removed at some time in $[t^8, t^{10}-1]$. But by (1.4) and (1.9), we know that y_i and x_i' in $\bullet[[t^2, t^{10}-1]]\bullet$, and by (1.7), we know that c_i is in $\bullet[[t^8, t^{10}-1]]\bullet$, so d_i's white pebble cannot be removed in $[t^8, t^{10}-1]$ due to its high in-degree. d_i must therefore contain a black pebble at t^8-1.

We therefore know that d_i must be empty at t^2 and $t^{10}-1$ and must contain a black pebble at some point in between. From the previous paragraph, we know that y_i and x_i' are in $\bullet[[t^2, t^{10}-1]]\bullet$. By the induction hypothesis, no nodes contain white pebbles at t^α. By (1.4), between t^α and t^1 the only nodes that are pebbled are y_i and the predecessors of y_i, and by (1.9) the only nodes pebbled between t^1 and t^2 are x_i' and the predecessors of x_i'. Therefore, there are no white pebbles on any nonsource nodes of the graph at t^2. Since d_i has $4i+1$ source nodes as predecessors, we can therefore apply Lemma 12 to conclude that d_i is uniquely black pebblable in $[t^2, t^{10}-1]$ and must be pebbled at t^3, $t^2 < t^3 < t^8-1$, where no nodes other than d_i's predecessors are pebbled between t^2 and t^3, so $d_i \in \oslash[[t^\alpha, t^3-1]]\oslash$. Also, since d_i is a predecessor of b_i, $d_i \in \bullet[[t^3, t^9-1]]\bullet$.

We now show that $d_i \in \oslash[[t^{10}-1, t^{14}_{g_i^{4i+1}}-1]]\oslash$. As mentioned already, d_i must be empty at $t^{10}-1$. We show that d_i cannot be pebbled at any time in $[t^{10}, t^{14}_{g_i^{4i+1}}-1]$. By (1.4) $y_i \in \bullet[[t^{10}, t^{14}_{g_i^{4i+1}}-1]]\bullet$. By (1.10) there is a pebble on either x_i or x_i' during every time unit in $[t^{10}, t^{14}_{g_i^{4i+1}}-1]$. By (1.5) $a_i \in \bullet[[t^{10}, t^{14}_{g_i^{4i+1}}-1]]\bullet$. Since d_i has in-degree $4i+1$, it is therefore impossible to black pebble it at any point in $[t^{10}, t^{14}_{g_i^{4i+1}}-1]$.

By the same argument, it is also impossible to remove a white pebble from d_i at any point in $[t^{10}, t^{14}_{g_i^{4i+1}}-1]$. So if d_i was white pebbled during $[t^{10}, t^{14}_{g_i^{4i+1}}-1]$, the pebble would have to remain on d_i at $t^{14}_{g_i^{4i+1}}-1$. But since g_i^{4i+1} has $4i+3$ predecessors, none of which is d_i, d_i must be empty at $t^{14}_{g_i^{4i+1}}-1$. It is therefore impossible to white pebble d_i during $[t^{10}, t^{14}_{g_i^{4i+1}}-1]$. Since d_i cannot be pebbled at all in $[t^{10}, t^{14}_{g_i^{4i+1}}-1]$ and is uniquely black pebblable in $[t^2, t^{10}-1]$, it is uniquely black pebblable in $[t^2, t^{14}_{g_i^{4i+1}}-1]$. Since it is pebbled at $t^3 < t^{10}-1$ and is empty at $t^{10}-1$, this implies that $d_i \in \oslash[[t^{10}-1, t^{14}_{g_i^{4i+1}}-1]]\oslash$.

- (1.12) $\bar{x}_i'$ *is in* $\oslash[[t^7, t^9-1]]\oslash$.
  By (1.4) $y_i \in \bullet[[t^7-1, t^9]]\bullet$. Also, h_i has $4i+2$ predecessors, none of which is y_i or $\bar{x}_i'$, so in order to black pebble h_i at t^7, $\bar{x}_i'$ must be empty at t^7-1. At t^9-1, $\bar{x}_i'$ must also be empty for essentially the same reasons. Between these points there are always at least 4 pebbles on the widget. From t^7 through t^9-1 there is a pebble on y_i by (1.4), on x_i' by (1.9), on h_i by (1.8), and on d_i by (1.11). Since $\bar{x}_i'$ has $4i$ source nodes as predecessors, it cannot be black pebbled during $[t^7, t^9-1]$. It also cannot be white pebbled during this interval and have the pebble removed by t^9-1. It must therefore remain empty during $[t^7, t^9-1]$.

- (1.13) $\bar{x}_i$ *must contain a black pebble at* $t^8 - 1$ *and must be empty at* $t^9 - 1$. In order to black pebble c_i at t^8, $\bar{x}_i$ must contain a pebble at $t^8 - 1$. Suppose, for the sake of contradiction, that $\bar{x}_i$ contains a white pebble at $t^8 - 1$. We show that this will force us to exceed the space bound. b_i is pebbled at t^9 and has $4i+2$ predecessors, none of which is $\bar{x}_i$. Also, by (1.4), y_i is clamped at this time and is also not a predecessor of b_i. $\bar{x}_i$ must therefore be empty at $t^9 - 1$. This means that $\bar{x}_i$'s white pebble must be removed at some point in $[t^8 - 1, t^9 - 1]$. This requires $\bar{x}'_i$ to contain a pebble at the moment the white pebble is removed from $\bar{x}_i$. But by (1.12) we know that $\bar{x}'_i$ cannot contain a pebble during $[t^8 - 1, t^9 - 1]$, so $\bar{x}_i$ must contain a black pebble at $t^8 - 1$.
- (1.14) $\bar{x}'_i$ *is uniquely pebblable in the interval* $[t^3, t^8 - 1]$, $\bar{x}'_i \in \oslash[[t^\alpha, t^4 - 1]]\oslash$, $\bar{x}'_i \in \bullet[[t^4, t^4]]\bullet$, *and* $\bar{x}'_i \in \oslash[[t^4 + 1, t^8 - 1]]\oslash$; $\bar{x}_i \in \oslash[[t^\alpha, t^4]]\oslash$ *and* $\bar{x}_i \in \bullet[[t^4 + 1, t^8 - 1]]\bullet$*, where no nodes other than* $\bar{x}'_i$*'s predecessors are pebbled between* t^3 *and* t^4*, so* $t^3 < t^4 < t^8$.
We first prove that $\bar{x}'_i \in \oslash[[t^\alpha, t^3]]\oslash$. $\bar{x}'_i$ is not a member of $B[\beta_i]$, so it is empty at t^α. From (1.4) we know that $\bar{x}'_i$ cannot be pebbled between t^α and t^1. By (1.9) we know that $\bar{x}'_i$ cannot be pebbled between t^1 and t^2. By (1.11) we know that $\bar{x}'_i$ cannot be pebbled between t^2 and t^3 and therefore must remain empty at t^3. By the same argument $\bar{x}_i \in \oslash[[t^\alpha, t^3]]\oslash$.
From (1.13) we know that $\bar{x}_i$ must contain a black pebble at $t^8 - 1$. In order to black pebble $\bar{x}_i$ by $t^8 - 1$, we must first pebble $\bar{x}'_i$ at some time t^4, $t^4 < t^8 - 1$. Suppose $\bar{x}'_i$ is white pebbled at this time. $\bar{x}'_i$ must be empty at $t^8 - 1$ since by (1.4) $y_i \in \bullet[[t^1, t^{14}_{g_i^{4i+3}} - 1]]\bullet$ and by (1.7) c_i is pebbled at t^8 and c_i has $4i + 2$ predecessors, none of which is y_i or $\bar{x}'_i$, so $\bar{x}'_i$'s white pebble is needed and must be removed between t^4 and $t^8 - 1$ at a time when there is already a pebble on $\bar{x}_i$. But by (1.4), (1.10), and (1.11) there must be three other nodes clamped during this time, so $\bar{x}'_i$'s white pebble cannot be removed between t^4 and $t^8 - 1$. $\bar{x}'_i$ therefore must be black pebbled at t^4.
We therefore know that $\bar{x}'_i$ must be empty at t^3 and $t^8 - 1$ and must contain a black pebble at some point in between. By (1.4), (1.9), and (1.11), we know that there are three nodes in $\bullet[[t^3, t^8 - 1]]\bullet$. By the induction hypothesis, there are no white pebbles on the graph at t^α. By (1.4), (1.9), and (1.11), the only nodes pebbled between t^α and t^3 are black pebbles on y_i and its predecessors, x'_i and its predecessors, and d_i and its predecessors. Therefore, there are no white pebbles on any nonsource nodes at t^3. Finally, $\bar{x}'_i$ has $4i$ source nodes as predecessors. We can therefore apply Lemma 12 to conclude that $\bar{x}'_i$ is uniquely black pebblable in $[t^3, t^8 - 1]$ and $\bar{x}'_i$ must be pebbled at t^4, $t^3 < t^4 < t^{14}_{g_i^1} - 1$, where no nodes other than $\bar{x}'_i$'s predecessors are pebbled between t^3 and t^4, so $\bar{x}'_i \in \oslash[[t^\alpha, t^4 - 1]]\oslash$ and $\bar{x}_i \in \oslash[[t^\alpha, t^4]]\oslash$.
Since $\bar{x}'_i$ must be empty at $t^8 - 1$, it must be removed before any of $\bar{x}'_i$'s other successors get pebbled after $t^{14}_{g_i^1} - 1$. This means that the pebble assignment to $\bar{x}'_i$ that begins at t^4 only supports pebble assignments to a single successor, $\bar{x}_i$, between t^4 and the time it is removed before $t^8 - 1$. We can therefore apply Lemma 14 to conclude that $\bar{x}'_i$'s pebble is slid up to $\bar{x}_i$ at $t^4 + 1$. So $\bar{x}'_i \in \bullet[[t^4, t^4]]\bullet$, $\bar{x}'_i \in \oslash[[t^4 + 1, t^8 - 1]]\oslash$, and $\bar{x}_i \in \bullet[[t^4 + 1, t^8 - 1]]\bullet$.
- (1.15) $x_i \in \oslash[[t^\alpha, t^4 + 1]]\oslash$.
x_i is not a member of $B[\beta_i]$, so it is empty at t^α. From (1.4) we know that x_i cannot be pebbled between t^α and t^1, and by (1.9) we know that x_i cannot be pebbled between t^1 and t^2. By (1.11) we know that x_i cannot be pebbled

between t^2 and t^3. By (1.14) we know that x_i cannot be pebbled between t^3 and t^4+1. It must therefore remain empty at t^4+1.

- (1.16) r_i^1 *is uniquely black pebblable in* $[t^4+1, t^8-1]$, $r_i^1 \in \bullet[[t^6, t^7-1]]\bullet$, *where* $t^6 < t^7$.
  In order to black pebble h_i we must first pebble r_i^1 at some time t^6 before t^7. By (1.14), r_i^1 is empty at t^4+1. By (1.4) y_i is in $\bullet[[t^1, t^{14}_{g_i^{4i+3}}-1]]\bullet$. By (1.9) x_i' is in $\bullet[[t^2, t^{10}-1]]\bullet$. By (1.11) d_i is in $\bullet[[t^3, t^9-1]]\bullet$. And by (1.14) $\bar{x}_i$ is in $\bullet[[t^4+1, t^8-1]]\bullet$. r_i^1 has $4i-1$ predecessors, none of which is y_i, x_i', d_i, or $\bar{x}_i$. By (1.1), (1.3), (1.5), (1.6), (1.7), and (1.8) no descendant of r_i^1 can contain a white pebble at any time in $[t^4+1, t^8-1]$. Finally, r_i^1 cannot contain a white pebble at t^8-1 because it must be empty then due to the clamping of y_i and c_i's in-degree of $4i+2$. We can therefore apply Lemma 11 to conclude that r_i^1 is uniquely black pebblable in $[t^4+1, t^8-1]$.
  Due to the fact that r_i^1 is a predecessor of h_i, r_i^1 must be pebbled at some time t^6, $t^6 < t^7$ and $r_i^1 \in \bullet[[t^6, t^7-1]]\bullet$.
- (1.17) G_{i-1} *must be simultaneously black pebbled at* $t^5 = t^6 - 1$ *when every member of* R_i *is empty and every member of* G_{i-1} *is in* $\oslash[[t^\alpha, t^4+1]]\oslash$, *where* $t^4+1 < t^5$.
  By (1.4) y_i is in $\bullet[[t^1, t^{14}_{g_i^{4i+3}}-1]]\bullet$. By (1.9) x_i' is in $\bullet[[t^2, t^{10}-1]]\bullet$. By (1.11) d_i is in $\bullet[[t^3, t^9-1]]\bullet$. And by (1.14) $\bar{x}_i$ is in $\bullet[[t^4+1, t^8-1]]\bullet$. By (1.16) r_i^1 is black pebbled at t^6. We can therefore apply Lemma 16 to conclude that G_{i-1} must be simultaneously black pebbled at $t^5 = t^6-1$ when every member of R_i is empty.
  (1.4), (1.9), (1.11), and (1.14) also tell us that every member of G_{i-1} is in $\oslash[[t^\alpha, t^4+1]]\oslash$, so $t^5 > t^4+1$.
- (1.18) *Every member of* R_i *is in* $\oslash[[t^\alpha, t^6-1]]\oslash$.
  Since no member of R_i is in $B[\beta_i]$, R_i is empty at t^α. By (1.17), every member of R_i also must be empty at t^5. Furthermore, the first time in $[t^\alpha, t^\omega]$ that any successor of any member of R_i is pebbled is after $t^6 = t^5+1$. It would therefore not be frugal to pebble any member of R_i between t^α and t^5, since the pebble would have to be removed before supporting another pebble placement. Since by (1.17) $t^5 = t^6-1$, every member of R_i is in $\oslash[[t^\alpha, t^6-1]]\oslash$.

The following relationships between the times at which certain nodes are pebbled are demonstrated by the points proved above.

1. By (1.3), $t^\omega = t^{14}_{g_i^{4i+3}}$, and for each j, $1 \le j \le 4i+2$, $t^{14}_{g_i^j} < t^{14}_{g_i^{j+1}}$.
2. By (1.5), $t^{10} < t^{14}_{g_i^1}$.
3. By (1.6), $t^9 < t^{10}$.
4. By (1.7), $t^8 < t^9$.
5. By (1.8), $t^7 < t^8$.
6. By (1.16), $t^6 < t^7$.
7. By (1.17), $t^4+1 < t^5$ and $t^5 < t^6$.
8. By (1.14), $t^3 < t^4$.
9. By (1.11), $t^2 < t^3$.
10. By (1.9), $t^1 < t^2$.
11. By (1.4), $t^\alpha < t^1$.

These inequalities produce the following ordering of times, which labels most of the x-axis of Figure 7: $t^\alpha < t^1 < t^2 < t^3 < t^4 < t^4+1 < t^5 < t^6 < t^7 < t^8 < t^9 < t^{10} < t^{14}_{g_i^1} < \cdots < t^{14}_{g_i^{4i+3}} = t^\omega$.

Figure 7 provides a summary of the points proved above. Every thin red line segment represents a time interval during which we know a specific node must be empty. Every thick blue segment represents a time interval during which we know a specific node must contain a black pebble. Each sequence of line segments is labeled to the right by the points which justify it. Of particular interest to us now is the sequence of line segments that are entering the region labeled as the "first induction." In particular, we know that every member of G_i is empty during $[t^4+1, t^5]$ since $[t^4+1, t^5]$ is a subinterval of $[t^\alpha, t^\omega]$ and, by (1.3), each member of G_i is uniquely black pebblable in $[t^\alpha, t^\omega]$ and must be pebbled after t^5. Similarly, by (1.8), h_i must be empty during $[t^4+1, t^5]$ because $[t^4+1, t^5]$ is a subinterval of $[t^1, t^{14}_{g_i^{4i+3}}-1]$ and h_i is uniquely black pebblable in $[t^1, t^{14}_{g_i^{4i+3}}-1]$ at some point after t^5. By (1.5), (1.6), and (1.7), the same is also true for a_i, b_i, and c_i. By (1.18), every member of R_i must also be empty during this interval. Therefore, no member of widget i which is a descendant of g^{4i-1}_{i-1} can be pebbled during the interval $[t^4+1, t^5]$. Also, by (1.4), (1.14), (1.11), and (1.9), exactly the nodes in $B[\beta_i] \cup \{y_i, \bar{x}_i, d_i, x_i'\} = B[\beta_i \cup \{\bar{x}_i\}]$ contain black pebbles at t^4+1 and are in $\bullet[[t^4+1, t^5]]\bullet$.

We can therefore apply the induction hypothesis to conclude that black pebbling G_{i-1} requires $\psi\lceil_{\beta_i\cup\{\bar{x}_i\}}$ to be in QSAT and that $\Omega(2^k)$ units of time must pass between t^4+1 and t^5, where k is the number of universally quantified variables among the innermost $i-1$ variables of ψ.

We now proceed with the second phase of the argument, which will show that $\psi\lceil_{\beta_i\cup\{x_i\}}$ must also be in QSAT and that a further $\Omega(2^k)$ units of time must pass between t^5 and t^ω.

