Last class:

SOS in simple case
where we have NO constraints
Just wanted to derive
SOS certificate for non-neg $f : \{0,1\}^n \to \mathbb{R}$

Today SOS as way
to tighten SDPs

Recap - SA degree d LP

Original LP: [ignore max $c^T x$]

$$\left.\begin{array}{l} Ax \geq b \;, \; 0 \leq x \leq 1 \\ 1 \geq 0 \end{array}\right\} K$$

add new variables y_S $\forall S \subseteq [n], |S| \leq d$

Impose constraints $\underbrace{\prod_{i \in S} x_i \cdot \prod_{i \in T} (1-x_i)}_{\substack{S \cap T = \phi \;\; |S \cup T| \leq d \\ \text{"junta"}}} \cdot (a^T x - b) \geq 0$ $\forall$ rows $a \in A$

New constraints:

lifted SA_d constraints (*)

$$y_\phi = 1$$

$$y_{\{i\}} = x_i$$

$$0 \leq y_S \leq 1$$

$$\sum_{T' \subseteq T} (-1)^{|T'|} \left(\sum_{i=1}^{n} a_i y_{S \cup T' \cup \{i\}} - b y_{S \cup T'} \right) \geq 0$$

$\forall$ rows $a \in A$

above constraints translated to linear inequalities using new y vars plus multilinearization ($x_i^2 = x_i$)

Each Feasible solution $y \in SA_d(K)$ corresponded to a Linear functional

$$E_y : \mathbb{R}[x_1 .. x_n]_d \Rightarrow \mathbb{R} \quad \text{s.t.}$$

$\forall E \in \mathcal{E}_d(\mathcal{H})$:

(1) $E(1) = 1$

(2) $E(Q) \geq 0$ $\forall$ nn-junta Q with degree$(Q) \leq d$

(3) $E(PQ) \geq 0$ for $P \in \mathcal{H}$, and nn-junta Q with degree$(PQ) \leq d$

nn-junta : $\alpha \cdot \prod_{i \in S} x_i \cdot \prod_{j \in T} (1 - x_j)$

$S \cup T = \phi$

non-neg coeff

$SA_d(K)$ empty (no feasible solns) iff $\exists$ a degree d SA refutation of K'

Today: Start with $K = \{x \in \mathbb{R}^n \mid Ax \geq b\}$

Degree t SOS SDP relaxation:

New variables $Y_{I \cup J}$, $|I|, |J| \leq t$
$I, J \subseteq [n]$

$$\begin{array}{c|c} & J \\ \hline I & Y_{I\cup J} \end{array}$$

Degree t SDP

$$\max \quad C \bullet Y$$

$$\text{s.t.} \quad \left(\sum_{i=1}^{n} A_{\ell i} Y_{I \cup J \cup \{i\}} - b_\ell Y_{I \cup J} \right)_{|I|, |J| \leq t} \succeq 0$$

$\forall \ell \in [m]$

$$Y \succeq 0$$

Original vars x_i are $Y_{\{i\}}$

Canonical form: $\max \quad C \bullet Y$ s.t.

$A_i \bullet Y = b_i$ $\quad A_i$ symm.

$Y \succeq 0$

Let $K = \{x \in \mathbb{R}^n \mid Ax \geq b\}$ ← Linear relaxation of optimization problem over $\{0,1\}^n$

the t^{th} level of sos hierarchy $SOS_t(K)$

is the set of vectors $y \in \mathbb{R}^{2^{[n]}}$ satisfying:

$$M_t(y) := \left(y_{J \cup J}\right)_{|J|, |J| \leq t} \succeq 0$$

moment matrix (of y)

$$M_t^{\ell}(y) := \left(\sum_{i=1}^{n} A_{\ell i}\, y_{J \cup J \cup \{i\}} - b_\ell\, y_{J \cup J}\right)_{|J|, |J| \leq t} \succeq 0$$

