Review

LP as a Proof System

Sound, complete proof system for linear inequalities over $\mathbb{R}$

LP: $\max \ c^T x$

s.t. $Ax \le b$ } linear constraints (*)

Decision version:

Is there a value of x satisfying (*) ?

Farkas' Lemma

(Soundness + completeness of Decision version)

A set $\{Ax - b \ge 0\}$ of linear inequalities is unsat over $\mathbb{R}$ iff

$\exists y \ge 0$ s.t. $y^T A = 0$, $y^T b = -1$

Sherali Adams (SA)

Some (equivalent) views

- LP tightening
- Pseudo distributions
- As a proof system

SA as LP tightening (degree d)

- Add new variables to represent all degree $\leq d$ terms
- This "lifts" polytope from n dimensions to $n^{O(d)}$ dimensions.
- Projection back to $x_1 \ldots x_n$ preserves all 0/1 solutions (+ removes some fractional ones)

SA degree d tightening

Original LP: [ignore max $c^T x$]

$$\underset{1 \geq 0}{Ax \geq b}, \quad 0 \leq x \leq 1$$

add new variables $y_S \quad \forall S \subseteq [n], \ |S| \leq d$

Impose constraints $\underbrace{\prod_{i \in S} x_i \cdot \prod_{i \in T} (1 - x_i)}_{\substack{S \cap T = \phi \ |S \cup T| \leq d \\ \text{"junta"}}} \cdot (a^T x - b) \geq 0$ $\quad \forall$ rows $a \in A$

New constraints:

lifted SA_d constraints (*)

$$y_\phi = 1$$

$$y_{\{i\}} = x_i$$

$$0 \leq y_S \leq 1$$

$$\sum_{T' \subseteq T} (-1)^{|T'|} \left(\sum_{i=1}^{n} a_i \, y_{S \cup T' \cup \{i\}} - b \, y_{S \cup T'} \right) \geq 0 \quad \forall \text{ rows } a \in A$$

above constraints translated to linear inequalities using new y vars plus multilinearization ($x_i^2 = x_i$)

Degree-d pseudoexpectations for SA

Let $\mathcal{H} = \{Ax - b \geq 0,\ x \geq 0,\ 1 \geq 0\}$

$\mathcal{E}_d(\mathcal{H})$ is a set of linear functionals

$$E : \mathbb{R}[x_1 .. x_n]_d \Rightarrow \mathbb{R} \quad \text{s.t.}$$

$$\forall E \in \mathcal{E}_d(\mathcal{H}) :$$

(1) $E(1) = 1$

(2) $E(Q) \geq 0$ $\forall$ nn-junta Q with degree$(Q) \leq d$

(3) $E(PQ) \geq 0$ for $P \in \mathcal{H}$, and nn-junta Q with degree$(PQ) \leq d$

nn-junta : $\alpha \cdot \prod_{i \in S} x_i \cdot \prod_{j \in T} (1 - x_j)$

$S \cup T = \phi$

non-neg coeff

Each $E \in \mathcal{E}_d(\mathcal{H})$ is called a degree-d "pseudo-distrib" for $\mathcal{H}$

Feasible solutions to degree-d SA polytope are exactly degree-d "pseudo-distributions"

Recall $E \in \mathcal{E}_d(\mathcal{H})$ is a <u>linear</u> functional s.t. :

$$(*)\begin{cases} \bullet\ E[1]=1 \\ \bullet\ E(Q)\geq 0 \text{ for any junta } Q \text{ of degree} \leq d \\ \bullet\ E(PQ)\geq 0 \text{ for } P\in S \text{ and junta } Q \\ \quad\quad \text{with } \deg(PQ)\leq d \end{cases}$$

A feasible soln α gives a value $\alpha(y_s)$ to every variable y_s (every degree $\leq d$ junta) such that all linear constraints are satisfied.

The corresponding pseudodistribution E_α is obtained in the obvious way. For any degree-d polynomial, we make it multilinear, and then use $\alpha(y_s)$ values.