- (1.19) *There exists a time t^{11}, $t^{10}+1 < t^{11} < t^{14}_{g_i^1}$, such that $\bar{x}_i'$ is black pebbled at t^{11} and $\bar{x}_i' \in \bullet[[t^{11}, t^{14}_{g_i^{4i}}-1]]\bullet$.*

  In order to black pebble g_i^1 at time $t^{14}_{g_i^1}$ we must pebble $\bar{x}_i'$ at some time t^{11}, $t^{11} < t^{14}_{g_i^1}$. Note that $\bar{x}_i'$ must be empty at t^{10} since by (1.4) y_i is clamped at that time and a_i has $4i+2$ other predecessors, none of which is $\bar{x}_i'$. By (1.10) $\bar{x}_i'$ must remain empty at $t^{10}+1$, since the move that transitions from t^{10} to $t^{10}+1$ places a black pebble on x_i and does not affect $\bar{x}_i'$. Also, $\bar{x}_i'$ must be empty again by $t^{14}_{g_i^{4i+1}}$ since g_i^{4i+1} has $4i+3$ predecessors, none of which is $\bar{x}_i'$. We will now show that $\bar{x}_i'$ must contain a black pebble at some point t^{11} between $t^{10}+1$ and $t^{14}_{g_i^{4i+1}}$.

  By (1.3) we know that g_i^{4i} is empty at $t^{14}_{g_i^{4i}}-1$ and is black pebbled at $t^{14}_{g_i^{4i}}$. By (1.3) we also know that there are $4i-1$ members of G_i in $\bullet[[t^{14}_{g_i^{4i}}-1, t^{14}_{g_i^{4i}}]]\bullet$. By (1.4), (1.10), and (1.5) the nodes y_i, x_i, and a_i are also in $\bullet[[t^{14}_{g_i^{4i}}-1, t^{14}_{g_i^{4i}}]]\bullet$. Since $\bar{x}_i'$ is another one of g_i^{4i}'s predecessors, we can apply Lemma 15 to conclude that $\bar{x}_i'$ must contain a black pebble at $t^{14}_{g_i^{4i}}-1$.

  We have therefore shown that $\bar{x}_i'$ is empty at $t^{10}+1$ and $t^{14}_{g_i^{4i+1}}$ and must contain a black pebble at some point t^{11} in between. By (1.4), (1.10), and (1.5) the nodes y_i, x_i, and a_i are all in $\bullet[[t^{10}+1, t^{14}_{g_i^{4i+1}}-1]]\bullet$. Thus at time $t^{11}-1$, the $4i$ source node predecessors of $\bar{x}_i'$ contain pebbles, and all other pebbles on the graph are black. Thus we can apply Lemma 12 to conclude that $\bar{x}_i'$ is uniquely black pebblable at t^{11} in $[t^{11}-1, t^{14}_{g_i^{4i+1}}-1]$. Furthermore, since g_i^{4i} is a successor of $\bar{x}_i'$ that by (1.3) is black pebbled at $t^{14}_{g_i^{4i}}$, we have $\bar{x}_i' \in \bullet[[t^{11}, t^{14}_{g_i^{4i}}-1]]\bullet$.

- (1.20) *Every member of* $\{h_i, c_i, b_i\}$ *is in* $\oslash[[t^{11}, t^{14}_{g_i^{4i+3}} - 1]]\oslash$.

  By (1.4), (1.10), and (1.5) y_i, a_i, and x_i are all in $\bullet[[t^{10}+1, t^{14}_{g_i^{4i}}]]\bullet$. Since $\bar{x}'_i$ has $4i+1$ source nodes as predecessors, we know that the rest of the nodes in widget i, including every member of $\{h_i, c_i, b_i\}$, must be empty at t^{11}, which by (1.19) is when $\bar{x}'_i$ is black pebbled.

  By (1.6), (1.7), and (1.8) every member of $\{h_i, c_i, b_i\}$ is uniquely black pebblable in $[t^1, t^{14}_{g_i^{4i+3}} - 1]$, and each is pebbled before t^{11}. It is therefore impossible to pebble them again before $t^{14}_{g_i^{4i+3}}$, so they must each be in $\oslash[[t^{11}, t^{14}_{g_i^{4i+3}} - 1]]\oslash$.
- (1.21) $\bar{x}_i$ *is empty at* t^{11}.

  By (1.19), $\bar{x}'_i$, which has $4i$ source node predecessors, is black pebbled at t^{11}. By (1.4), (1.10), and (1.5), y_i, x_i, and a_i are clamped at $t^{11}-1$. This means that there is not a single free pebble available for $\bar{x}_i$ at $t^{11}-1$. It remains empty at t^{11} because $\bar{x}'_i$ is black pebbled at t^{11}, and this cannot affect $\bar{x}_i$.
- (1.22) *Every member of* R_i *and every member of* G_{i-1} *is empty at* t^{11}.

  This proof is essentially the same as that of (1.21). (In (1.21) we argued that $4i+3$ pebbles are on specific nodes at time t^{11}, and therefore there are no pebbles left to put on R_i or G_{i-1}.)
- (1.23) r_i^{4i-1} *is uniquely black pebblable in* $[t^{11}, t^{14}_{g_i^{4i}} - 1]$, $r_i^{4i-1} \in \oslash[[t^{11}, t^{13}_{r_i^{4i-1}} - 1]]\oslash$, *and* $r_i^{4i-1} \in \bullet[[t^{13}_{r_i^{4i-1}}, t^{14}_{g_i^{4i-1}} - 1]]\bullet$, *where* $t^{13}_{r_i^{4i-1}} < t^{14}_{g_i^1}$; *and for each* j, $1 \le j \le 4i-2$, r_i^j *is uniquely black pebblable in* $[t^{11}, t^{14}_{g_i^{4i}} - 1]$, $r_i^j \in \oslash[[t^{11}, t^{13}_{r_i^j} - 1]]\oslash$, *and* $r_i^j \in \bullet[[t^{13}_{r_i^{4i-1}}, t^{14}_{g_i^1} - 1]]\bullet$, *where* $t^{11} < t^{13}_{r_i^j} < t^{13}_{r_i^{j+1}} < t^{14}_{g_i^1}$.

  In order to black pebble g_i^1, every member of R_i must first contain a pebble at some time t^{13} before $t^{14}_{g_i^1}$. By (1.22) every member of R_i is empty at t^{11}. Since, by (1.19), $t^{11} < t^{14}_{g_i^1}$ and every member of R_i is empty at t^{11}, we know that $t^{14}_{g_i^1} > t^{13} > t^{11}$. We prove that for all j, $1 \le j \le 4i-1$, r_i^j is uniquely black pebblable in $[t^{11}, t^{14}_{g_i^{4i}} - 1]$.

  The proof is by induction from $j = 4i-1$ down to $j = 1$ in which we apply Lemma 11 during each round. Consider r_i^j, $1 \le j \le 4i-1$. By (1.4), (1.10), (1.5), and (1.19), the four nodes y_i, x_i, a_i, and $\bar{x}'_i$ are all in $\bullet[[t^{11}, t^{14}_{g_i^{4i}} - 1]]\bullet$. g_{i-1}^j has $4i-1$ other predecessors that by (1.19) are empty at t^{11}. By (1.1), (1.3), (1.20), and (1.5) and the induction hypothesis that each r_i^l, where $l > j$, is uniquely black pebblable in $[t^{11}, t^{14}_{g_i^{4i}} - 1]$, no descendant of r_i^j can contain a white pebble at any time in $[t^{11}, t^{14}_{g_i^{4i}} - 1]$. Finally, we must show that r_i^j itself cannot contain a white pebble at $t^{14}_{g_i^{4i}} - 1$. By (1.3), g_i^{4i} is black pebbled at $t^{14}_{g_i^{4i}}$, and due to g_i^{4i}'s in-degree of $4i+3$ and since r_i^j is not a predecessor of g_i^{4i}, we know that r_i^j must be empty at $t^{14}_{g_i^{4i}} - 1$ and therefore cannot contain a white pebble at that time. We can therefore apply Lemma 11 to conclude that r_i^j is uniquely black pebblable in $[t^{11}, t^{14}_{g_i^{4i}} - 1]$.

  Furthermore, since r_i^j is a predecessor of each r_i^l, such that $l > j$, it must be black pebbled before r_i^l can be. Therefore, r_i^j must be pebbled at $t^{13}_{r_i^j}$, $t^{13}_{r_i^j} < t^{13}_{r_i^{j+1}}$. Also, since every member of R_i must be pebbled in order to black pebble g_i^1 at $t^{14}_{g_i^1}$, we have $t^{13}_{r_i^j} < t^{14}_{g_i^1}$. Since r_i^j is empty at t^{11}, is uniquely black pebblable in $[t^{11}, t^{14}_{g_i^{4i}} - 1]$, and is first pebbled at $t^{13}_{r_i^j}$, we can conclude that $r_i^j \in \oslash[[t^{11}, t^{13}_{r_i^j} - 1]]\oslash$.

Since each member of R_i is uniquely black pebblable in $[t^{11}, t^{14}_{g_i^{4i}} - 1]$ and t^{13} is the earliest time that every member of R_i contains a black pebble, $t^{13} = t^{13}_{r_i^{4i-1}}$. Since every member of r_i^j is a predecessor of g_i^1, we have $t^{13} < t^{14}_{g_i^1}$.

- (1.24) g_{i-1}^{4i-1} *is uniquely black pebblable in* $[t^{11}, t^{14}_{g_i^{4i}} - 1]$, $g_{i-1}^{4i-1} \in \oslash[[t^{11}, t^{12}_{g_{i-1}^{4i-1}} - 1]]\oslash$, *and* $g_{i-1}^{4i-1} \in \bullet[[t^{12}_{g_{i-1}^{4i-1}}, t^{14}_{g_i^{4i-1}} - 1]]\bullet$, *where* $t^{12}_{g_{i-1}^{4i-1}} = t^{12} < t^{14}_{g_i^1}$; *and for each* j, $1 \le j \le 4i-2$, g_{i-1}^j *is uniquely black pebblable in* $[t^{11}, t^{14}_{g_i^{4i}} - 1]$, $g_{i-1}^j \in \oslash[[t^{11}, t^{12}_{g_{i-1}^j} - 1]]\oslash$, *and* $g_{i-1}^j \in \bullet[[t^{12}_{g_{i-1}^j}, t^{13}_{r_i^j} - 1]]\bullet$, *where* $t^{11} < t^{12}_{g_{i-1}^j} < t^{12}_{g_{i-1}^{j+1}} < t^{13}_{r_i^j}$.

  In order to black pebble r_i^1, every member of G_{i-1} must first contain some pebble at some time t^{12}, $t^{12} < t^{13}_{r_i^1}$. By (1.22) every member of G_{i-1} is empty at t^{11}. Since, by (1.19), $t^{11} < t^{14}_{g_i^1}$ and every member of G_{i-1} is empty at t^{11}, we know that $t^{12} > t^{11}$. We now prove that for all j, $1 \le j \le 4i-1$, g_{i-1}^j is uniquely black pebblable in $[t^{11}, t^{14}_{g_i^{4i}} - 1]$.

  The proof is by induction from $j = 4i-1$ down to $j = 1$ in which we apply Lemma 11 during each round. Consider g_{i-1}^j, $1 \le j \le 4i-1$. By (1.4), (1.10), (1.5), and (1.19), the 4 nodes y_i, x_i, a_i, and $\bar{x}'_i$ are all in $\bullet[[t^{11}, t^{14}_{g_i^{4i}} - 1]]\bullet$. g_{i-1}^j has $4i-1$ other predecessors that by (1.19) are empty at t^{11}. By (1.1), (1.3), (1.20), (1.5), and the induction hypothesis that each g_{i-1}^l, where $l > j$, is uniquely black pebblable in $[t^{11}, t^{14}_{g_i^{4i}} - 1]$, no descendant of g_{i-1}^j can contain a white pebble at any time in $[t^{11}, t^{14}_{g_i^{4i}} - 1]$. Finally, we must show that g_{i-1}^j itself cannot contain a white pebble at $t^{14}_{g_i^{4i}} - 1$. By (1.3), g_i^{4i} is black pebbled at $t^{14}_{g_i^{4i}}$, and due to g_i^{4i}'s in-degree of $4i+3$ and since g_{i-1}^j is not a predecessor of g_i^{4i}, we know that g_{i-1}^j must be empty at $t^{14}_{g_i^{4i}} - 1$ and therefore cannot contain a white pebble at that time. We therefore can apply Lemma 11 to conclude that g_{i-1}^j is uniquely black pebblable in $[t^{11}, t^{14}_{g_i^{4i}} - 1]$.

  Furthermore, since g_{i-1}^j is a predecessor of each g_{i-1}^l, such that $l > j$, it must be black pebbled before g_{i-1}^l can be. Therefore, g_{i-1}^j must be pebbled at $t^{12}_{g_{i-1}^j}$, $t^{12}_{g_{i-1}^j} < t^{12}_{g_{i-1}^{j+1}}$. Also since every g_{i-1}^p, $p \ge j$, is a predecessor of every r_i^q, $q \le j$, each g_{i-1}^p must be pebbled in order to black pebble any r_i^q. Therefore, $t^{13}_{r_i^j} > t^{12}_{g_{i-1}^j}$. Since g_{i-1}^j is empty at t^{11}, is uniquely black pebblable in $[t^{11}, t^{14}_{g_i^{4i}} - 1]$, and is first pebbled at $t^{12}_{g_{i-1}^j}$, we can conclude that $g_{i-1}^j \in \oslash[[t^{11}, t^{12}_{g_{i-1}^j} - 1]]\oslash$.

  Since each member of G_{i-1} is uniquely black pebblable in $[t^{11}, t^{14}_{g_i^{4i}} - 1]$ and t^{12} is the earliest time that every member of G_{i-1} contains a black pebble, $t^{12} = t^{12}_{g_{i-1}^{4i-1}}$. Since every member of g_{i-1}^j is a predecessor of r_i^1, we have $t^{12} < t^{13}_{r_i^1}$.

The following relationships between the times at which certain nodes are pebbled are demonstrated by the points proved above.

12. By (1.19), $t^{10} + 1 < t^{11}$.
13. By (1.24), $t^{11} < t^{12}_{g^1_{i-1}}$, $t^{12}_{g^{4i-1}_{i-1}} < t^{14}_{g^1_i}$, and for each j, $1 \le j \le 4i-2$, $t^{12}_{g^j_{i-1}} < t^{12}_{g^{j+1}_{i-1}}$.
14. By (1.23), $t^{12}_{g^{4i-1}_{i-1}} < t^{13}_{r^1_i}$, $t^{13}_{r^{4i-1}_i} < t^{14}_{g^1_i}$, and for each j, $1 \le j \le 4i-2$, $t^{13}_{r^j_i} < t^{13}_{r^{j+1}_i}$.

These inequalities produce the following ordering of times, which labels all of the x-axis of Figure 7: $t^\alpha < t^1 < t^2 < t^3 < t^4 < t^4 + 1 < t^5 < t^6 < t^7 < t^8 < t^9 < t^{10} < t^{11} < t^{12}_{g^1_{i-1}} < \cdots < t^{12}_{g^{4i-1}_{i-1}} = t^{12} < t^{13}_{r^1_i} < \cdots < t^{13}_{r^{4i-1}_i} = t^{13} < t^{14}_{g^1_i} < \cdots < t^{14}_{g^{4i+3}_i} = t^\omega$.

As before, Figure 7 provides a summary of the points proved above. Of particular interest to us now is the sequence of line segments that are entering the region labeled as the "second induction." Particularly, we know that every member of G_i is empty during $[t^{11}, t^{12}]$ since $[t^{11}, t^{12}]$ is a subinterval of $[t^\alpha, t^\omega]$ and each member of G_i is uniquely black pebblable in $[t^\alpha, t^\omega]$ and must be pebbled after t^{12}. By (1.20) every member of $\{h_i, c_i, b_i\}$ must be empty during $[t^{11}, t^{12}]$. By (1.23), every member of R_i must be empty during $[t^{11}, t^{12}]$. Also, a_i is clamped in $\bullet[[t^{10}, t^{14}_{g^{4i+2}_i} - 1]]\bullet$, which contains $[t^{11}, t^{12}]$ as a subinterval. Therefore, no member of widget i which is a descendant of g^{4i-1}_{i-1} can be pebbled during the interval $[t^{11}, t^{12}]$. Also every node in $B[\beta_i] \cup \{y_i, \bar{x}'_i, a_i, x_i\} = B[\beta_i \cup \{x_i\}]$ is in $\bullet[[t^{11}, t^{12}]]\bullet$.

We can therefore apply our induction hypothesis to conclude that $\psi\lceil_{\beta_i \cup \{x_i\}}$ must be in QSAT and that $\Omega(2^k)$ units of time must pass between t^{11} and t^{12}, where k is the number of universally quantified variables among the innermost $i-1$ variables of ψ.

Thus we have shown that a $4n+3$ pebbling must black pebble G_{i-1} twice between t^α and t^ω, once implying that $\psi\lceil_{\beta_i \cup \{\bar{x}_i\}}$ is in QSAT, and once implying that $\psi\lceil_{\beta_i \cup \{x_i\}}$ is in QSAT. Each time requires $\Omega(2^k)$ time, where k is the number of universally quantified variables among the innermost $i-1$ variables of ψ. Therefore, black pebbling G_i requires time $\Omega(2^{k+1})$ and implies that $\psi\lceil_{\beta_i}$ is in QSAT.

Case 2. Q_i is an existential quantifier. We will show that in order to black pebble G_i, we must necessarily pass through a number of all-black partial configurations, including simultaneously black pebbling G_{i-1}, either with black pebbles on y_i, x'_i, d_i, and $\bar{x}_i$ (the false configuration), or with black pebbles on y_i, $\bar{x}'_i$, d_i, and x_i (the true configuration). The proof will split into two cases, Case 2a, in which we consider the false configuration, and Case 2b, in which we consider the true configuration. But first we prove a few general points that are true for both cases.