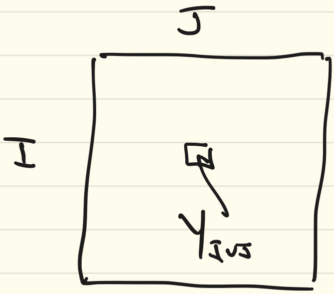
$\forall \ell \in [m]$

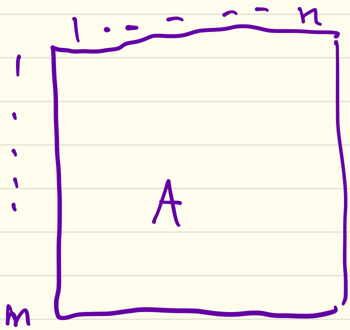
moment matrix of slacks

$$y_\phi = 1$$

$$y_I = 0 \quad \forall I,\ |I| > 2t+1$$

$M_t(y)$:





Let $SOS_t^{proj}(K) := \{ y_{\{1\}}, \dots, y_{\{n\}} \mid y \in SOS_t(K) \}$

projection of the vectors in $SOS_t(K)$ to original variables $x_1 \dots x_n$

$M_t(y) \succeq 0$ enforces vars are consistent

$y_{\{1,2\}} \in [y_{\{1\}} + y_{\{2\}} - 1, \min\{y_{\{1\}}, y_{\{2\}}\}]$

$M_t^{\ell}(y) \succeq$ guarantees that y satisfies ℓ^{th} linear constraint (and more)

- The set of PSD matrices forms a nonpolyhedral cone, so $SOS_t(K)$ is a convex set.

Separation problem for $SOS_t(K)$ is in polytime.

$\therefore$ If $m = \#$ of constraints, can optimize over $SOS_t(K)$ in time $n^{O(t)}\,\text{poly}(m)$ up to numerical errors.

Moment Matrix:

	ϕ	1	2	3			
ϕ	$y_\phi = 1$	y_1	y_2	y_3	y_4		
1	y_1	y_1			$y_{1,4}$		
2	y_2		y_2				
3	y_3			y_3			
	y_4				y_4		

Lemma Let $K = \{x \in \mathbb{R}^n \mid Ax \geq b\}$ and let $y \in SOS_t(K)$ for $t \geq 0$. Then:

(a) $K \cap \{0,1\}^n \subseteq SOS_t^{proj}(K)$

integral convex hull inside $SOS_t^{proj}(K)$ convex set

(b) $0 \leq y_I \leq 1 \quad \forall I, |I| \leq t$

(c) $0 \leq y_J \leq y_I \leq 1 \quad \forall I \subseteq J, 0 \leq |I| \leq |J| \leq t$

(d) $|y_{I \cup J}| \leq \sqrt{y_I \cdot y_J} \quad \forall I, J \ (|I|, |J| \leq t$

(e) $SOS_t^{proj}(K) \subseteq K$ (easy)

(f) $SOS_0(K) \supseteq SOS_1(K) \supseteq \cdots \supseteq SOS_n(K)$ (easy)

-its a tightening and by (a) contains integral hull

(a) Let $x \in K \cap \{0,1\}^n$ be a feasible integral soln.

Let $y_I = \prod_{i \in I} x_i$. then $y \in SOS_t(K)\ \forall t \geq 0$

① $M_t(y)$ is a submatrix of the rank-1 psd matrix yy^T since $\forall$ entry I, J:

$$(M_t(y))_{I,J} = y_{I \cup J} = \prod_{i \in I \cup J} x_i = \Big(\prod_{i \in I} x_i\Big)\Big(\prod_{j \in J} x_j\Big)$$

$$= y_I \cdot y_J = (yy^T)_{I,J}$$

since $x_i^2 = x_i\ \forall x_i \in \{0,1\}$

so $M_t(y)$ is PSD

② Consider a constraint $ax \geq \beta$ in $Ax \geq b$

The slack moment matrix for this inequality is a submatrix of $(ax-\beta)\cdot yy^T$ which is again PSD, rank-1

(since $ax - \beta \geq 0$ because x is a feas. soln)

$$\sum_{i=1}^{n} a_i y_{I \cup J \cup i} - \beta y_{I \cup J} = (ax-\beta)\cdot y_I \cdot y_J$$