- (2.1) *No descendant of any node in $G_i \cup G_{i-1} \cup H_i \cup R_i \cup \{a_i\}$ that is outside of widget i contains a white pebble at any time during $[t^\alpha, t^\omega]$.*
  This argument is the same as the argument for (1.1).
- (2.2) *Every node of widget i other than members of G_i is in $\oslash[[t^\omega, t^\omega]]\oslash$.*
  This argument is the same as the argument for (1.2).
- (2.3) *g^{4i+3}_i is uniquely black pebblable in $[t^\alpha, t^\omega]$, $g^{4i+3}_i \in \oslash[[t^\alpha, t^{13}_{g^{4i+3}_i} - 1]]\oslash$, and $g^{4i+3}_i \in \bullet[[t^{13}_{g^{4i+3}_i}, t^\omega]]\bullet$, where $t^{13}_{g^{4i+3}_i} = t^\omega$; and for each j, $1 \le j \le 4i+2$, g^j_i is uniquely black pebblable in $[t^\alpha, t^\omega]$, $g^j_i \in \oslash[[t^\alpha, t^{13}_{g^j_i} - 1]]\oslash$, and $g^j_i \in \bullet[[t^{13}_{g^j_i}, t^\omega]]\bullet$, where $t^{13}_{g^j_i} < t^{13}_{g^{j+1}_i}$.*
  This argument is the same as the argument for (1.3).
- (2.4) *x'_i is uniquely black pebblable in the interval $[t^\alpha, t^\omega]$, and there exists some time t^* such that $x'_i \in \oslash[[t^\alpha, t^1 - 1]]\oslash$, $x'_i \in \bullet[[t^1, t^*]]\bullet$, and $x_i \in$*

$\bullet[[t^*+1, t^{13}_{g_i^{4i+3}}-1]]\bullet$, *where no nodes other than predecessors of* x_i' *are pebbled between* t^α *and* t^1, *so* $t^1 > t^\alpha$.

In order to black pebble g_i^{4i+3} at time $t^{13}_{g_i^{4i+3}}$, we must slide a black pebble up from x_i since, by (2.3), the other $4i+2$ pebbles are clamped on previous members of G_i until t^ω. Therefore x_i must be black pebbled at some time t^*, $t^* < t^{13}_{g_i^{4i+3}}$. In order to black pebble x_i we must first pebble x_i' at some time t^1, $t^1 < t^*$. We will now show that x_i' must be black pebbled at t^1.

Since every node of widget i is empty at t^α, x_i' is empty at t^α. By (2.2), x_i' is empty again at t^ω. But x_i''s pebble can be removed no earlier than t^*. From t^* to $t^{13}_{g_i^{4i+3}} = t^\omega$ there is always at least one pebble on x_i or in G_i, so x_i's pebble could not be removed if it were white. x_i' must therefore be black pebbled at t^1, and $t^\alpha < t^1 < t^\omega$. We know that there are no white pebbles on the graph at time t^α. Since x_i' has $4i+3$ source nodes as predecessors, we can apply Lemma 12 to conclude that x_i' is uniquely black pebblable in $[t^\alpha, t^\omega]$ and must be pebbled at t^1, $t^\alpha < t^1 < t^*$, where no nodes other than x_i''s predecessors are pebbled between t^α and t^1, so $x_i' \in \oslash[[t^\alpha, t^1-1]]\oslash$.

Since x_i' is uniquely black pebblable in $[t^\alpha, t^\omega]$ and must contain a pebble until x_i is black pebbled at t^*, $x_i' \in \bullet[[t^1, t^*-1]]\bullet$ and $x_i \in \bullet[[t^*, t^{13}_{g_i^{4i+3}}-1]]\bullet$.

- (2.5) $y_i \in \oslash[[t^{13}_{g_i^1}-1, t^\omega]]\oslash$.

  Since y_i is not one of g_i^1's $4i+3$ predecessors, it must be empty at $t^{13}_{g_i^1}-1$. For y_i to contain a pebble after $t^{13}_{g_i^1}-1$, it must therefore be pebbled at some time during $[t^{13}_{g_i^1}, t^\omega]$. By (2.3) and (2.4), we know that g_i^1 and either x_i or x_i' must each contain a pebble from $t^{13}_{g_i^1}$ through $t^{13}_{g_i^{4i+3}}-1$. There are therefore not enough free pebbles to black pebble y_i at any time during $[t^{13}_{g_i^1}, t^\omega]$ or to remove a white pebble from y_i at any time during $[t^{13}_{g_i^1}, t^\omega]$.

  By (2.2) we know that y_i must be empty by t^ω, so if y_i were white pebbled during $[t^{13}_{g_i^1}, t^\omega]$, it would also have to have its white pebble removed during $[t^{13}_{g_i^1}, t^\omega]$. It is therefore not possible to pebble y_i at all in this interval and it must remain empty. Therefore $y_i \in \oslash[[t^{13}_{g_i^1}-1, t^\omega]]\oslash$.

- (2.6) a_i *is uniquely black pebblable in* $[t^\alpha, t^\omega]$, $a_i \in \oslash[[t^\alpha, t^{11}-1]]\oslash$, *and* $a_i \in \bullet[[t^{11}, t^{13}_{g_i^{4i+1}}-1]]\bullet$, *where* $t^{11} < t^{13}_{g_i^1}$.

  Let t^{11} be the earliest time that a_i is pebbled during $[t^\alpha, t^\omega]$. We must show that a_i must be black pebbled at t^{11} and that a_i cannot be pebbled again after t^{11}. We first prove that $a_i \in \bullet[[t^{11}, t^{13}_{g_i^1}-1]]\bullet$. This is because members of G_i are the only successors of a_i, so a_i must contain a pebble until at least the first of these is black pebbled; otherwise the assignment to a_i would not be necessary.

  Therefore, if a_i were white pebbled at t^{11}, by (2.2) its pebble would have to be removed sometime after $t^{13}_{g_i^1}-1$ but before t^ω.

  Since g_i^1 has $4i+3$ predecessors, none of which is y_i or one of y_i's predecessors, or is in H_i or R_i, all of these nodes must be empty at $t^{13}_{g_i^1}-1$, so there is an unblocked path ρ from each of y_i's predecessors to a_i at $t^{13}_{g_i^1}-1$. By Lemma 13, removing the white pebble from a_i after $t^{13}_{g_i^1}-1$ and then removing all of the pebbles from ρ before t^ω requires y_i to be repebbled during $[t^{13}_{g_i^1}, t^\omega]$, which is not possible by (2.5). We can therefore conclude that a_i must be black pebbled at t^{11}.

The same argument can be made to show that a_i cannot be pebbled at all again after $t^{13}_{g_i^1} - 1$. Since t^{11} is the earliest time in $[t^\alpha, t^\omega]$ that a_i is pebbled, we can conclude that a_i is uniquely black pebblable in $[t^\alpha, t^\omega]$ and therefore must contain a pebble until the last time one of its successors is pebbled in $[t^\alpha, t^\omega]$, which, by (2.3), occurs at $t^{13}_{g_i^{4i+1}} - 1$, so $a_i \in \bullet[[t^{11}, t^{13}_{g_i^{4i+1}} - 1]]\bullet$.

Since a_i is uniquely black pebblable in $[t^\alpha, t^\omega]$ and is black pebbled at t^{11}, $a_i \in \oslash[[t^\alpha, t^{11} - 1]]\oslash$.

At this point our proof splits into two cases: either a black pebble is on $\bar{x}'_i$ at t^{11} or it is not. The first of these cases implies that $\psi\lceil_{\beta_i \cup \{\bar{x}_i\}}$ is in QSAT, and the second implies that $\psi\lceil_{\beta_i \cup \{x_i\}}$ is in QSAT.

Case 2a (see Figure 8). Suppose there is no black pebble on $\bar{x}'_i$ at t^{11}. Then there are two subcases to consider. In Subcase (i) there is a white pebble on $\bar{x}'_i$ at t^{11}, and in Subcase (ii) there is no pebble at all on $\bar{x}'_i$.

First, we consider **Subcase (i)**: there is a white pebble on $\bar{x}'_i$ at t^{11}. We show that this subcase is impossible.

To start with, we prove that y_i must be empty at $t^{11} - 1$ because all the free pebbles are needed in other places. a_i has $4i + 1$ predecessors which must all be pebbled at $t^{11} - 1$. By (2.4), either x_i or x'_i must contain a pebble at this time. $\bar{x}'_i$ must also because we are assuming that it contains a pebble at t^{11}. There are therefore not enough pebbles to allow one to remain on y_i at $t^{11} - 1$. Also, a pebble is placed on a_i at t^{11}, not on y_i, so it must remain empty at t^{11}.

We cannot pebble y_i after t^{11} because by (2.6) and (2.4) both a_i and either x_i or x'_i contain pebbles from t^{11} through $t^{13}_{g_i^{4i+1}} - 1$. By (2.5), it is also not possible to pebble y_i after $t^{13}_{g_i^{4i+1}} - 1$. We therefore cannot black pebble y_i or remove a white pebble from y_i at any time in $[t^{11} - 1, t^\omega]$ because y_i has $4i + 2$ source node predecessors.

But since y_i is a predecessor of $\bar{x}'_i$, removing $\bar{x}'_i$'s white pebble along with all the rest of the widget's pebbles outside of G_i by t^ω requires that y_i is pebbled after t^{11}. Since this is impossible, we can therefore conclude that $\bar{x}'_i$ cannot contain a white pebble at t^{11}.

Second, we consider **Subcase (ii)**: there is no pebble at all on $\bar{x}'_i$ at t^{11}.

- (2a.1) *$\bar{x}'_i$ must be black pebbled at some time t^{12}, $t^{11} < t^{12} < t^{13}_{g_i^1}$ and $\bar{x}'_i \in \bullet[[t^{12}, t^{13}_{g_i^{4i+2}} - 1]]\bullet$.*

  In order to black pebble g_i^1 at time $t^{13}_{g_i^1}$, $t^{13}_{g_i^1} > t^{11}$, we must first pebble $\bar{x}'_i$ at some time t^{12} before $t^{13}_{g_i^1 - 1}$. Since, by the assumption of Subcase (ii), $\bar{x}'_i$ is empty at t^{11}, we have $t^{11} < t^{12} < t^{13}_{g_i^1} - 1$.

  $\bar{x}'_i$ cannot be black pebbled after $t^{13}_{g_i^1} - 1$ since, by (2.5), y_i must stay empty after this time and y_i is a predecessor of $\bar{x}'_i$. This also precludes $\bar{x}'_i$ from having a white pebbled removed from it during $[t^{13}_{g_i^1}, t^\omega]$, which, by (2.2), must happen by t^ω. Therefore, we can conclude that $\bar{x}'_i$ cannot be pebbled at all in $[t^{13}_{g_i^1}, t^\omega]$. This also means that $\bar{x}'_i$ must be black pebbled at t^{12}, since otherwise its white pebble would have to be removed during $[t^{13}_{g_i^1}, t^\omega]$. Therefore, $\bar{x}'_i \in \bullet[[t^{12}, t^{13}_{g_i^{4i+2}} - 1]]\bullet$.
- (2a.2) $x'_i \in \bullet[[t^1, t^{12}]]\bullet$ *and* $x_i \in \bullet[[t^{12} + 1, t^{13}_{g_i^{4i+3}} - 1]]\bullet$.

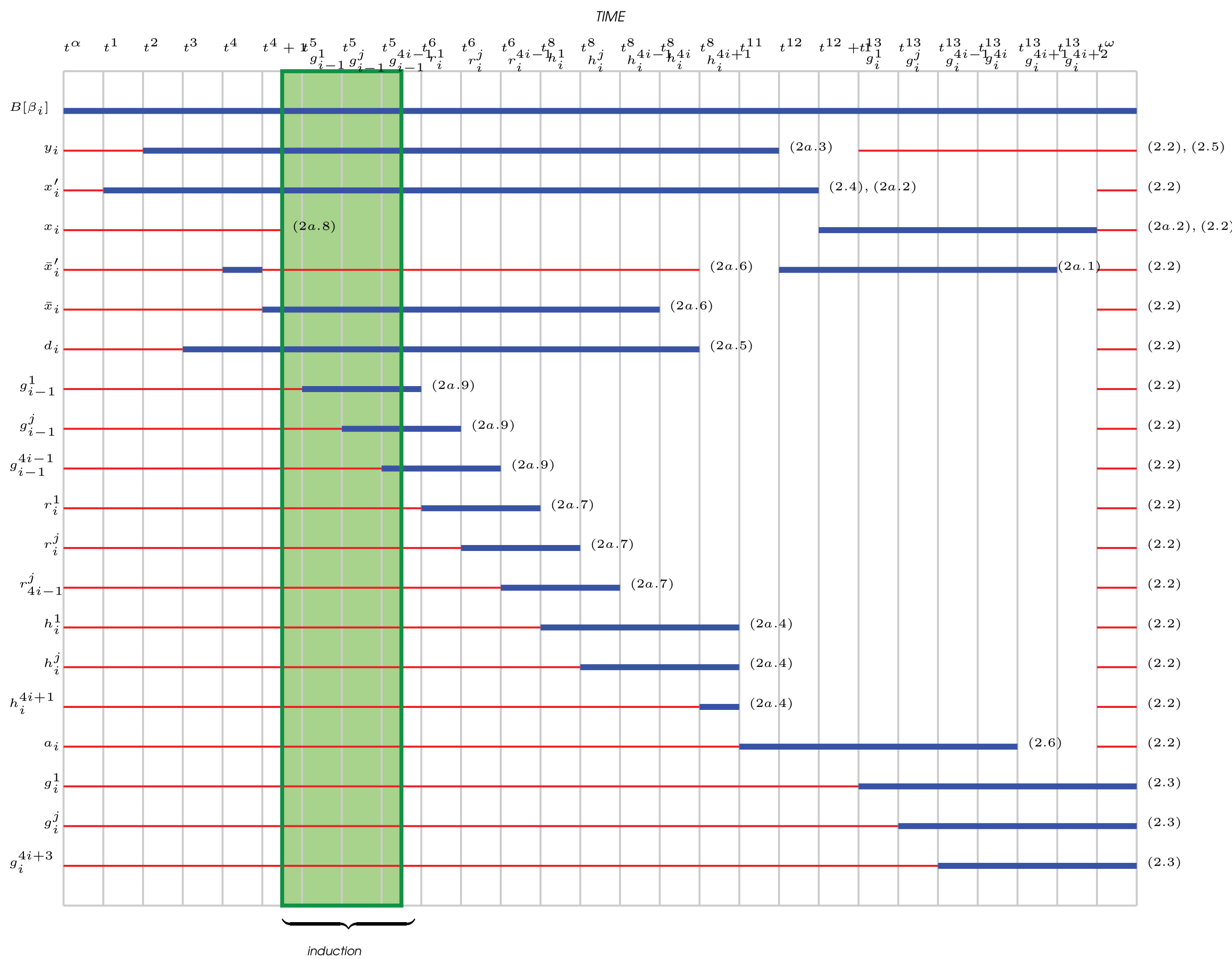

FIG. 8. *Subintervals of* $[t^\alpha, t^\omega]$ *during which nodes of the existential widget must be empty or must contain a black pebble in Case 2a. Intervals during which a node must contain a black pebble are shown as a thick blue line, while intervals during which a node must be empty are shown as a thinner red line.*

The above argument implies that x'_i will not have to support another pebble assignment to $\bar{x}'_i$ after t^{12}. Since x_i is x'_i's only other successor, we can apply Lemma 14 to conclude that x'_i's pebble is slid up to x_i at $t^{12}+1$. Therefore, $x'_i \in \bullet[[t^1, t^{12}]]\bullet$ and $x_i \in \bullet[[t^{12}+1, t^{13}_{g_i^{4i+3}} - 1]]\bullet$.

- (2a.3) *y_i is uniquely black pebblable in $[t^1, t^\omega]$, $y_i \in \oslash[[t^\alpha, t^2-1]]\oslash$, and $y_i \in \bullet[[t^2, t^{12}-1]]\bullet$, where no nodes other than y_i's predecessors are pebbled between t^1 and t^2, where $t^2 > t^1$.*

  In order to black pebble $\bar{x}'_i$ at t^{12} we must first pebble y_i at some time t^2, $t^2 < t^{12}$. By (2.4) y_i is in $\oslash[[t^\alpha, t^1]]\oslash$, so $t^2 > t^1$. Suppose y_i contains a white pebble at $t^{12}-1$. Then by (2.5), this pebble must be removed by $t^{13}_{g_i^1}$. But it cannot be removed by then because it has $4i+2$ predecessors and by (2a.1) and (2a.2) there are not enough free pebbles. y_i must therefore contain a black pebble at $t^{12}-1$. By the induction hypothesis and (2.4) we know that no nonsource nodes contain white pebbles at t^1. Also y_i has $4i+2$ predecessors and by (2a.2) $x'_i \in \bullet[[t^1, t^{12}]]\bullet$. We can therefore apply Lemma 12 to conclude that y_i is uniquely black pebblable in $[t^1, t^{12}-1]$, $y_i \in \oslash[[t^\alpha, t^2-1]]\oslash$, and $y_i \in \bullet[[t^2, t^{12}-1]]\bullet$, where no nodes other than y_i's predecessors are pebbled between t^1 and t^2.

  By (2.6), $a_i \in \bullet[[t^{11}, t^{13}_{g_i^{4i+1}} - 1]]\bullet$; we can therefore make the same argument as in (2.5) to conclude that y_i cannot be pebbled at any time in $[t^{11}, t^\omega]$. This means that y_i is uniquely black pebblable in $[t^1, t^\omega]$.

- (2a.4) *h_i^{4i+1} is uniquely black pebblable in $[t^2, t^{12}-1]$, $h_i^{4i+1} \in \oslash[[t^\alpha, t^8_{h_i^{4i+1}} - 1]]\oslash$, and $h_i^{4i+1} \in \bullet[[t^8_{h_i^{4i+1}}, t^{11}-1]]\bullet$, where $t^8_{h_i^{4i+1}} < t^{11}$; and for each j, $1 \le j \le 4i$, h_i^j is uniquely black pebblable in $[t^2, t^{12}-1]$, $h_i^j \in \oslash[[t^\alpha, t^8_{h_i^j} - 1]]\oslash$, and $h_i^j \in \bullet[[t^8_{h_i^j}, t^{11}-1]]\bullet$, where $t^8_{h_i^j} < t^8_{h_i^{j+1}}$.*

  We first prove that each member of H_i is in $\oslash[[t^\alpha, t^2]]\oslash$. No member of H_i is a member of $B[\beta_i]$, so they are all empty at t^α. From (2.4) we know that no member of H_i can be pebbled between t^α and t^1, and by (2a.3) we know that no member of H_i can be pebbled between t^1 and t^2 and all must therefore remain empty at t^2.

  By (2.1), (2.3), and (2.6) we know that none of the descendants of h_i^{4i+1} can contain a white pebble at any time during $[t^2, t^{12}-1]$. In order to pebble a_i at time t^{11}, h_i^{4i+1} must contain a pebble at time $t^{11}-1$. By (2.6), (2a.2), and (2a.3), a_i, x'_i, and y_i are in $\bullet[[t^{11}, t^{12}-1]]\bullet$ and $\bar{x}'_i$ has $4i$ other predecessors, none of which is in H_i, so h_i^{4i+1} must be empty at $t^{12}-1$. Since h_i^{4i+1} has $4i+1$ predecessors, none of which is x'_i or y_i, we can apply Lemma 11 to conclude that h_i^{4i+1} is uniquely black pebblable in $[t^2, t^{12}-1]$. Let $t^8_{h_i^{4i+1}}$ be the time at which h_i^{4i+1} is black pebbled. Since h_i^{4i+1} is a predecessor of a_i and can be pebbled only once before t^{11}, $t^8_{h_i^{4i+1}} < t^{11}$, $h_i^{4i+1} \in \oslash[[t^\alpha, t^8_{h_i^{4i+1}} - 1]]\oslash$, and $h_i^{4i+1} \in \bullet[[t^8_{h_i^{4i+1}}, t^{11}-1]]\bullet$.

  We can now prove by induction from $j = 4i$ down to $j = 1$ that h_i^j is uniquely black pebblable in $[t^2, t^{12}-1]$ and is clamped in $\bullet[[t^8_{h_i^j}, t^{11}-1]]\bullet$, where $t^8_{h_i^j} < t^8_{h_i^{j+1}}$.

  By (2.1), (2.3), (2.6), and the induction hypothesis we know that none of the descendants of h_i^j can contain a white pebble at any time during $[t^2, t^{12}-1]$.

In order to pebble a_i at time t^{11}, h_i^j must contain a pebble at time $t^{11}-1$. By (2.6), (2a.2), and (2a.3), a_i, x_i', and y_i are in $\bullet[[t^{11}, t^{12}-1]]\bullet$ and $\bar{x}_i'$ has $4i$ other predecessors, none of which is in H_i, so h_i^j must be empty at $t^{12}-1$. Since h_i^j has $4i+1$ predecessors, none of which is x_i' or y_i, we can apply Lemma 11 to conclude that h_i^j is uniquely black pebblable in $[t^2, t^{12}-1]$. Let $t^8_{h_i^j}$ be the time at which h_i^j is black pebbled. Since h_i^j is a predecessor of h_i^{j+1} and both must be pebbled before t^{11}, we have $t^8_{h_i^j} < t^8_{h_i^{j+1}}$. Since h_i^j is a predecessor of a_i and can be pebbled only once before t^{11}, $h_i^j \in \oslash[[t^\alpha, t^8_{h_i^j}-1]]\oslash$ and $h_i^j \in \bullet[[t^8_{h_i^j}, t^{11}-1]]\bullet$.

- (2a.5) *d_i is uniquely black pebblable in $[t^2, t^{11}]$, $d_i \in \oslash[[t^\alpha, t^3-1]]\oslash$, and $d_i \in \bullet[[t^3, t^8_{h_i^{4i+1}}-1]]\bullet$, where no nodes other than d_i's predecessors can be pebbled between t^2 and t^3, where $t^3 > t^2$.*
We first prove that d_i is in $\oslash[[t^\alpha, t^2]]\oslash$. d_i is not a member of $B[\beta_i]$, so it is empty at t^α. From (2.4) we know that d_i cannot be pebbled between t^α and t^1 and by (2a.3) we know that d_i cannot be pebbled between t^1 and t^2 and must therefore remain empty at t^2.
In order to black pebble h_i^1 at $t^8_{h_i^1}$, we must first pebble its predecessor d_i at some time $t^3 < t^8_{h_i^1}$. Since $d_i \in \oslash[[t^\alpha, t^2]]\oslash$, we have $t^3 > t^2$. Also, d_i must be empty at t^{11}, since a_i has $4i+1$ predecessors, none of which is d_i, y_i, or x_i', and by (2a.3) and (2a.2) both y_i and x_i' are in $\bullet[[t^2, t^{11}]]\bullet$.
We also show that d_i must contain a black pebble at $t^8_{h_i^{4i+1}}-1$ since, as mentioned, both y_i and x_i' are clamped until t^{11}. By (2a.4), so are a_i's $4i$ other predecessors from H_i. There are therefore $4i+2$ nodes clamped in $\bullet[[t^8_{h_i^{4i+1}}-1, t^{11}-1]]\bullet$, so d_i's pebble must slide from d_i to h_i^{4i+1} at $t^8_{h_i^{4i+1}}$. d_i must therefore contain a black pebble at $t^8_{h_i^{4i+1}}-1$.
By the induction hypothesis, (2.4), and (2a.3), we know that no nonsource nodes contain white pebbles at t^2. Finally, since d_i has $4i+1$ source nodes as predecessors, we can apply Lemma 12 to conclude that d_i is uniquely black pebblable in $[t^2, t^{11},]$, $d_i \in \bullet[[t^3, t^8_{h_i^{4i+1}}-1]]\bullet$, and no nodes other than d_i's predecessors can be pebbled between t^2 and t^3, so $d_i \in \oslash[[t^\alpha, t^3-1]]\oslash$.
- (2a.6) *$\bar{x}_i'$ is uniquely black pebblable in $[t^3, t^8_{h_i^{4i+1}}-1]$, $\bar{x}_i' \in \oslash[[t^\alpha, t^4-1]]\oslash$, $\bar{x}_i' \in \bullet[[t^4, t^4]]\bullet$, and $\bar{x}_i' \in \oslash[[t^4+1, t^8_{h_i^{4i+1}}-1]]\oslash$; $\bar{x}_i \in \oslash[[t^\alpha, t^4]]\oslash$ and $\bar{x}_i \in \bullet[[t^4+1, t^8_{h_i^{4i}}-1]]\bullet$, where no nodes other than $\bar{x}_i'$'s predecessors can be pebbled between t^3 and t^4, where $t^4 > t^3$.*
We first prove that $\bar{x}_i'$ is in $\oslash[[t^\alpha, t^3]]\oslash$. $\bar{x}_i'$ is not a member of $B[\beta_i]$, so it is empty at t^α. From (2.4) we know that $\bar{x}_i'$ cannot be pebbled between t^α and t^1. By (2a.3) we know that $\bar{x}_i'$ cannot be pebbled between t^1 and t^2. By (2a.5) we know that $\bar{x}_i'$ cannot be pebbled between t^2 and t^3 and must therefore remain empty at t^3.
In order to pebble h_i^{4i}, $\bar{x}_i$ must contain a pebble at $t^8_{h_i^{4i}}-1$. We now show that this pebble must be black. By (2a.3), (2a.2), and (2a.4) we know that y_i, x_i', and h_i^1 through h_i^{4i-1} are in $\bullet[[t^8_{h_i^{4i}}-1, t^{11}]]\bullet$. There are therefore $4i+2$ pebbles clamped in $[t^8_{h_i^{4i}}-1, t^{11}]$. This means that $\bar{x}_i$'s pebble must move from $\bar{x}_i$ at $t^8_{h_i^{4i}}-1$ to h_i^{4i} at $t^8_{h_i^{4i}}$. The only way this can happen is if $\bar{x}_i'$'s

pebble is black and is slid to $\bar{x}_i$. Therefore, $\bar{x}_i$ must contain a black pebble at $t^8_{h_i^{4i}} - 1$. By (2a.5) $\bar{x}_i$ must be empty at t^3, so we know that $\bar{x}_i$ must be black pebbled at some time $t^\#$, $t^3 < t^\# \le t^8_{h_i^{4i}} - 1$, and that $\bar{x}_i \in \bullet[[t^\#, t^8_{h_i^{4i}} - 1]]\bullet$. In order to black pebble $\bar{x}_i$ at $t^\#$, we must first pebble $\bar{x}'_i$ at some time t^4, $t^4 < t^\#$. Since $\bar{x}'_i$ is in $\oslash[[t^\alpha, t^3]]\oslash$, $t^4 > t^3$. Also, $\bar{x}'_i$ must be empty at t^{11} because a_i has $4i+1$ predecessors, none of which is $\bar{x}'_i$, y_i, or x'_i, and by (2a.3) and (2a.2) we know that both y_i and x'_i are in $\bullet[[t^2, t^8_{h_i^{4i}} - 1]]\bullet$. We will now show that $\bar{x}'_i$ must contain a black pebble at $t^\# - 1$.

Suppose for the sake of contradiction that $\bar{x}'_i$ contains a white pebble at $t^\# - 1$. The same argument that shows that $\bar{x}'_i$ must be empty at t^{11} can also be made to show that $\bar{x}'_i$ must be empty at $t^8_{h_i^{4i}} - 1$. So its white pebble must be removed at some time during $[t^\#, t^8_{h_i^{4i}} - 1]$. But by (2a.3), (2a.2), and (2a.5) we know that y_i, x'_i, and d_i are in $\bullet[[t^2, t^8_{h_i^{4i}} - 1]]\bullet$, and by the argument above we have $\bar{x}_i \in \bullet[[t^\#, t^8_{h_i^{4i}} - 1]]\bullet$. This means that there are never enough free pebbles available during $[t^\#, t^8_{h_i^{4i}} - 1]$ to remove $\bar{x}'_i$'s white pebble. $\bar{x}'_i$ must therefore contain a black pebble at $t^\# - 1$.

By the induction hypothesis, (2.4), (2a.3), and (2a.5), we can argue that no nonsource nodes contain white pebbles at t^3. Also, $\bar{x}'_i$ has $4i$ source nodes as predecessors, and by (2a.3), (2a.2), and (2a.5) we know that y_i, x'_i, and d_i are in $\bullet[[t^2, t^8_{h_i^{4i+1}} - 1]]\bullet$. Thus we can apply Lemma 12 to conclude that $\bar{x}'_i$ is uniquely black pebblable in $[t^3, t^8_{h_i^{4i+1}} - 1]$, where no nodes other than $\bar{x}'_i$'s predecessors can be pebbled between t^3 and t^4, so $\bar{x}'_i \in \oslash[[t^\alpha, t^4 - 1]]\oslash$ and $\bar{x}_i \in \oslash[[t^\alpha, t^4]]\oslash$. Furthermore, $\bar{x}'_i$'s pebble must stay in place until the last time one of its successors is pebbled during $[t^3, t^8_{h_i^{4i+1}} - 1]$.

We already know that there is no pebble on $\bar{x}'_i$ at t^{11} and by (2.3) and (2.6) every member of G_i is pebbled only after t^{11}. So the only one of $\bar{x}'_i$'s successors that is pebbled in $[t^3, t^8_{h_i^{4i+1}} - 1]$ is $\bar{x}_i$. We can therefore apply Lemma 14 to conclude that $\bar{x}'_i$'s pebble is slid to $\bar{x}_i$ at time t^4+1. So $\bar{x}'_i \in \bullet[[t^4, t^4]]\bullet$ and $\bar{x}_i \in \bullet[[t^4+1, t^8_{h_i^{4i}} - 1]]\bullet$. Also, since $\bar{x}'_i$ is uniquely black pebblable in $[t^3, t^8_{h_i^{4i+1}} - 1]$, is pebbled at t^4, and is empty again at $t^4 + 1$, $\bar{x}'_i \in \oslash[[t^4 + 1, t^8_{h_i^{4i+1}} - 1]]\oslash$.

- (2a.7) r_i^{4i-1} *is uniquely black pebblable in* $[t^4+1, t^8_{h_i^{4i}} - 1]$*,* $r_i^{4i-1} \in \oslash[[t^\alpha, t^6_{r_i^{4i-1}} - 1]]\oslash$*, and* $r_i^{4i-1} \in \bullet[[t^6_{r_i^{4i-1}}, t^8_{h_i^{4i-1}} - 1]]\bullet$*, where* $t^6_{r_i^{4i-1}} < t^8_{h_i^1}$*; and for each* j*,* $1 \le j \le 4i-2$*,* r_i^j *is uniquely black pebblable in* $[t^4+1, t^8_{h_i^j} - 1]$*,* $r_i^j \in \oslash[[t^\alpha, t^6_{r_i^j} - 1]]\oslash$*, and* $r_i^j \in \bullet[[t^6_{r_i^j}, t^8_{h_i^j} - 1]]\bullet$*, where* $t^6_{r_i^j} < t^6_{r_i^{j+1}}$*.*

We first prove that each member of R_i is in $\oslash[[t^\alpha, t^4 + 1]]\oslash$. No member of R_i is a member of $B[\beta_i]$, so they are all empty at t^α. From (2.4) we know that no member of R_i can be pebbled between t^α and t^1, and by (2a.3) we know that no member of R_i can be pebbled between t^1 and t^2. By (2a.5) we know that no member of R_i can be pebbled between t^2 and t^3. By (2a.6) we know that no member of R_i can be pebbled between t^3 and $t^4 + 1$, so each member of R_i is in $\oslash[[t^\alpha, t^4 + 1]]\oslash$.

By (2.1), (2.3), (2.6), and (2a.4) we know that none of the descendants of r_i^{4i-1} can contain a white pebble at any time during $[t^4 + 1, t^8_{h_i^{4i}} - 1]$. In order to pebble h_i^{4i-1} at time $t^8_{h_i^{4i-1}}$, r_i^{4i-1} must contain a pebble at time $t^8_{h_i^{4i-1}} - 1$. By (2a.2), (2a.3), (2a.5), and (2a.6), x'_i, y_i, d_i, and $\bar{x}_i$ are in $\bullet[[t^4+1, t^8_{h_i^{4i}} - 1]]\bullet$. h_i^{4i} has $4i-1$ other predecessors, none of which is in R_i, so r_i^{4i-1} must be empty at $t^8_{h_i^{4i}} - 1$. Since r_i^{4i-1} has $4i-1$ predecessors other than x'_i, y_i, d_i, or $\bar{x}_i$, we can apply Lemma 11 to conclude that r_i^{4i-1} is uniquely black pebblable in $[t^4+1, t^8_{h_i^{4i}} - 1]$. Let $t^6_{r_i^{4i-1}}$ be the time at which r_i^{4i-1} is pebbled in $[t^4+1, t^8_{h_i^{4i}} - 1]$. So $r_i^{4i-1} \in \oslash[[t^\alpha, t^6_{r_i^{4i-1}} - 1]]\oslash$. By (2a.4), the only time before t^{11} at which h_i^1 can be black pebbled is $t^8_{h_i^1}$, so $t^6_{r_i^{4i-1}} < t^8_{h_i^1}$. Since r_i^{4i-1} is a predecessor of h_i^{4i-1} and can be pebbled only once before $t^8_{h_i^{4i-1}}$, $r_i^{4i-1} \in \bullet[[t^6_{r_i^{4i-1}}, t^8_{h_i^{4i-1}} - 1]]\bullet$.

We can now prove by induction from $j = 4i-2$ down to $j = 1$ that r_i^j is uniquely black pebblable in $[t^4+1, t^8_{h_i^{4i}} - 1]$ and is clamped in $\bullet[[t^6_{r_i^j}, t^8_{h_i^j} - 1]]\bullet$, where $t^6_{r_i^j} < t^6_{r_i^{j+1}}$.

By (2.1), (2.3), (2.6), (2a.4), and the induction hypothesis we know that none of the descendants of r_i^j can contain a white pebble at any time during $[t^4+1, t^8_{h_i^{4i}} - 1]$. In order to pebble h_i^j at time $t^8_{h_i^j}$, r_i^j must contain a pebble at time $t^8_{h_i^j} - 1$. By (2a.2), (2a.3), (2a.5), and (2a.6), x'_i, y_i, d_i, and $\bar{x}_i$ are in $\bullet[[t^4+1, t^8_{h_i^{4i}} - 1]]\bullet$. h_i^{4i} has $4i-1$ other predecessors, none of which is in R_i, so r_i^j must be empty at $t^8_{h_i^{4i}} - 1$. Since r_i^j has $4i-1$ predecessors other than x'_i, y_i, d_i, or $\bar{x}_i$, we can apply Lemma 11 to conclude that r_i^j is uniquely black pebblable in $[t^4+1, t^8_{h_i^{4i}} - 1]$. Let $t^6_{r_i^j}$ be the time at which r_i^j is pebbled in $[t^4+1, t^8_{h_i^{4i}} - 1]$. So $r_i^j \in \oslash[[t^\alpha, t^6_{r_i^j} - 1]]\oslash$. Since r_i^j must be black pebbled before r_i^{j+1} can be pebbled, $t^6_{r_i^j} < t^6_{r_i^{j+1}}$. Since r_i^j is a predecessor of h_i^j and can be pebbled only once before $t^8_{h_i^j}$, $r_i^j \in \bullet[[t^6_{r_i^j}, t^8_{h_i^j} - 1]]\bullet$.