(b) Let $|I| \leq T$. The determinant of the principle submatrix of $M_t(y)$ indexed by (ϕ, I) is

$$\det \begin{array}{c|cc} & \phi & I \\ \hline \phi & 1_\phi & Y_I \\ I & Y_I & Y_I \end{array} = Y_I(1-Y_I) \geq 0$$

so $0 \leq Y_I \leq 1$

(c) Similar to (b), det of principle submatrix of $M_t(y)$ indexed by (I,J)

$$\det \begin{array}{c|cc} & I & J \\ \hline I & Y_I & Y_J \\ J & Y_J & Y_J \end{array} = Y_J(Y_I - Y_J) \geq 0$$

Using $I \cup J = J$

Since $Y_J \cdot Y_J \geq 0$ (by part (b)),

it must be that $Y_I \geq Y_J$

(d) Look at det of principal submatrix of $M_t(y)$ indexed by $\{I, J\}$

$$\det \begin{array}{c|cc} & I & J \\ \hline I & y_I & y_{I\cup J} \\ J & y_{I\cup J} & y_J \end{array} = y_I y_J - y_{I\cup J}^2 \geq 0$$

so $|y_{I\cup J}| \leq \sqrt{y_I y_J}$

Now we want to show that feasible solutions $y \in SOS_t(K)$ can be viewed as locally consistent distributions

Theorem (See Lauria lecture notes Thm 8 or Rothvoss lecture notes)

Let $y \in SOS_t(K)$. Then $\forall S \subseteq [n], |S| \leq t$ there is a distribution $\mathcal{D}(S)$ over $\{0,1\}^S$ s.t. $\forall I \subseteq S$

$$\Pr_{z \sim \mathcal{D}(S)} \left[\bigwedge_{i \in I} z_i = 1 \right] = y_I$$

* We proved similar statement for SA.

Any feasible solution $y \in SOS_t(K)$ corresponds to a linear functional $E_y : \mathbb{R}[x_1 .. x_n]_t \Rightarrow \mathbb{R}$

satisfying:

$$E_y(1) = 1$$

$E_y(Q) \geq 0$ for any $Q \in SOS$

$E_y(P \cdot Q)$ for any $Q \in SOS$, and $P \in \mathcal{A}$

* Note: this is stronger than SA functionals where Q's were non-neg juntas

Note: a junta is a SOS:

$$[x_1(1-x_2)(x_3)]^2$$

$$= x_1(1-x_2)(x_3)$$

Writing $y \in SOS_t(K)$ as convex combination

For any $y \in SOS_t(K)$ and any variable i where y_i not integral, we can write y as convex combination of $z^{(1)}$ and $z^{(0)}$ s.t.

the i^{th} variable in $z^{(0)}$ is 0

the i^{th} variable in $z^{(1)}$ is 1

and $z^{(0)}$, $z^{(1)}$ are in $SOS_{t-1}(K)$

More precisely:

Lemma Let $y \in SOS_t(K)$, $i \in [n]$ s.t. $0 < y_i < 1$.

Let $z_I^{(1)} = \dfrac{y_{I \cup \{i\}}}{y_i}$ $\quad z_I^{(0)} = \dfrac{y_I - y_{I \cup \{i\}}}{1 - y_i}$

Then $y = y_i \cdot z^{(1)} + (1 - y_i) \cdot z^{(0)}$, with $z^{(0)}, z^{(1)} \in SOS_{t-1}(K)$ and $z_i^{(0)} = 0$, $z_i^{(1)} = 1$

Iterating the previous lemma we get:

Corollary

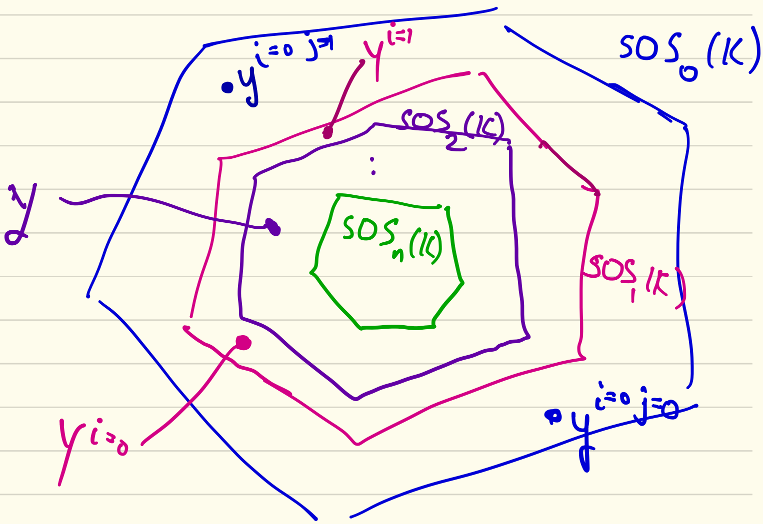
Let $y \in SOS_t(K)$, $S \subseteq [n]$, $|S| \le t$

Then $y \in \text{conv}\{ z \in SOS_{t-|S|}(K) \mid z_i \in \{0,1\} \; \forall i \in S \}$

So after n iterations we have

$$SOS_n^{proj}(K) = \text{conv}(K \cap \{0,1\}^n)$$

Picture:



<u>Proof</u> (Rothvoss gives 2 proofs)

Clearly $z_i^{(1)} = \frac{y_i}{y_i} = 1$, $z_i^{(0)} = \frac{y_i - y_i}{1-y_i} = 0$

and $y_i \cdot z_I^{(1)} + (1-y_i) z_I^{(0)} = y_{J \cup \{i\}} + (y_J - y_{J\cup\{i\}}) = y_I$

so we just have to show $z^{(0)}, z^{(1)} \in SOS_{t-1}(K)$

Since $M_t(y) \succeq 0$, $\exists$ vectors v_I

s.t. $\langle v_I, v_J \rangle = y_{I \cup J}$ $\forall I, J$ $|I|, |J| \le t$

<u>Choose</u> $v_I^{(1)} = \frac{1}{\sqrt{y_i}} v_{I \cup \{i\}}$

Can check that $\langle v_I^{(1)}, v_J^{(1)} \rangle = \frac{1}{y_i} \langle v_{I\cup\{i\}}, v_{J \cup \{i\}} \rangle$

$= \frac{y_{I \cup J \cup \{i\}}}{y_i} = z_{I \cup J}^{(1)}$ for I, J $|I|, |J| \le t$

$\therefore M_{t-1}(z^{(1)}) \succeq 0$.

<u>Similarly</u> <u>choose</u> $v_I^{(0)} = \frac{1}{\sqrt{1-y_i}} \langle v_I - v_{I \cup \{i\}} \rangle$

+ again can check it equals $z_{I \cup J}^{(0)}$

$\therefore M_{t-1}(z^{(0)}) \succeq 0$

Similarly consider ℓ^{th} constraint $ax \geq \beta$

Since $M_t^\ell(y) \succeq 0$ $\exists$ $\tilde{v}_I$ vectors

s.t. $\langle \tilde{v}_I, \tilde{v}_J \rangle = \sum_{j=1}^{n} a_j \, y_{I \cup J \cup \{j\}} - \beta_{I \cup J}$

Let $\tilde{v}_I^{(1)} = \frac{1}{\sqrt{y_i}} v_{I \cup \{i\}}$.

then can ~~check~~ check

$$\langle \tilde{v}_I^{(1)}, \tilde{v}_J^{(1)} \rangle = M_{t-1}^\ell (z^{(1)})_{I,J}$$

and thus $M_{t-1}^\ell(z^{(1)}) \succeq 0$

and similarly can show $M_{t-1}^\ell(z^{(0)}) \succeq 0$

<u>Claim</u> any $y \in SOS_t(K)$

satisfies $E_y(Q) \geq 0$ where $Q = f^2$, $\deg(f) \leq t$

$Y \succeq 0$ so $Y = U^T U$ (cholesky decomp)