- (2a.8) $x_i \in \oslash[[t^\alpha, t^4+1]]\oslash$.
  x_i is not a member of $B[\beta_i]$, so it is empty at t^α. From (2a.2) we know that x_i cannot be pebbled between t^α and t^1, and by (2.4) we know that x_i cannot be pebbled between t^1 and t^2. By (2a.5) we know that x_i cannot be pebbled between t^2 and t^3. By (2a.6) we know that x_i cannot be pebbled between t^3 and t^4+1. It must therefore remain empty at t^4+1.
- (2a.9) g_{i-1}^{4i-1} *is uniquely black pebblable in* $[t^4+1, t^8_{h_i^{4i}} - 1]$, $g_{i-1}^{4i-1} \in \oslash[[t^\alpha, t^5_{g_{i-1}^{4i-1}} - 1]]\oslash$, *and* $g_{i-1}^{4i-1} \in \bullet[[t^5_{g_{i-1}^{4i-1}}, t^6_{r_i^{4i-1}} - 1]]\bullet$, *where* $t^4+1 < t^5_{g_{i-1}^{4i-1}} = t^5 < t^6_{r_i^1}$; *and for each* j, $1 \le j \le 4i-2$, g_{i-1}^j *is uniquely black pebblable in* $[t^4+1, t^8_{h_i^{4i}} - 1]$, $g_{i-1}^j \in \oslash[[t^\alpha, t^5_{g_{i-1}^j} - 1]]\oslash$, *and* $g_{i-1}^j \in \bullet[[t^5_{g_{i-1}^j}, t^6_{r_i^j} - 1]]\bullet$, *where* $t^4+1 < t^5_{g_{i-1}^j} < t^5_{g_{i-1}^{j+1}}$.
  This argument is almost the same as the argument for (2a.7).

The following relationships between the times at which certain nodes are pebbled are demonstrated by the points proved above.

- By (2.3), $t^\omega = t^{13}_{g_i^{4i+3}}$, and for each j, $1 \le j \le 4i+2$, $t^{13}_{g_i^j} < t^{13}_{g_i^{j+1}}$.
- By (2.6), $t^{11} < t^{13}_{g_i^1}$.

- By (2a.4), $t^8_{h_i^{4i+1}} < t^{11}$, and for each j, $1 \le j \le 4i$, $t^8_{h_i^j} < t^8_{h_i^{j+1}}$.
- By (2a.7), $t^6_{r_i^{4i-1}} < t^8_{h_i^1}$, and for each j, $1 \le j \le 4i-2$, $t^6_{r_i^j} < t^6_{r_i^{j+1}}$.
- By (2a.9), $t^4+1 < t^5_{g_i^1}$, $t^5_{g_i^{4i-1}} < t^6_{r_i^1}$, and for each j, $1 \le j \le 4i-2$, $t^5_{g_i^j} < t^5_{g_i^{j+1}}$.
- By (2a.6), $t^3 < t^4$.
- By (2a.5), $t^2 < t^3$.
- By (2a.3), $t^1 < t^2$.
- By (2.4), $t^\alpha < t^1$.

These inequalities produce the following ordering of times, which labels most of the x-axis of Figure 8: $t^\alpha < t^1 < t^2 < t^3 < t^4 < t^4+1 < t^5_{g_{i-1}^1} < \cdots < t^5_{g_{i-1}^{4i-1}} = t^5 < t^6_{r_i^1} < \cdots < t^6_{r_i^{4i-1}} < t^8_{h_i^1} < \cdots < t^8_{h_i^{4i+1}} < t^{11} < t^{12} < t^{13}_{g_i^1} < \cdots < t^{13}_{g_i^{4i+3}} = t^\omega$.

As in the universal case, the ordering allows us to produce a figure which summarizes the points proved above (Figure 8). Of particular interest to us now is the sequence of line segments that are entering the region labeled as the "induction." Particularly, we know that every member of G_i is empty during $[t^4+1, t^5]$ since $[t^4+1, t^5]$ is a subinterval of $[t^\alpha, t^\omega]$ and each member of G_i is uniquely black pebblable in $[t^\alpha, t^\omega]$ and must be pebbled after t^5. a_i must also be empty during $[t^4+1, t^5]$ since it is also uniquely black pebblable in $[t^\alpha, t^\omega]$ and must be pebbled at t^{11} after t^5. The same is true for H_i and R_i since both are uniquely black pebblable in $[t^4+1, t^8_{h_i^{4i}} - 1]$ and must be pebbled after t^5. Therefore, no member of widget i which is a descendant of g_{i-1}^{4i-1} can be pebbled during the interval $[t^4+1, t^5]$. Also every node in $B[\beta_i] \cup \{y_i, \bar{x}_i, d_i, x_i'\} = B[\beta_i \cup \{\bar{x}_i\}]$ is in $\bullet[[t^4+1, t^5]]\bullet$.

We can therefore apply the induction hypothesis to conclude that black pebbling G_{i-1} requires $\psi\lceil_{\beta_i \cup \{\bar{x}_i\}}$ to be in QSAT and that $\Omega(2^k)$ units of time must pass between t^4+1 and t^5, where k is the number of universally quantified variables among the innermost $i-1$ variables of ψ.

Case 2b (see Figure 9). Suppose, on the other hand, that there is a black pebble on $\bar{x}_i'$ at t^{11}. We will now show that $\psi\lceil_{\beta_i \cup \{\bar{x}_i\}}$ must be in QSAT. Our first step is to show that only nodes in $B[\beta_i] \cup \{x_i', d_i, y_i\}$ can be pebbled when we pebble $\bar{x}_i'$ for the last time before t^{11}, call the time t^4.

- (2b.1) $y_i \in \oslash[[t^{11}-1, t^\omega]]\oslash$.
  By (2.4) we know that either x_i or x_i' contains a pebble during each time step from t^1 through $t^{13}_{g_i^{4i+3}} - 1$, and by the assumption of this case $\bar{x}_i' \in \bullet[[t^4, t^{11}-1]]\bullet$. Furthermore, a_i has $4i+1$ predecessors which are not x_i, x_i', $\bar{x}_i'$, or y_i. y_i must therefore be empty at $t^{11}-1$.
  By (2.6) a_i contains a pebble during each time step from t^{11} through $t^{13}_{g_i^{4i+1}} - 1$. Combined with (2.4), this means that there are therefore at most $4i+1$ free pebbles at all times from t^{11} through $t^{13}_{g_i^1} - 1$. It is therefore impossible to black pebble y_i at any time in $[t^{11}, t^{13}_{g_i^1} - 1]$. By (2.5), y_i's pebble would have to be removed by $t^{13}_{g_i^1} - 1$ if it was white pebbled at some time in $[t^{11}, t^{13}_{g_i^1} - 1]$, which is also impossible. y_i therefore cannot be pebbled at all during $[t^{11}, t^{13}_{g_i^1} - 1]$. Since it must be empty at $t^{11}-1$, it must remain empty until $t^{13}_{g_i^1} - 1$. By (2.5) y_i must remain empty until t^ω, so $y_i \in \oslash[[t^{11}-1, t^\omega]]\oslash$.
- (2b.2) $\bar{x}_i'$ *must contain a black pebble at* $t^{13}_{g_i^{4i+2}} - 1$ *and must be empty at* $t^{13}_{g_i^{4i+2}}$.
  By (2.3) g_i^{4i+2} is empty at $t^{13}_{g_i^{4i+2}} - 1$ and is black pebbled at $t^{13}_{g_i^{4i+2}}$. By (2.3)

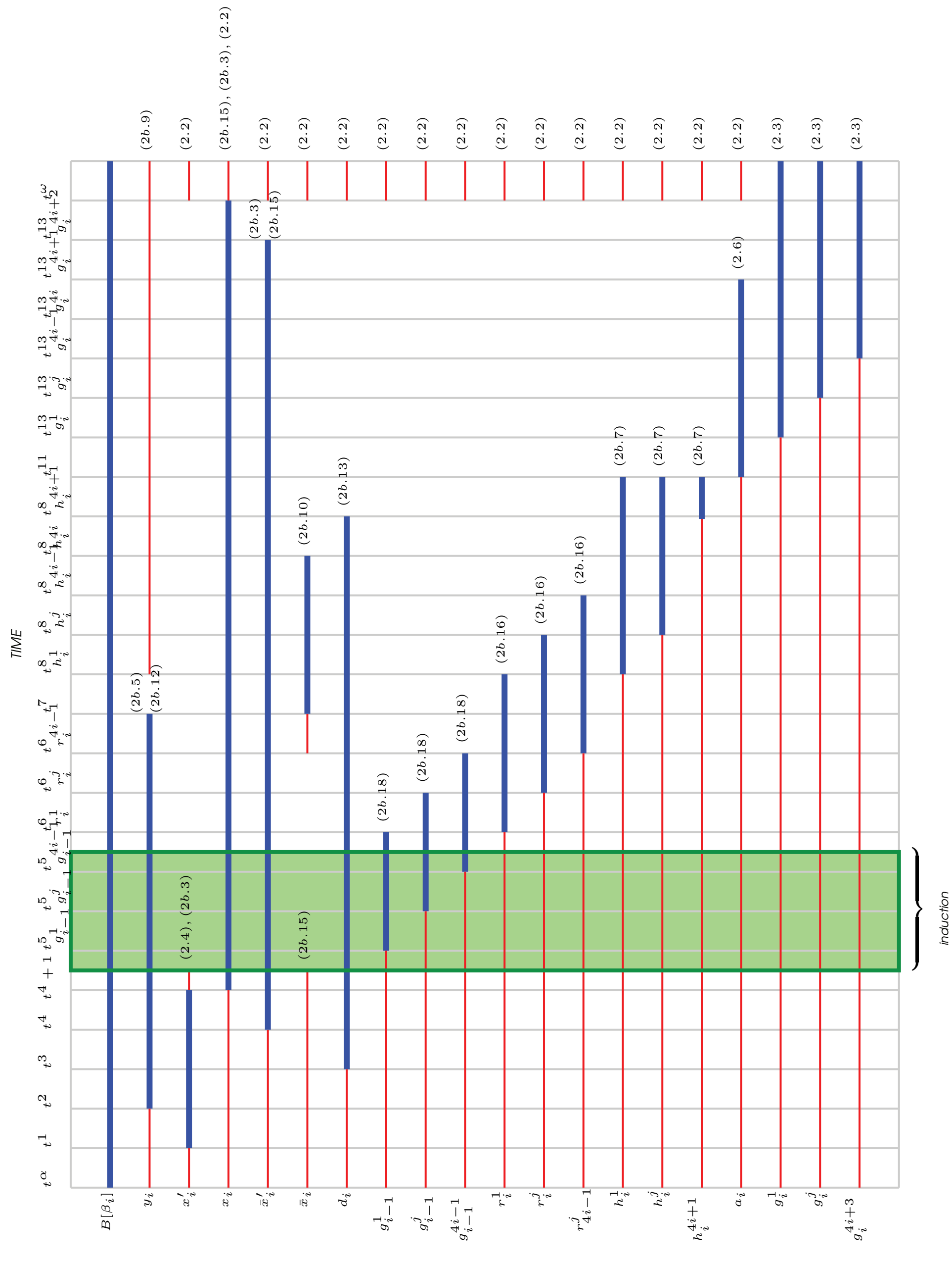

FIG. 9. *Subintervals of $[t^\alpha, t^\omega]$ during which nodes of the existential widget must be empty or must contain a black pebble in Case 2b. Intervals during which a node must contain a black pebble are shown as a thick blue line, while intervals during which a node must be empty are shown as a thinner red line.*

we know that there are $4i+1$ members of G_i in $\bullet[[t^{13}_{g_i^{4i+2}}-1, t^{13}_{g_i^{4i+2}}]]\bullet$. By (2.4) we know that either x_i or x'_i is in $\bullet[[t^{13}_{g_i^{4i+2}}-1, t^{13}_{g_i^{4i+2}}]]\bullet$. Since $\bar{x}'_i$ is another predecessor of g_i^{4i+2}, we can apply Lemma 15 to conclude that $\bar{x}'_i$ must contain a black pebble at $t^{13}_{g_i^{4i+2}}-1$ and must be empty at $t^{13}_{g_i^{4i+2}}$.

- (2b.3) $\bar{x}'_i \in \bullet[[t^4, t^{13}_{g_i^{4i+2}}-1]]\bullet$*,* $x'_i \in \bullet[[t^1, t^4]]\bullet$*,* $x'_i \in \oslash[[t^4+1, t^4+1]]\oslash$*, and* $x_i \in \bullet[[t^4+1, t^{13}_{g_i^{4i+3}}-1]]\bullet$*.*
By the conditions of this case, we know that $\bar{x}'_i \in \bullet[[t^4, t^{11}]]\bullet$. Also, by (2b.1), $y_i \in \oslash[[t^{11}-1, t^\omega]]\oslash$. Therefore $\bar{x}'_i$ cannot be black pebbled after t^{11} since y_i is a predecessor of $\bar{x}'_i$. This also precludes the removal of any white pebble from $\bar{x}'_i$ after t^{11}. Since $\bar{x}'_i$ is a predecessor of g_i^{4i+2}, this means that $\bar{x}'_i \in \bullet[[t^4, t^{13}_{g_i^{4i+2}}-1]]\bullet$. Also, x'_i must be empty at $t^{13}_{g_i^1}-1$, since g_i^1 has $4i+3$ predecessors, none of which is x'_i.
Therefore, between t^4 and the time x'_i's pebble is removed before $t^{13}_{g_i^1}$, the pebble assignment to x'_i will not support another assignment to $\bar{x}'_i$. Since x'_i has only one other successor, x_i, we can apply Lemma 14 to conclude that x'_i's pebble can be slid to x_i at t^4+1. So $x'_i \in \bullet[[t^1, t^4]]\bullet$, $x'_i \in \oslash[[t^4+1, t^4+1]]\oslash$, and $x_i \in \bullet[[t^4+1, t^{13}_{g_i^{4i+3}}-1]]\bullet$.
- (2b.4) y_i *must contain a black pebble at* t^4-1*.*
Since y_i is a predecessor of $\bar{x}'_i$, it must contain some pebble at t^4-1. Suppose for the sake of contradiction that y_i contains a white pebble at t^4-1. By (2b.3), $\bar{x}'_i \in \bullet[[t^4+1, t^{11}]]\bullet$ and $x_i \in \bullet[[t^4+1, t^{11}]]\bullet$.
Therefore y_i's pebble must be removed at some time $t^{\#}$ before $t^{11}-1$, since a_i has $4i+1$ predecessors, none of which is y_i, $\bar{x}'_i$, or x_i. Also because $\bar{x}'_i$ and x_i are in $\bullet[[t^4+1, t^{11}]]\bullet$ there are at most $4i+1$ free pebbles at any time during this interval and only $4i$ free pebbles between t^4+1 and $t^{\#}$ because of the white pebble on y_i. It is therefore impossible to remove a white pebble from y_i at any time in $[t^4, t^{11}]$. Therefore, y_i must contain a black pebble at t^4-1.
- (2b.5) y_i *is uniquely black pebblable in* $[t^1, t^\omega]$*,* $y_i \in \oslash[[t^\alpha, t^2-1]]\oslash$*, and* $y_i \in \bullet[[t^2, t^4-1]]\bullet$*, where no nodes are pebbled between* t^1 *and* t^2 *except* y_i*'s predecessors, where* $t^2 > t^1$*.*
y_i is not a member of $B[\beta_i]$, so it is empty at t^α. From (2.4) we know that y_i cannot be pebbled between t^α and t^1, so $y_i \in \oslash[[t^\alpha, t^1]]\oslash$.
In order to black pebble $\bar{x}'_i$ at t^4, we must first pebble y_i at some time t^2, $t^2 < t^4$. By (2.4) we know that y_i must be empty at t^1, so $t^2 > t^1$. By (2b.4) y_i must contain a black pebble at t^4-1. Since y_i has $4i+2$ source nodes as predecessors, and by (2.4) we know that there are no white pebbles on nonsource nodes at t^1, and by (2b.3) $x'_i \in \bullet[[t^1, t^4-1]]\bullet$, we can apply Lemma 12 to conclude that y_i is uniquely black pebblable in $[t^1, t^4-1]$ and $y_i \in \bullet[[t^2, t^4-1]]\bullet$, where no nodes other than y_i's predecessors can be pebbled from t^1 through t^2, so $y_i \in \oslash[[t^\alpha, t^2-1]]\oslash$.
Furthermore, by (2.4) we know that either x_i or x'_i contains a pebble during each time step from t^1 through $t^{13}_{g_i^{4i+3}}-1$, and by the assumption of this case $\bar{x}'_i \in \bullet[[t^4, t^{11}-1]]\bullet$. It is therefore impossible to black pebble y_i at any time in $[t^4, t^{11}-1]$. By (2b.1), y_i must be empty at $t^{11}-1$, so if y_i were white pebbled at some time in $[t^4, t^{11}-1]$, its pebble would have to be removed by then. But this is also impossible during $[t^4, t^{11}-1]$. Finally, by (2b.1) y_i

cannot be pebbled at t^{11} or later. It is therefore impossible to black pebble y_i again after t^4, and it is also impossible to remove a white pebble from y_i after t^4, which would be necessary by (2.2) if y_i were white pebbled in this time. We can therefore extend the interval during which y_i is uniquely black pebblable from $[t^1, t^4 - 1]$ to $[t^1, t^\omega]$.

- (2b.6) *Every member of H_i is in $\oslash[[t^\alpha, t^4 + 1]]\oslash$.*
  Suppose, for the sake of contradiction, that some member h_i^j of H_i is pebbled a some time $t^\#$ in $[t^\alpha, t^4 - 1]$. Then h_i^j or another member of H_i must still contain a pebble during every time unit in $[t^\#, t^{11} - 1]$. This is because every member of H_i has successors only in $H_i \cup \{a_i\}$, and a_i is pebbled only later at t^{11}. So if there were a time in $[t^\#, t^{11} - 1]$ at which no member of H_i contains a pebble, then a pebble would have had to have been removed before it supported any other assignment, and therefore it could not have been necessary.
  Since $\bar{x}_i'$ has $4i + 2$ predecessors, none of which is in H_i, only one member of H_i, namely h_i^j, can contain a pebble at $t^4 - 1$. Also, d_i must be empty at $t^4 - 1$, because $\bar{x}_i'$ has $4i + 2$ predecessors and h_i^j contains the last free pebble at $t^4 - 1$. This means that d_i must also be empty at t^4 since d_i cannot be affected by the move which pebbles $\bar{x}_i'$ at t^4.
  We first show that this implies that d_i must be entirely empty during $[t^4 - 1, t^{13}_{g_i^1}]$. This will then allow us to prove that a_i cannot be black pebbled at t^{11}, which is a contradiction with (2.6) and will allow us to conclude that no member of H_i can be pebbled during $[t^\alpha, t^4]$.
  By (2b.3) and the fact that there must be a pebble somewhere in H_i at all times during $[t^4-1, t^{11}-1]$, we know that there would have to be three pebbles on nodes which are not predecessors of d_i at all times in $[t^4 - 1, t^{11} - 1]$. And then from (2b.3) and (2.6) we know that there would have to be three pebbles on nodes which are not predecessors of d_i at all times in $[t^{11}, t^{13}_{g_i^1}]$. This means that d_i cannot be black pebbled and cannot have a white pebbled removed from it at any time in $[t^4 - 1, t^{13}_{g_i^1}]$, since it has in-degree $4i + 1$. And note that if d_i contained a white pebble at any time before $t^{13}_{g_i^1} - 1$, it would have to be removed before $t^{13}_{g_i^1} - 1$ because g_i^1 has $4i + 3$ predecessors, none of which is d_i. d_i can therefore never contain a white pebble at any time in $[t^4 - 1, t^{13}_{g_i^1}]$. Since d_i is empty at t^4 and cannot be black or white pebbled during $[t^4 - 1, t^{13}_{g_i^1}]$, d_i must be entirely empty during $[t^4 - 1, t^{13}_{g_i^1}]$.
  This will now allow us to prove that a_i cannot be black pebbled at t^{11}. In order to black pebble a_i at t^{11}, every member of H_i must contain a pebble at $t^{11} - 1$. Since only one member of H_i can contain a pebble at t^4, a pebble must be placed on some member of H_i at some point in $[t^4 + 1, t^{11} - 1]$. This pebble cannot be black, because black pebbling a member of H_i would require d_i to contain a pebble during the preceding time unit, which we have already shown to be impossible. The member of H_i must therefore be white pebbled at some point in $[t^4 + 1, t^{11} - 1]$. But then its white pebble would have to be removed by $t^{13}_{g_i^1} - 1$, since g_i^1 has $4i + 3$ predecessors, none of which is in H_i. But removing the white pebble by $t^{13}_{g_i^1} - 1$ would also require that d_i contain a pebble at some point in $[t^4 - 1, t^{13}_{g_i^1}]$, which is impossible. It is therefore

impossible to pebble any member of H_i at any point in $[t^4+1, t^{11}-1]$. This means that it will be impossible to black pebble a_i at t^{11}.

This is a contradiction with (2.6), so our original assumption must be false and we can conclude that no member of H_i can be pebbled during $[t^\alpha, t^4]$. By (2b.3), x_i is pebbled at t^4+1. This move cannot affect any member of H_i, so no member of H_i can be pebbled during $[t^\alpha, t^4+1]$. Since every member of H_i is empty at t^α, every member of H_i is in $\oslash[[t^\alpha, t^4+1]]\oslash$.

- (2b.7) h_i^{4i+1} *is uniquely black pebblable in* $[t^4+1, t^{13}_{g_i^{4i+2}}-1]$, $h_i^{4i+1} \in \oslash[[t^\alpha, t^8_{h_i^{4i+1}}]]\oslash$, *and* $h_i^{4i+1} \in \bullet[[t^8_{h_i^j}, t^{11}-1]]\bullet$, *where* $t^4+1 < t^8_{h_i^{4i+1}} < t^{11}$; *and for each* j, $1 \leq j \leq 4i$, h_i^j *is uniquely black pebblable in* $[t^4+1, t^{13}_{g_i^{4i+2}}-1]$, $h_i^j \in \oslash[[t^\alpha, t^8_{h_i^j}]]\oslash$, *and* $h_i^j \in \bullet[[t^8_{h_i^j}, t^{11}-1]]\bullet$, *where* $t^4+1 < t^8_{h_i^j} < t^8_{h_i^{j+1}}$.

  By (2.1), (2.3), and (2.6) we know that none of the descendants of h_i^{4i+1} can contain a white pebble at any time during $[t^4+1, t^{13}_{g_i^{4i+2}}-1]$. In order to pebble a_i at time t^{11}, h_i^{4i+1} must contain a pebble at time $t^{11}-1$. By (2b.3), x_i and $\bar{x}'_i$ are in $\bullet[[t^4+1, t^{13}_{g_i^{4i+2}}-1]]\bullet$. g_i^{4i+2} has $4i+3$ predecessors, none of which is in H_i, so h_i^{4i+1} must be empty at $t^{13}_{g_i^{4i+2}}-1$. Since h_i^{4i+1} has $4i+1$ predecessors, none of which is x_i or $\bar{x}'_i$, we can apply Lemma 11 to conclude that h_i^{4i+1} is uniquely black pebblable in $[t^4+1, t^{13}_{g_i^{4i+2}}-1]$. We can combine this with (2b.6) to conclude that h_i^{4i+1} is uniquely black pebblable in $[t^\alpha, t^{13}_{g_i^{4i+2}}-1]$, so $h_i^{4i+1} \in \oslash[[t^\alpha, t^8_{h_i^{4i+1}}]]\oslash$. Since h_i^{4i+1} is a predecessor of a_i and can be pebbled only once before t^{11}, $h_i^{4i+1} \in \bullet[[t^8_{h_i^{4i+1}}, t^{11}-1]]\bullet$.

  We can now prove by induction from $j = 4i$ down to $j = 1$ that h_i^j is uniquely black pebblable in $[t^4+1, t^{13}_{g_i^{4i+2}}-1]$ and is clamped in $\bullet[[t^8_{h_i^j}, t^{11}-1]]\bullet$, where $t^8_{h_i^j} < t^8_{h_i^{j+1}}$.

  By (2.1), (2.3), (2.6), and the induction hypothesis we know that none of the descendants of h_i^j can contain a white pebble at any time during $[t^4+1, t^{13}_{g_i^{4i+2}}-1]$. In order to pebble a_i at time t^{11}, h_i^j must contain a pebble at time $t^{11}-1$. By (2b.3), x_i and $\bar{x}'_i$ are in $\bullet[[t^4+1, t^{13}_{g_i^{4i+2}}-1]]\bullet$. g_i^{4i+2} has $4i+3$ predecessors, none of which is in H_i, so h_i^j must be empty at $t^{13}_{g_i^{4i+2}}-1$. Since h_i^j has $4i+1$ predecessors, none of which is x_i or $\bar{x}'_i$, we can apply Lemma 11 to conclude that h_i^j is uniquely black pebblable in $[t^4+1, t^{13}_{g_i^{4i+2}}-1]$. We can combine this with (2b.6) to conclude that h_i^j is uniquely black pebblable in $[t^\alpha, t^{13}_{g_i^{4i+2}}-1]$, so $h_i^j \in \oslash[[t^\alpha, t^8_{h_i^j}]]\oslash$. Since h_i^j must be pebbled before h_i^{j+1} can be pebbled, $t^8_{h_i^j} < t^8_{h_i^{j+1}}$. Since h_i^j is a predecessor of a_i and can be pebbled only once before t^{11}, $h_i^j \in \bullet[[t^8_{h_i^j}, t^{11}-1]]\bullet$.

- (2b.8) d_i *must contain a black pebble at* $t^8_{h_i^{4i+1}}-1$ *and must be empty at* $t^8_{h_i^{4i+1}}$.

  By (2b.7), h_i^{4i+1} is empty at $t^8_{h_i^{4i+1}}-1$ and is black pebbled at $t^8_{h_i^{4i+1}}$. By (2b.7) there are $4i$ members of H_i in $\bullet[[t^8_{h_i^{4i+1}}-1, t^8_{h_i^{4i+1}}]]\bullet$. By (2b.3), we know that $\bar{x}'_i \in \bullet[[t^8_{h_i^{4i+1}}-1, t^8_{h_i^{4i+1}}]]\bullet$ and $x'_i \in \bullet[[t^8_{h_i^{4i+1}}-1, t^8_{h_i^{4i+1}}]]\bullet$. Since d_i is another predecessor of h_i^{4i+1}, we can apply Lemma 15 to conclude that d_i must contain a black pebble at $t^8_{h_i^{4i+1}}-1$ and must be empty at $t^8_{h_i^{4i+1}}$.

- (2b.9) $y_i \in \oslash[[t^8_{h^1_i} - 1, t^\omega]]\oslash$.
  By (2b.6), $t^4 + 1 < t^8_{h^1_i} - 1$. Therefore, by (2b.3) x_i and $\bar{x}'_i$ are clamped at $t^8_{h^1_i} - 1$. Since h^1_i has $4i+1$ predecessors other than x_i and $\bar{x}'_i$, none of which is y_i, y_i must be empty at $t^8_{h^1_i} - 1$. Furthermore, by (2b.5), since y_i is uniquely black pebblable and was pebbled at t^2, before $t^4 + 1$, y_i cannot be pebbled again before t^ω, so $y_i \in \oslash[[t^8_{h^1_i} - 1, t^\omega]]\oslash$.
- (2b.10) $\bar{x}_i \in \bullet[[t^7, t^8_{h^{4i}_i} - 1]]\bullet$, $t^7 < t^8_{h^1_i}$.
  Since $\bar{x}_i$ is a predecessor of h^1_i, it must be pebbled at some last time before $t^8_{h^1_i}$, call the time t^7, so clearly $t^7 < t^8_{h^1_i}$. As proved in the argument for (2b.6), at any time in $[t^8_{h^1_i}, t^{11} - 1]$, there must be some pebble in H_i, and by (2b.9) y_i must be empty at all times in $[t^8_{h^1_i} - 1, t^{11} - 1]$.
  Suppose that the pebble is removed from $\bar{x}_i$ at some time during the interval $[t^8_{h^1_i}, t^8_{h^{4i+i}_i} - 1]$. Then it must be pebbled again at some time before $t^8_{h^{4i+1}_i}$, since it is a predecessor of h^{4i+1}_i. Since y_i must be empty at every time unit during $[t^8_{h^1_i} - 1, t^{11} - 1]$, it is not possible to black pebble $\bar{x}_i$ at that time. $\bar{x}_i$ must therefore be white pebbled. But $\bar{x}_i$ must be empty again by $t^{11} - 1$, since by (2b.3) x_i and $\bar{x}'_i$ are in $\bullet[[t^4 + 1, t^{11}]]\bullet$, and since a_i has $4i + 1$ other predecessors, none of which is $\bar{x}_i$. But removing a white pebble from $\bar{x}_i$ by $t^{11} - 1$ requires y_i to contain a pebble at some point after $t^8_{h^1_i} - 1$, which is impossible. $\bar{x}_i$'s pebble can therefore not be removed before $t^8_{h^{4i}_i} - 1$, so $\bar{x}_i \in \bullet[[t^7, t^8_{h^{4i}_i} - 1]]\bullet$.
- (2b.11) $t^4 < t^7$.
  Since two different nodes are pebbled at t^4 and t^7, $t^4 \neq t^7$. Therefore, suppose for the sake of contradiction that $t^4 > t^7$. Since, by (2b.10), $\bar{x}_i \in \bullet[[t^7, t^8_{h^{4i}_i} - 1]]\bullet$ and by (2b.6) $t^8_{h^{4i}_i} - 1 > t^4 + 1$, the pebble must remain on $\bar{x}_i$ at t^4, when $\bar{x}'_i$ is black pebbled. $\bar{x}'_i$ has $4i + 2$ other predecessors, so d_i must be empty at t^4.
  By (2b.3) both x_i and $\bar{x}'_i$ are clamped from $t^4 + 1$ through $t^8_{h^{4i+1}_i} - 1$. And by (2b.10), $\bar{x}_i$ is clamped through $t^8_{h^{4i}_i} - 1$ and by (2b.7), h^1_i is clamped from $t^8_{h^{4i}_i} - 1$ through $t^8_{h^{4i+1}_i} - 1$. This means that there are not enough free pebbles to black pebble d_i at any time before $t^8_{h^{4i+1}_i}$. But by (2b.8) d_i must contain a black pebble at $t^8_{h^{4i+1}_i} - 1$. This is a contradiction, which allows us to conclude that $t^4 < t^7$.
- (2b.12) $y_i \in \bullet[[t^2, t^7 - 1]]\bullet$.
  By (2b.5) we already know that y_i is uniquely black pebblable in $[t^1, t^\omega,]$ and must be pebbled at time $t^2 < t^7$ and $y_i \in \bullet[[t^2, t^4 - 1]]\bullet$. Now we will show that $y_i \in \bullet[[t^2, t^7 - 1]]\bullet$. This is simply because y_i is a predecessor of $\bar{x}_i$ which, by (2b.10), is pebbled at t^7, $t^1 < t^7 < t^\omega$. Since y_i cannot be repebbled before t^ω and $\bar{x}_i$ is a successor of y_i, we conclude that $y_i \in \bullet[[t^2, t^7 - 1]]\bullet$.
- (2b.13) *d_i is uniquely black pebblable in $[t^2, t^{11}]$, $d_i \in \oslash[[t^\alpha, t^3 - 1]]\oslash$, and $d_i \in \bullet[[t^3, h^{4i+1}_i - 1]]\bullet$, where $t^2 < t^3 < t^4$ and no pebbles are placed on any node between t^2 and t^3, except onto d_i's predecessors.*
  We first prove that $d_i \in \oslash[[t^\alpha, t^2]]\oslash$. d_i is not a member of $B[\beta_i]$, so it is empty at t^α. From (2.4) we know that d_i cannot be pebbled between t^α and

t^1. By (2b.5), d_i cannot be pebbled between t^1 and t^2, so $d_i \in \oslash[[t^\alpha, t^2]]\oslash$.
In order to black pebble any member of H_i, we must first pebble d_i. d_i must be empty by $t^{11}-1$, since by (2b.3) $\bar{x}'_i$ and x_i are clamped during $t^{11}-1$ and a_i has $4i+1$ other predecessors, none of which is d_i. By (2b.8), d_i must contain a black pebble at $t^8_{h_i^{4i+1}}-1$.
By (2b.12) and (2b.3), at least three nodes y_i, $\bar{x}'_i$, and x'_i must be clamped in $\bullet[[t^4, t^4]]\bullet$; by (2b.12), (2b.3), and (2b.11) three nodes y_i, $\bar{x}'_i$, and x_i clamped in $\bullet[[t^4+1, t^7-1]]\bullet$; by (2b.10) and (2b.3) three nodes $\bar{x}_i$, $\bar{x}'_i$, and x_i clamped in $\bullet[[t^7, t^8_{h_i^1}-1]]\bullet$; by (2b.7) and (2b.3) three nodes h_i^1, $\bar{x}'_i$, and x_i clamped in $\bullet[[t^8_{h_i^1}, t^{11}-1]]\bullet$; and by (2.6) and (2b.3) three nodes a_i, $\bar{x}'_i$, and x_i clamped in $\bullet[[t^{11}-1, t^{11}]]\bullet$. So d_i cannot be black pebbled at any time from t^4 until $t^{11}-1$. So d_i must contain a black pebble at t^4.
By (2b.5) d_i must be empty at t^2. Therefore, d_i must be black pebbled at some time t^3, $t^2 < t^3 < t^4$. By (2b.3) and (2b.12) x'_i and y_i are in $\bullet[[t^2, t^4]]\bullet$. By the induction hypothesis, (2.4), and (2b.5), we know that no nonsource nodes contain white pebbles at t^2. Also d_i has $4i+1$ source nodes as predecessors. Thus we can apply Lemma 12 to conclude that d_i is uniquely black pebblable in $[t^2, t^4]$, where no pebbles are placed on any node between t^2 and t^3, except onto d_i's predecessors, so $d_i \in \oslash[[t^\alpha, t^3-1]]\oslash$. By the argument above, we extend the interval during which d_i is uniquely black pebblable to $[t^2, t^{11}]$. Since d_i is a predecessor of h_i^{4i+1}, $d_i \in \bullet[[t^3, t^8_{h_i^{4i+1}}-1]]\bullet$.