Let z be coefficient vector of f

(one entry for each degree $\leq t$ monomial)

Then $z^T U^T U z$

$= (Uz)^T (Uz)$

$= [f(y)]^2$

≥ 0 since $z^T U^T U z$

$= z^T Y z$ & Y is PSD

<u>Similarly can show</u>

$E_y(PQ) \geq 0$ where $Q = SOS$, $P \in K$

SOS_t Proof System

Let $\{P_1 \geq 0 \ldots P_m \geq 0\} = \mathcal{P}$ be poly inequalities in $x_1 \ldots x_n$

An SOS_t derivation of $P_0 \geq 0$ is

$$\sum_i P_i Q_i = P_0$$

where $P_i \in \mathcal{P}_1$ Q_i a SOS

By previous claims

If there is some feasible Y in $SOS_t(\mathcal{P})$, then for any $SOS_t(\mathcal{P})$ derivation $\sum P_i Q_i = P_0$,

$E_Y(\sum P_i Q_i) \geq 0$

Since $\sum P_i Q_i = P_0$, $E_Y(P_0) \geq 0$

Example: <u>Max Cut</u>

<u>LP</u>: $\max \sum_{u \in V} z_{ij}$

$$\text{s.t.} \quad \left.\begin{aligned} x_i - x_j &\le z_{ij} \\ x_j - x_i &\le z_{ij} \\ z_{ij} &\le x_i + x_j \\ z_{ij} &\le 2 - x_i - x_j \quad \forall (i,j) \in E \end{aligned}\right\} K_g$$

It turns out [gw] that degree 3 SOS SDP has the property that $\forall g$, and for all feasible $Y \in SOS_3(K_g)$

$$\text{val}(Y) \le \frac{\text{opt}(g)}{.878}$$

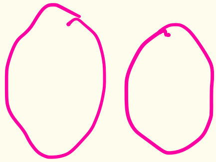
ie. can prove in degree 3 $SOS(K_s)$

$$\sum_i Q_i P_i = P_0$$

where P_0 : $\text{opt}(g) - .878\left(\sum_{u \in V} z_{ij}\right)$

says degree 3 SOS never worse than $\frac{\text{opt}(g)}{.878}$

Example: Max Cut



LP: $$\max \sum_{u \in V} z_{ij}$$

$$\left.\begin{aligned}
\text{s.t.}\quad & x_i - x_j \le z_{ij} \\
& x_j - x_i \le z_{ij} \\
& z_{ij} \le x_i + x_j \\
& z_{ij} \le 2 - x_i - x_j \qquad \forall (i,j) \in E
\end{aligned}\right\} K_g$$

In contrast, the original LP and even linear degree SA has integrality gap ½!

on complete graph all ½ pt has value $\frac{n \cdot (n-1)}{2}$ vs maxcut $\frac{n}{2} \cdot \frac{n}{2}$ so $I_g = \frac{1}{2}$

Thus SOS "finds" the clever GW SDP constraints

SOS as a universal algorithm

- Given an approx problem, take the standard LP and tighten via degree $d \sim 4$ SOS. Then solve SDP
- yields in many cases the best known approx. algorithm (for small $d \sim 4$)
- Takes guess-work out of approx algorithms
- We'll see can also be used to obtain unsupervised learning algs
- Raghavendra: Under UGC, degree 4 SOS ~is optimal for all CSPs

So far we proved SOS <u>soundness</u>

(if $\exists$ a feasible solution, then the corresponding pseudo-distrib. satisfies any degree d SOS proof)

<u>Completeness</u>: If no feasible solns in $SOS_d(K)$ then there is a degree d SOS refutation

requires strong duality which doesn't hold in general for SDPs but does hold on hypercube (as well as most other cases of interest to us)

SDP standard forms

① PRIMAL

$$\min \quad b^T y$$

$$\text{s.t.} \quad \sum_i y_i A_i \succeq C \qquad i = 1..m$$

equivalent

② DUAL

$$\max \quad C \bullet X$$

$$\text{s.t.} \quad A_i \bullet X = b_i \quad \forall i$$

$$X \succeq 0$$

this is the SDP relaxation

①a equivalently

$$\min \ b^T y$$

$$\text{s.t.} \quad \sum_{i=1}^{m} y_i A_i + S = C$$

$$S \succeq 0$$

Thm (Weak Duality)

If y is feasible for PRIMAL SDP and X is feasible for dual SDP then $C \bullet X \geq b^T y$

Pf $C \bullet X = \left(\sum y_i A_i + S \right) \bullet X$

$$= \sum y_i (A_i \bullet X) + (S \bullet X)$$

$$= \sum y_i b + (S \bullet X)$$

since S PSD, we know $S \bullet X \geq 0$ $\forall X$ PSD

& since X PSD $\Rightarrow$ $S \bullet X \geq 0$

$\therefore$ $C \bullet X - b^T y = S \bullet X \geq 0$

so $C \bullet X \geq b^T y$

STRONG DUALITY FOR SDP'S FAILS:

Aaron's somewhat easier ex

Primal: Min x_2 s.t.

$$\begin{bmatrix} x_1 & 1 \\ 1 & x_2 \end{bmatrix} \succeq 0$$

Dual: Max $2y$ s.t.

$$\begin{bmatrix} 0 & y \\ y & 0 \end{bmatrix} \preceq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Duality demonstration:

$$\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & y \\ y & 0 \end{bmatrix} \right) \bullet \begin{bmatrix} x_1 & 1 \\ 1 & x_2 \end{bmatrix} =$$

$$0 - y - y + x_2 = x_2 - 2y \geq 0$$

Dual has opt value 0, but 0
not attainable in primal
(can get arbitrarily close)

Next example gives a gap –
primal has opt = 1, dual = 0

Here we show a gap (so can't get arbitrarily close to primal)

Example [Lovasz]

$$\min\ y_1$$

$$\text{s.t.} \begin{pmatrix} 0 & y_1 & 0 \\ y_1 & y_2 & 0 \\ 0 & 0 & y_1+1 \end{pmatrix} \succeq 0$$

since entry 11 is 0, SDP-ness forces top row/col to all 0, so feasible solns are $(y_1 = 0 \ y_2 \geq 0)$. So primal opt is 0

Dual Form

$$y_1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + y_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \succeq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

gives

$$\max\ -X_{33}$$

$$\text{s.t.}\ X_{12} + X_{21} + X_{33} = 1$$

$$X_{22} = 0$$

$$X \succeq 0$$

forces $X_{12} = X_{21} = 0,\ X_{33} = 1$

so opt is -1

Note if we change PRIMAL so 11 entry is some very small $\varepsilon > 0$, then primal OPT becomes -1. $(x_1 = \varepsilon,\ x_2 = \frac{1}{x_1})$

STRONG DUALITY FOR SDP

STRONG DUALITY holds if the program is at all robust (Slater's condition is satisfied) or <u>either</u> the primal or dual is feasible and bo<u>unde</u>d (any very large point violates the constraints)

★ SDP over hypercube satisfies boundedness

Proof uses Farka's Lemma
extended to SDP's:

$C=0$ case

$\boxed{\text{SDP Farkas Lemma}}$ Let A_i be symmetric matrices. Then $\sum_i y_i A_i \succ 0$ has no soln iff $\exists X \succeq 0$, $X \neq 0$ s.t. $A_i \bullet X = 0 \quad \forall i$

$\boxed{\text{SDP Farka's Lemma, Derivational form}}$

Let A_i, C be symmetric matrices.
Then $\sum_i y_i A_i \succeq C$ has no soln iff
$\exists X \succeq 0, X \neq 0$ s.t. $A_i X = 0 \;\forall i$ and $C \circ X \geq 0$

Pf if $\sum y_i A_i \succ 0$ infeas then $\exists X$
follows by hyperplane separation thm

other direction: then $\left(\sum_i y_i A\right) \circ X > 0$ by lemma from SOS prelims
but $A_i \bullet X = 0$ so ⨳