- (2b.14) $\bar{x}'_i$ *is uniquely black pebblable in* $[t^3, t^{13}_{g_i^{4i+2}}-1]$ *and* $\bar{x}'_i \in \oslash[[t^\alpha, t^4-1]]\oslash$, *where no pebbles are placed on any node between* t^3 *and* t^4*, except onto* $\bar{x}'_i$*'s predecessors.*
$\bar{x}'_i$ is not a member of $B[\beta_i]$, so it is empty at t^α. From (2.4) we know that $\bar{x}'_i$ cannot be pebbled between t^α and t^1. By (2b.5), $\bar{x}'_i$ cannot be pebbled between t^1 and t^2. By (2b.13) we know that $\bar{x}'_i$ must be empty at t^3, so $\bar{x}'_i \in \oslash[[t^\alpha, t^3]]\oslash$.
By (2b.3) we know that $x'_i \in \bullet[[t^3, t^4]]\bullet$, by (2b.12) we know that $y_i \in \bullet[[t^3, t^4]]\bullet$, and by (2b.6) and (2b.13) we know that $d_i \in \bullet[[t^3, t^4]]\bullet$. Furthermore, by the conditions of Case 2, we know that $\bar{x}'_i$ contains a black pebble at t^4. Since $\bar{x}'_i$ has $4i$ source node predecessors, and since we can argue (by the induction hypothesis, (2.4), (2b.5), and (2b.13)) that no nonsource nodes contain white pebbles at t^3, we can apply Lemma 12 to conclude that $\bar{x}'_i$ is uniquely black pebblable in $[t^3, t^4]$, where no pebbles are placed on any node between t^3 and t^4, except onto $\bar{x}'_i$'s predecessors. Therefore $\bar{x}'_i \in \oslash[[t^\alpha, t^4-1]]\oslash$.
By (2b.3) we know that $\bar{x}'_i \in \bullet[[t^4, t^{13}_{g_i^{4i+2}}-1]]\bullet$, so we can extend the interval during which $\bar{x}'_i$ is uniquely black pebblable to $[t^3, t^{13}_{g_i^{4i+2}}-1]$.
- (2b.15) $\bar{x}_i \in \oslash[[t^\alpha, t^4+1]]\oslash$, $x_i \in \oslash[[t^\alpha, t^4]]\oslash$, $\bar{x}'_i \in \oslash[[t^\alpha, t^4-1]]\oslash$.
No member of $\{\bar{x}_i, \bar{x}'_i, x_i\}$ is a member of $B[\beta_i]$, so they are all empty at t^α. From (2b.3) we know that no member of $\{\bar{x}_i, \bar{x}'_i, x_i\}$ can be pebbled between t^α and t^1, and by (2.4) we know that no member of $\{\bar{x}_i, \bar{x}'_i, x_i\}$ can be pebbled between t^1 and t^2. By (2b.13) we know that no member of $\{\bar{x}_i, \bar{x}'_i, x_i\}$ can be pebbled between t^2 and t^3. By (2b.14) we know that no member of $\{\bar{x}_i, \bar{x}'_i, x_i\}$ can be pebbled between t^3 and t^4-1 and therefore they must all be empty. Furthermore, by (2b.3), neither $\bar{x}_i$ nor x_i can be pebbled at t^4, and $\bar{x}_i$ cannot be pebbled between t^4 and t^4+1.

- (2b.16) *r_i^{4i-1} is uniquely black pebblable in $[t^4+1, t^8_{h_i^{4i}}-1]$, $r_i^{4i-1} \in \oslash[[t^\alpha, t^6_{r_i^{4i-1}}-1]]\oslash$, and $r_i^{4i-1} \in \bullet[[t^6_{r_i^{4i-1}}, t^8_{h_i^{4i-1}} - 1]]\bullet$, where $t^4 + 1 < t^6_{r_i^{4i-1}} < t^8_{h_i^1}$; and for each j, $1 \le j \le 4i-2$, r_i^j is uniquely black pebblable in $[t^4+1, t^8_{h_i^{4i}} - 1]$, $r_i^j \in \oslash[[t^\alpha, t^6_{r_i^j} - 1]]\oslash$, and $r_i^j \in \bullet[[t^6_{r_i^j}, t^8_{h_i^j} - 1]]\bullet$, where $t^4 + 1 < t^6_{r_i^j} < t^6_{r_i^{j+1}}$.*

  No member of R_i is a member of $B[\beta_i]$, so they are all empty at t^α. From (2b.3) we know that no member of R_i can be pebbled between t^α and t^1, and by (2.4) we know that no member of R_i can be pebbled between t^1 and t^2. By (2b.13) we know that no member of R_i can be pebbled between t^2 and t^3. By (2b.14) we know that no member of R_i can be pebbled between t^3 and t^4 and therefore they must all be empty. Furthermore, by (2b.3), no member of R_i can be pebbled between t^4 and t^4+1. Therefore, every member of R_i is in $\oslash[[t^\alpha, t^4+1]]\oslash$.

  By (2.1), (2.3), (2.6), and (2b.7) we know that none of the descendants of r_i^{4i+1} can contain a white pebble at any time during $[t^\alpha, t^8_{h_i^{4i}}-1]$. I n order to pebble h_i^{4i-1} at time $t^8_{h_i^{4i-1}}$, r_i^{4i-1} must contain a pebble at time $t^8_{h_i^{4i-1}}-1$. By (2b.3) and (2b.13), x_i, $\bar{x}'_i$, and d_i are in $\bullet[[t^4+1, t^8_{h_i^{4i}} - 1]]\bullet$. h_i^{4i} has $4i$ other predecessors, none of which is in R_i, so r_i^{4i-1} must be empty at $t^8_{h_i^{4i}}-1$. Since r_i^{4i-1} has $4i$ predecessors, none of which is x_i, $\bar{x}'_i$, or d_i, we can apply Lemma 11 to conclude that r_i^{4i-1} is uniquely black pebblable in $[t^4+1, t^8_{h_i^{4i-1}} - 1]$, so $r_i^{4i-1} \in \oslash[[t^\alpha, t^6_{r_i^{4i-1}} - 1]]\oslash$. Since r_i^{4i-1} is a predecessor of h_i^{4i-1} and can be pebbled only once before $t^8_{h_i^{4i}}$, $r_i^{4i-1} \in \bullet[[t^6_{r_i^{4i-1}}, t^8_{h_i^{4i-1}} - 1]]\bullet$.

  We can now prove by induction from $j = 4i-2$ down to $j = 1$ that r_i^j is uniquely black pebblable in $[t^4+1, t^8_{h_i^{4i-1}}-1]$ and is clamped in $\bullet[[t^6_{r_i^j}, t^8_{h_i^j}-1]]\bullet$, where $t^6_{r_i^j} < t^6_{r_i^{j+1}}$.

  By (2.1), (2.3), (2.6), (2b.7), and the induction hypothesis we know that none of the descendants of r_i^j can contain a white pebble at any time during $[t^4+1, t^8_{h_i^{4i}} - 1]$. In order to pebble h_i^j at time $t^8_{h_i^j}$, r_i^j must contain a pebble at time $t^8_{h_i^j}-1$. By (2b.3) and (2b.13), x_i, $\bar{x}'_i$, and d_i are in $\bullet[[t^4+1, t^8_{h_i^{4i}}-1]]\bullet$ and h_i^{4i} has $4i$ other predecessors, none of which is in R_i, so r_i^j must be empty at $t^8_{h_i^{4i}} - 1$. Since r_i^j has $4i$ predecessors, none of which is x_i, $\bar{x}'_i$, or d_i, we can apply Lemma 11 to conclude that r_i^j is uniquely black pebblable in $[t^4+1, t^8_{h_i^{4i}} - 1]$, $r_i^j \in \oslash[[t^\alpha, t^6_{r_i^j} - 1]]\oslash$. Since r_i^j must be pebbled before r_i^{j+1} can be pebbled, $t^6_{r_i^j} < t^6_{r_i^{j+1}}$. Since r_i^j is a predecessor of h_i^j and can be pebbled only once before $t^8_{h_i^{4i}}$, $r_i^j \in \bullet[[t^6_{r_i^j}, t^8_{h_i^j} - 1]]\bullet$.

- (2b.17) $t^6_{r_i^{4i-1}} < t^7 < t^8_{h_i^1}$.

  By (2b.10) and (2b.11), we already know that $t^4 < t^7 < t^8_{h_i^1}$. We must now show that $t^6_{r_i^{4i-1}} < t^7$. We do not have to consider the case when $t^7 = t^6_{r_i^{4i-1}}$ since two nodes cannot be pebbled at the same time, and by (2b.10) $\bar{x}_i$ is black pebbled at t^7, and by (2b.16) r_i^{4i-1} is black pebbled at $t^6_{r_i^{4i-1}}$.

  Suppose for the sake of contradiction that $t^6_{r_i^{4i-1}} > t^7$. That means that $\bar{x}_i$ must contain a pebble at $t^6_{r_i^{4i-1}} - 1$, because removing it before then is not frugal since all of $\bar{x}_i$'s successors are pebbled only after $t^6_{r_i^{4i-1}} - 1$. But by (2b.3) and (2b.13), x_i, $\bar{x}'_i$, and d_i are in $\bullet[[t^4+1, t^8_{h_i^{4i}}-1]]\bullet$ and therefore also contain pebbles at $t^6_{r_i^{4i-1}} - 1$. This means that there are at most $4i-1$ free pebbles available at $t^6_{r_i^{4i-1}} - 1$, but r_i^{4i-1} has $4i$ predecessors, none of which is $\bar{x}_i$, x_i, $\bar{x}'_i$, or d_i, so it cannot be black pebbled at $t^6_{r_i^{4i-1}} - 1$. This contradiction allows us to conclude that $t^6_{r_i^{4i-1}} < t^7$.

- (2b.18) *g^{4i-1}_{i-1} is uniquely black pebblable in $[t^4+1, t^7-1]$, $g^{4i-1}_{i-1} \in \oslash[[t^\alpha, t^5_{g^{4i-1}_{i-1}} - 1]]\oslash$, and $g^{4i-1}_i \in \bullet[[t^5_{g^{4i-1}_{i-1}}, t^6_{r_i^{4i-1}} - 1]]\bullet$, where $t^4+1 < t^5_{g^{4i-1}_{i-1}} < t^6_{r_i^1}$; and for each j, $1 \leq j \leq 4i-2$, g^j_{i-1} is uniquely black pebblable in $[t^4+1, t^7-1]$, $g^j_{i-1} \in \oslash[[t^\alpha, t^5_{g^j_{i-1}} - 1]]\oslash$, and $g^{4i-1}_i \in \bullet[[t^5_{g^j_{i-1}}, t^6_{r_i^j} - 1]]\bullet$, where $t^4+1 < t^5_{g^j_{i-1}} < t^5_{g^{j+1}_{i-1}}$.*
No member of G_{i-1} is a member of $B[\beta_i]$, so they are all empty at t^α. From (2b.3) we know that no member of G_{i-1} can be pebbled between t^α and t^1 and by (2.4) we know that no member of G_{i-1} can be pebbled between t^1 and t^2. By (2b.13) we know that no member of G_{i-1} can be pebbled between t^2 and t^3. By (2b.14) we know that no member of G_{i-1} can be pebbled between t^3 and t^4 and therefore they must all be empty. Furthermore, by (2b.3), no member of G_{i-1} can be pebbled between t^4 and t^4+1. Therefore, every member of G_{i-1} is in $\oslash[[t^\alpha, t^4+1]]\oslash$.
In order to black pebble r_i^1 at $t^6_{r_i^1}$, every member of G_{i-1} must first contain a pebble at some time t^5. We will now show that every member of G_{i-1} must contain a black pebble at t^5. By (2b.14) we know that $t^5 > t^4$. By (2b.3) we know that $t^5 > t^4+1$.
This proof will be inductive, like the proof of (2a.9), but we will first prove noninductively that every member of G_{i-1} must be empty at t^7-1. By (2b.16) and (2b.17) each of R_i's $4i-1$ members is in $\bullet[[t^7, t^8_{h_i^1} - 1]]\bullet$. By (2b.3) $\bar{x}'_i$ and x_i are in $\bullet[[t^7, t^8_{h_i^1} - 1]]\bullet$. By (2b.13) $d_i \in \bullet[[t^7, t^8_{h_i^1} - 1]]\bullet$. And by (2b.10) $\bar{x}_i \in \bullet[[t^7, t^8_{h_i^1} - 1]]\bullet$. Since this sums to $4i+3$ clamped nodes at t^7, every node in G_{i-1} must be empty at t^7. Since no member of G_{i-1} is a predecessor of $\bar{x}_i$, and $\bar{x}_i$ is pebbled at t^7, every member of G_{i-1} must also be empty at t^7-1.
By (2.1), (2.3), (2.6), (2b.7), and (2b.16) we know that none of the descendants of g^{4i-1}_{i-1} can contain a white pebble at any time during $[t^4+1, t^7-1]$. In order to pebble r_i^{4i-1} at time $t^6_{r_i^{4i-1}}$, g^{4i-1}_{i-1} must contain a pebble at time $t^5_{g^{4i-1}_{i-1}} - 1$. We already know from the argument above that g^{4i-1}_{i-1} must be empty at t^7-1. Since g^{4i-1}_{i-1} has $4i-1$ predecessors other than $\bar{x}'_i$, x_i, d_i, or y_i, we can apply Lemma 11 to conclude that g^{4i-1}_{i-1} is uniquely black pebblable in $[t^4+1, t^7-1]$. Let $t^5_{g^{4i-1}_{i-1}}$ be the time at which g^{4i-1}_{i-1} is pebbled. Then $g^{4i-1}_{i-1} \in \oslash[[t^\alpha, t^5_{g^{4i-1}_{i-1}} - 1]]\oslash$. By (2b.7), (2b.16), and (2b.17), the only time before t^7-1 at which r_i^1 can be black pebbled is $t^6_{r_i^1}$, so $t^5_{g^{4i-1}_{i-1}} < t^6_{r_i^1}$. Since g^{4i-1}_{i-1} is a predecessor of r_i^{4i-1} and can be pebbled only once before $t^6_{r_i^{4i-1}}$, $g^{4i-1}_i \in \bullet[[t^5_{g^{4i-1}_{i-1}}, t^6_{r_i^{4i-1}} - 1]]\bullet$.

We can now prove by induction from $j = 4i-2$ down to $j = 1$ that g^j_{i-1} is uniquely black pebblable in $[t^4+1, t^7-1]$ and is clamped in $\bullet[[t^5_{g^j_{i-1}}, t^6_{r^j_i}-1]]\bullet$, where $t^5_{g^j_{i-1}} < t^5_{g^{j+1}_{i-1}}$.

By (2.1), (2.3), (2.6), (2b.7), (2b.16), and the induction hypothesis we know that none of the descendants of g^j_{i-1} can contain a white pebble at any time during $[t^4+1, t^7-1]$. In order to pebble r^j_i at time $t^6_{r^j_i}$, g^j_{i-1} must contain a pebble at time $t^6_{r^j_i}-1$. We already know from the argument above that g^j_{i-1} must be empty at t^7-1. Since g^j_{i-1} has $4i-1$ predecessors other than $\bar{x}'_i$, x_i, d_i, or y_i, we can apply Lemma 11 to conclude that g^j_{i-1} is uniquely black pebblable in $[t^4+1, t^7-1]$. Let $t^5_{g^j_{i-1}}$ be the time at which g^j_{i-1} is pebbled. Then $g^j_{i-1} \in \oslash[[t^\alpha, t^5_{g^j_{i-1}}-1]]\oslash$. Since g^j_{i-1} must be black pebbled before g^{j+1}_{i-1} can be pebbled, $t^5_{g^j_{i-1}} < t^5_{g^{j+1}_{i-1}}$. Since g^j_{i-1} is a predecessor of r^j_i and can be pebbled only once before $t^6_{r^j_i}$, $g^j_{i-1} \in \bullet[[t^5_{g^j_{i-1}}, t^6_{r^j_i}-1]]\bullet$.

Finally, it is clear that since every member of G_{i-1} is uniquely black pebblable and all are clamped until at least $t^6_{r^1_i}-1$, $t^5_{g^{4i-1}_{i-1}} = t^5$.

The following relationships between the times at which certain nodes are pebbled are demonstrated by the points proved above.

- By (2.3), $t^\omega = t^{13}_{g^{4i+3}_i}$, and for each j, $1 \le j \le 4i+2$, $t^{13}_{g^j_i} < t^{13}_{g^{j+1}_i}$.
- By (2.6), $t^{11} < t^{13}_{g^1_i}$.
- By (2b.7), $t^8_{h^{4i+1}_i} < t^{11}$, and for each j, $1 \le j \le 4i$, $t^8_{h^j_i} < t^8_{h^{j+1}_i}$.
- By (2b.10), $t^7 < t^8_{h^1_i}$.
- By (2b.17), $t^6_{r^{4i-1}_i} < t^7$
- By (2b.16), for each j, $1 \le j \le 4i-2$, $t^6_{r^j_i} < t^6_{r^{j+1}_i}$.
- By (2b.18), $t^4+1 < t^5_{g^1_i}$, $t^5_{g^{4i-1}_i} < t^6_{r^1_i}$, and for each j, $1 \le j \le 4i-2$, $t^5_{g^j_i} < t^5_{g^{j+1}_i}$.
- By (2b.13), $t^3 < t^4$.
- By (2b.13), $t^2 < t^3$.
- By (2b.5), $t^1 < t^2$.
- By (2.4), $t^\alpha < t^1$.