Strong Duality

Primal:

$$\min \left\{ b^T y \;\middle|\; \sum_i y_i A_i \succeq C \right\}$$

Dual

$$\max \left\{ C \bullet X \;\middle|\; A_i \bullet X = b_i \;\forall i,\; X \succeq 0 \right\}$$

Strong Duality

Assume both PRIMAL + DUAL are feasible. If primal strictly feasible then DUAL opt is achieved + primal/DUAL opt are the same

1. From SDP

make degree d SOS SDP

2. Solns to degree d SOS SDP are "pseudodistributions" with a vector view [give assignments to monomials satisfying fxnl eqns]

3. $\nexists$ a feasible solution to degree d SOS SDP iff $\exists$ degree d SOS refutation

$\Rightarrow$ $\exists$ a feas soln then plug in this functional to evaluate any SOS ref. to ≥ 0

any SOS will evaluate to ≥ 0

$\Leftarrow$ No feas soln, then by SDP duality,
$\exists X \succeq 0 \; X \neq 0$ s.t. $A_i \bullet X = 0 \; \forall i$

$X \succeq 0 \Rightarrow$ there's a SOS

this should convert to an SOS refutation via SDP $\Leftrightarrow$ sum-of-squares connection

Example of SOS Power

Knapsack instance: n items, each weight 1
knapsack capacity 1.9

$$K = \{ x \in \mathbb{R}^n_{\geq 0} \mid \sum_{i=1}^{n} x_i \leq 1.9 \}$$

$OPT = 1$

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consider 2-round soln $y \in SOS_2(K)$

choose y_i s.t. $0 < y_i < 1$

Then $y = y_i \cdot y^{(1)} + (1 - y_i) y^{(0)}$

where $y^{(1)}, y^{(0)} \in SOS_1(K)$

Since y a convex combination,
at least one of $y^{(1)}, y^{(0)}$ has
opt $\geq$ opt of y

<u>Case 1</u> $y^{(1)}$ is better (larger) than $y^{(0)}$

if $y^{(1)}$ not ~~integral~~ completely integral,
$\exists j$ s.t. $y_j > 0$, so

$y^{(1)}$ is a convex comb of $y_j^{1,1}$, $y_j^{1,0}$

but then $y_j^{11} \in SOS_0(K)$ has value 2
which is not feasible.

<u>so</u> $y^{(1)}$ must be completely integral
& have value <u>1</u>.

<u>ow</u> $y^{(0)}$ is bigger soln, then we have a
problem - since we used 1 of our 2
rounds just to make <u>one</u> out of n
variables integral.

this is behind SA $1.9 - O(\frac{1}{n})$ integrality
gap for this instance

<u>Note</u>: this shows that for any $y \in SOS_2(K)$,
and every $i \neq j$ $Y_{i \cup j} = 0$.

ie. Prob $Y_i = Y_j = 1 = 0$ [pseudo-exp.]

Smarter SOS_2 analysis

Since for any $y \in SOS_2(K)$, $Y_{\{i,j\}} = 0 \;\forall\, i \neq j$

we have $v_i \cdot v_j = Y_{\{i,j\}} = 0$

so the vectors $v_1 \ldots v_n$ representing different items are orthogonal.

So we get

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \|v_i\|_2^2 = \left\langle v_\phi, \frac{v_i}{\|v_i\|_2} \right\rangle^2$$

$$\leq \|v_\phi\|_2^2$$

$$= 1$$

since $v_i v_i = v_i v_\phi$

Pythagorus Thm

so this means $SOS_2^{proj}(K) = \text{conv}(K \cap \{0,1\}^n)$

so $y \in SOS_2(K) \Rightarrow$ value of $Y = 1$

More generally

Knapsack instance: n items, each weight 1

$$K = \{ x \in \mathbb{R}^n_{\geq 0} \mid \sum_{i=1}^n x_i \leq k + .9 \}$$

$k \in \mathbb{N}$

(before $k = 1$)

Then $SOS^{proj}_{k+1}(K) = conv(K \cap \{0,1\}^n)$

so any $y \in SOS_{k+1}(K)$ has value k