These inequalities produce the following ordering of times, which labels the x-axis of Figure 9: $t^\alpha < t^1 < t^2 < t^3 < t^4 < t^4+1 < t^5_{g^1_{i-1}} < \cdots < t^5_{g^{4i-1}_{i-1}} = t^5 < t^6_{r^1_i} \cdots < t^6_{r^{4i-1}_i} < t^7 < t^8_{h^1_i} \cdots < t^8_{h^{4i+1}_i} < t^{11} < t^{13}_{g^1_i} \ldots t^{13}_{g^{4i+3}_i} = t^\omega$.

As in the universal case, the ordering allows us to produce a figure (Figure 9) which summarizes the points proved above. Of particular interest to us now is the sequence of line segments that are entering the region labeled as the "induction." Particularly, we know that every member of G_i is empty during $[t^4+1, t^5]$ since $[t^4+1, t^5]$ is a subinterval of $[t^\alpha, t^\omega]$ and each member of G_i is uniquely black pebblable in $[t^\alpha, t^\omega]$ and must be pebbled after t^5. a_i must also be empty during $[t^4+1, t^5]$ since it is also uniquely black pebblable in $[t^\alpha, t^\omega]$ and must be pebbled at t^{11} after t^5. The same is true for H_i and R_i since both are uniquely black pebblable in $[t^4+1, t^9-1]$ and must be pebbled after t^5. Therefore, no member of widget i which is a descendant of g^{4i-1}_{i-1} can be pebbled during the interval $[t^4+1, t^5]$. Also every node in $B[\beta_i] \cup \{y_i, \bar{x}'_i, d_i, x_i\} = B[\beta_i \cup \{x_i\}]$ is in $\bullet[[t^4+1, t^5]]\bullet$.

We can therefore apply the induction hypothesis to conclude that black pebbling G_{i-1} requires $\psi\lceil_{\beta_i \cup \{x_i\}}$ to be in QSAT and that $\Omega(2^k)$ units of time must pass between t^4+1 and t^5, where k is the number of universally quantified variables among the innermost $i-1$ variables of ψ.

We have shown that regardless of whether widget i corresponds to a universal quantifier or an existential quantifier, simultaneously black pebbling G_i using no more than $4n+3$ pebbles requires that $\psi\lceil_{\beta_i}$ be in QSAT and requires $\Omega(2^k)$ units of time between t^α and t^ω, where k is the number of universal quantifiers among the i innermost quantifiers. This completes the proof of Lemma 18. ∎

This brings us to the main result of this section.

THEOREM 19 (main theorem). *The black-white pebbling game on DAGs is PSPACE-complete.*

Proof. By Lemma 2 we know that the black-white pebbling game is in PSPACE. By Lemmas 4 and 18 ψ is in QSAT iff $\mathcal{G}$ can be black-white pebbled using $4n+3$ pebbles, and the black-white pebbling game is therefore PSPACE-hard. ∎

6. Exponential time/space trade-off for black-white pebbling and the PSPACE-completeness of the symmetric game.

THEOREM 20. *There exists an infinite family of graphs such that any minimal space black-white pebbling of these graphs requires exponential time, but they can be pebbled in linear time with the use of two additional pebbles.*

Proof. Let $\mathcal{G}$ be the DAG corresponding to the formula $\psi = \forall x_n \forall x_{n-1} \dots \forall x_1 (x_1 \vee \bar{x}_1 \vee x_2) \wedge (x_2 \vee \bar{x}_2 \vee x_3) \wedge \cdots \wedge (x_n \vee \bar{x}_n \vee x_1)$. This formula is clearly QSAT, since its 3CNF part is a tautology. Also, since ψ has n universally quantified variables, by Lemma 17, the minimal $4n+3$ pebbling strategy for $\mathcal{G}$ requires time 2^n to execute. We can pebble $\mathcal{G}$ in linear time using exactly two extra pebbles by following our upper bound strategy except that in each universal widget, we keep a pebble on $\bar{x}'_i$ and one on R_i, which will allow us to pebble the universal widget just once without any repebbling. More concretely, we make the following changes to the strategy through the universal widget: We modify step 2 (see the proof of Lemma 5, Case 1) so that instead of sliding a black pebble to $\bar{x}_i$, we place a black pebble on $\bar{x}_i$, and continue to keep a black pebble on $\bar{x}'_i$. Similarly we modify step 5 so that instead of sliding a black pebble from r_i^1 to h_i, we place a black pebble on h_i and continue to keep a pebble on r_i^1. Modify step 9 by removing all pebbles from widget i except for those on a_i, x'_i, y_i, $\bar{x}'_i$, and R_i. Skip steps 10 and 11. Finally, modify step 12 to use the i-slide strategy to slide all of R_i's pebbles to $\{g_i^1, \dots, g_i^{4i-1}\}$. The rest of the steps stay unchanged. ∎

6.1. PSPACE-completeness of the symmetric black-white pebbling game.

Here we show that our PSPACE-completeness result is robust in the sense that it holds for both variants of black-white pebbling.

6.2. Definition of symmetric black-white pebbling.

The symmetric black-white pebbling game is very similar to the black-white pebbling game. The only difference is in the ending condition. The symmetric black-white pebble game ends successfully if the player can place either a black or a white pebble on s and then remove every pebble from $\mathcal{G}$, including the one on s, without ever exceeding a given bound on the number of pebbles simultaneously placed on $\mathcal{G}$. As before we are interested in pebbling s, using a minimum number of pebbles, but now we no longer need to end with a black pebble on s as a witness.

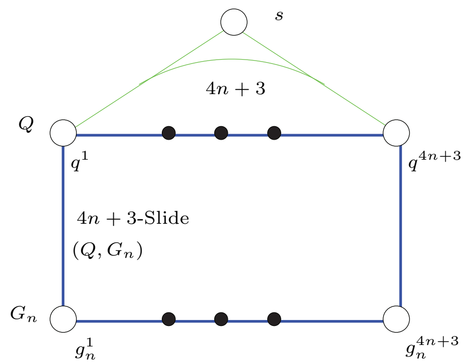

FIG. 10. *Modification made to $\mathcal{G}$ to form $\mathcal{G}_{sym}$.*

DEFINITION 6.1 (symmetric black-white pebbling strategy). *Let $\mathcal{G} = (V, E)$ be a DAG with distinguished output node s. A symmetric black-white pebbling strategy for $\mathcal{G}$ is a sequence of black-white pebbling configurations $\mathcal{M}[t^{start}], \dots, \mathcal{M}[t^{end}]$, such that $\mathcal{M}[t^{start}] = (\emptyset, \emptyset)$; $\mathcal{M}[t^{end}] = (\emptyset, \emptyset)$ for all t, $t^{start} \leq t \leq t^{end} - 1$; $\mathcal{M}_{t+1}$ follows from $\mathcal{M}_t$ by a legal black-white pebbling move; and for some t, $t^{start} < t < t^{end}$, $s \in B[t]$ or $s \in W[t]$.*

The symmetric black-white pebbling game is more general than the standard version, since every k-pebbling strategy can be converted to a symmetric k-strategy simply by adding one more move which removes the black pebble from s.

The symmetric black-white pebbling game was introduced in [CS76]. It was proved in [CS76, Hei81] that every symmetric k-pebbling strategy is reversible, meaning that if you transform the strategy by playing the moves in reverse after turning every white pebble into a black pebble and vice versa, then the resulting strategy is still a symmetric k-pebbling strategy. We will use this fact extensively to transform our PSPACE-completeness proof of black-white pebbling into a PSPACE-completeness proof of symmetric black-white pebbling.

6.2.1. Modification to the definition of frugality. Since we can now place either a black or a white pebble on s, whereas we could place only a black pebble on s in the asymmetric version of the game, we must modify our definition of frugality. Instead of asserting that the last black pebble assignment to s is necessary, we now say that the last pebble assignment to s, regardless of the color, is necessary.

6.2.2. Modification to the black-white pebbling reduction. In order to prove the PSPACE-completeness of the symmetric black-white pebbling game we must make one very minor modification to our construction $\mathcal{G}$. Instead of making s the successor of every node in G_n, we will add an intermediate set of $4n+3$ vertices $Q = \{q^1, \dots, q^{4n+3}\}$ such that (Q, G_n) forms a $4n+3$-slide and s is the successor of every node in Q. Since (Q, G_n) forms a $4n+3$-slide, q^i is a predecessor of every q^j, $j > i$. Note that each node in $\{s\} \cup Q \cup G_n$ has in-degree $4n+3$. The modification is shown in Figure 10. We call the modified construction $\mathcal{G}_{sym}$.

6.3. Proof that symmetric black-white pebbling is PSPACE-complete. We will show that $\mathcal{G}_{sym}$ is symmetric black-white pebblable iff ψ is in QSAT. Clearly the upper bound remains almost entirely unchanged. We still pebble s using the same strategy, except that we must use the i-slide strategy one more time to move the black pebbles from G_n to Q before finally pebbling s. We then remove all the black pebbles to empty the graph entirely.

The lower bound also remains very similar. Since symmetric strategies are reversible, we will no longer constrain our strategy to be purely black. Instead, when ψ is in QSAT, $\mathcal{G}_{sym}$ is $4n+3$ pebblable by both a pure black strategy and a pure white strategy but by no mixed strategies. (By a pure black strategy we mean that only black pebbles are placed on nodes other than source nodes, and, similarly, by a pure white strategy we mean that only white pebbles are placed on nodes, excluding source nodes.) We define the dual of a black clamped interval (Definition 3.4), a white clamped interval, which is required in our lower bound. As before, we will assume that all strategies are frugal.

DEFINITION 6.2 (white clamped interval). *For any node v and any time units t^α, t^ω such that $t^\alpha \leq t^\omega$, we say that v is* white clamped *in the interval $[t^\alpha, t^\omega]$, denoted as $v \in \circ[[t^\alpha, t^\omega]]\circ$, if v contains a white pebble during every configuration from $\mathcal{M}[t^\alpha]$ through $\mathcal{M}[t^\omega]$, i.e., for all t^*, $t^\alpha \leq t^* \leq t^\omega$, $v \in W[t^*]$.*

The first step of the proof is to prove the following statement.

LEMMA 21. *Let $P(s, t^2, t^3)$ be the last pebble assignment to s. Pebbling s at time t^2 and removing the pebble afterwards at t^3 without exceeding the space bound of $4n+3$ pebbles requires either that G_n be simultaneously black pebbled at some time t^1, $t^1 < t^2$ and $Q \cup \{s\} \subseteq \oslash[[t^{start}, t^1]]\oslash$, or that G_n be simultaneously white pebbled at some time t^4, $t^4 > t^3$ and $Q \cup \{s\} \subseteq \oslash[[t^4, t^{end}]]\oslash$.*

Proof.

Case 1. Suppose $P(s, t^2, t^3)$ is a black pebble assignment. Since q^1 is a predecessor of s, either there is a black pebble on q^1 at time $t^2 - 1$, or there is a white pebble on q^1 at time $t^2 - 1$.

Case 1a. Suppose the pebble on q^1 at $t^2 - 1$ is black and was placed on q^1 at time $t^* < t^2$, so $q^1 \in \bullet[[t^*, t^2 - 1]]\bullet$. Also suppose for the sake of contradiction that there is some white pebble on a node v of G_n at time $t^* - 1$. This white pebble must be removed from v at some time between t^* and t^2, because s has $4n+3$ predecessors, none of which is v. But v also has $4n+3$ predecessors, and $q^1 \in \bullet[[t^*, t^2 - 1]]\bullet$. This means that we cannot remove v's white pebble during $[t^*, t^2 - 1]$ without violating the space bound; thus there can be no white pebbles in G_n at time $t^* - 1$, so G_n must have been simultaneously black pebbled at time t^1, $t^1 = t^* - 1 < t^2$.

Now suppose there is a pebble of either color on some node of $Q \cup \{s\}$ during any time unit in $[t^{start}, t^1]$. We know that the pebble cannot be on s, since only the last pebble assignment to s is necessary, and that assignment begins only at t^2, $t^2 > t^1$. The pebble must therefore be on some member of Q. Consider the member of Q, q^j, which contains a pebble during $[t^{start}, t^1]$ such that j is maximum over all the members of Q that contain a pebble during $[t^{start}, t^1]$. By the definition of j, no successor of q^j in Q is pebbled during $[t^\alpha, t^1]$. Also since s is not pebbled during $[t^\alpha, t^1]$, we can conclude that no successor of q^j is pebbled until after t^1. But since G_n has $4n+3$ members and each contains a pebble at t^1, q^j must be empty at t^1, before a pebble is ever placed on one of its successors. Therefore, the pebble assignment to q^j was unnecessary, which contradicts our general assumption that all of our pebbling strategies are frugal. We can therefore conclude that $Q \cup \{s\} \subseteq \oslash[[t^{start}, t^1]]\oslash$.

Case 1b. Suppose, on the other hand, that q^1 is white pebbled at time $t^* < t^2$. Then q^1's white pebble must be removed at some time t^4 after t^2, so $q^1 \in \circ[[t^*, t^4 - 1]]\circ$. Since q^1 has $4n+3$ predecessors, none of which is s, $t^3 < t^4$. Since G_n is empty at t^2, all of q^1's predecessors (all the nodes of G_n) must be repebbled before t^4. Since each node of G_n has in-degree $4n+3$ and $q^1 \in \circ[[t^2, t^4-1]]\circ$, no member of G_n can be black pebbled during $[t^2, t^4-1]$. So q^1's white pebble must be slid down to some node of G_n, and G_n must then be simultaneously white pebbled at time t^4, $t^4 > t^3$.

Now suppose that there is a pebble of either color on some node of $Q \cup \{s\}$ during any time unit in $[t^4, t^{end}]$. Clearly, each member of $Q \cup \{s\}$ must be empty at t^4, since G_n has size $4n+3$ and every member of G_n contains a pebble at t^4. So some node in $Q \cup \{s\}$ is repebbled during $[t^4, t^{end}]$. We know that s cannot be repebbled, since $P(s, t^2, t^3)$ was the last pebble assignment to s. The pebble must therefore be placed on some member of Q. Consider the member of Q, q^j, which contains a pebble during $[t^4, t^{end}]$ such that j is maximum over all the members of Q that contain a pebble during $[t^4, t^{end}]$. By the definition of j, no successor of q^j in Q is pebbled during $[t^4, t^{end}]$, and since s is not pebbled again at all, we can conclude that no successor of q^j is ever pebbled. But the entire graph must be empty by t^{end}, so q^j's pebble must be removed before a pebble is ever placed on one of its successors. Therefore, the pebble assignment to q^j was unnecessary, which contradicts our general assumption that all of our pebbling strategies are frugal. We can therefore conclude that $Q \cup \{s\} \subseteq \oslash[[t^4, t^{end}]]\oslash$.

Case 2. Suppose $P(s, t^2, t^3)$ is a white pebble assignment. This case follows from Case 1 by Meyer auf der Heide's duality lemma. ∎

We now formalize a dual statement to Lemma 18 which reverses Lemma 18 by the method described in [Hei81].

LEMMA 22. *For all i, $1 \le i \le n$, and for all $\beta_i \in A_i$, if there exist times t^α, t^ω such that*

1. *every member of G_i contains a white pebble at t^α,*
2. *the only members of $\mathcal{G}$ which contain pebbles at t^ω are members of $W[\beta_i]$,*
3. *$W[\beta_i] \subseteq \circ[[t^\alpha, t^\omega]]\circ$,*
4. *there are never more than $4n+3$ pebbles on the graph at any time during $[t^\alpha, t^\omega]$, and*
5. *no pebble is placed on any descendant of g_i^{4i+3} during the interval $[t^\alpha, t^\omega]$,*

then $\psi\lceil_{\beta_i}$ is in QSAT and requires $\Omega(2^k)$ units of time between t^α and t^ω, where k is the number of universal quantifiers among the i innermost quantifiers.

We can appeal to Lemma 18 as a proof of Lemma 22. If we assume for the sake of contradiction that the statement of Lemma 22 is false and a $4n+3$ black-white pebbling strategy could remove the white pebbles from G_i without violating our conditions even when $\psi\lceil_{\beta_i}$ is not in QSAT, then we could reverse this $4n+3$ strategy using Meyer auf der Heide's duality lemma to simultaneously black pebble G_i using no more than $4n+3$ pebbles, even when $\psi\lceil_{\beta_i}$ is not in QSAT. This is a contradiction to Lemma 18.

By Lemma 21, it is therefore sufficient to appeal to Lemmas 18 and 22 together to conclude that $\mathcal{G}_{sym}$ is symmetric black-white pebblable only if ψ is in QSAT.

7. Conclusion and open problems. A preliminary version of this paper appeared in *Proceedings of the* 49*th Annual IEEE Symposium on Foundations of Computer Science* (FOCS 2008). In that version we claimed that the PSPACE-completeness of black-white pebbling can be used to prove a similar PSPACE-completeness result for determining the minimal space required to refute an unsatisfiable formula in resolution. We still conjecture that this is the case.

An intriguing open problem is the complexity of approximating the pebbling number of a graph. As far as we are aware, there are no results known here, even for black pebbling. In particular, what is the best polynomial-time approximation algorithm for black pebbling, or black-white pebbling, a graph? Can we get to within a constant factor of the optimal in polynomial time?

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