Ex. $f = -x_1x_2x_4 + x_7 - 3x_8x_1$

$$E_\alpha[f] = -\alpha(y_{124}) + \alpha(y_7) - 3y_{18}$$

The degree-d SA linear constraints
(in y_S vars) convert to properties (*)
when we replace y_S by corresponding
monomials

For each feasible point α of degree-d
SA LP, the functional E_α defines
a "pseudodistribution" in the sense that
for every set S of $\leq d$ original
variables, E_α gives a probability distrib.
to all $\{0,1\}$ assignments to S
<u>and</u> $\forall S', S$ s.t. $S' \subseteq S$, and $|S| \leq d$,
the marginal distrib. on S' [w.r.t.
distribution over S) is equal to the
distrib on $\{0,1\}$ asg's to S'

SA as a Refutation System

Let $\mathcal{F}$ be a set of polynomial equalities
(includes $x_i^2 - x_i = 0$)

Let $\mathcal{H}$ be a set of poly inequalities [$Ax - b \geq 0$]
(includes $1 \geq 0$)

A degree-d SA derivation of -1 from $(\mathcal{F}, \mathcal{H})$

[witnessing no feasible solution]

is $(g_1, g_2, .. g_m, P_1, P_2, .. P_s)$ such that:

$$\sum_{i=1}^{m} g_i f_i + \sum_{l=1}^{s} P_l h_l = -1$$

arbitrary poly's

juntas ie. $\prod_{i \in S} x_i \cdot \prod_{j \in T} (1 - x_j)$

$S \cap T = \phi$

$h_l \in \mathcal{H}$

Degree d:

max degree of $\{g_i f_i, P_l h_l\}$ is d

* Note $\mathcal{F}$ includes $x_i^2 - x_i = 0$

<u>SA Lemma</u> Let $\mathcal{H} = \{Ax \geq 0, 1 \geq 0\}$

Then the degree-d SA LP for $\mathcal{H}$, $\mathcal{I} = \{x_i^2 - x_i = 0\}$ no feasible solution iff there is a degree-d SA refutation of $\mathcal{H}, \mathcal{I}$

<u>Pf</u>

(1) degree-d SA LP has a feas. soln $\Rightarrow$ there is no degree-d SA refutation

plug in feas. soln into alleged SA ref. The LHS will evaluate to something ≥ 0

(2) degree-d SA LP has no feas soln $\Rightarrow$ $\exists$ a degree-d SA refutation

By Farkas Lemma, $\exists$ nonneg linear comb of inequalities (*) that is -1

They <u>convert</u> to a degree-d SA refutation — need $x_i^2 - x_i$ to multilinearize

Last class: We did indep. set example.
and saw that degree $d=2$ SA LP
derives all odd-cycle constraints

Unfortunately, there are easy
examples of ind. set where
$d=2$ SA performs very badly:

Ex the complete graph.

Have: $\forall (i,j) \quad x_i + x_j \leq 1$

Cannot derive in low degree

$$\sum_{i=1}^{n} x_i \leq 1$$

SA degree d tightening [Alternative & equivalent way]

Original LP: $\max \; c^T x$

s.t. $Ax \geq b$, $0 \leq x \leq 1$

$1 \geq 0$

add new variables $J_{S,T}$ $\forall S, T \subseteq [n]$, $S \cap T = \phi$

$|S| + |T| \leq d$

$J_{S,T}$ represents the junta $\prod_{i \in S} x_i \prod_{j \in T} (1 - x_j)$

New constraints:

$$J_{\phi,\phi} = 1$$

$$J_{\{i\},\phi} = x_i \qquad J_{\phi,\{i\}} = (1 - x_i)$$

$$Y_{S,T} \left(\sum a_i x_i \right) \geq Y_{S,T} \cdot b$$

converted to a linear constraint using

$$Y_{S \cup i, T} = x_i \cdot Y_{S,T}$$

$$Y_{S, T \cup i} = (1 - x_i) \cdot Y_{S,T}$$

SA as a Derivation System

Let $\mathcal{J}$ be a set of polynomial equalities
(includes $x_i^2 - x_i = 0$)

Let $\mathcal{H}$ be a set of poly inequalities $[Ax - b \geq 0]$
(includes $1 \geq 0$)

A degree-d SA derivation of from $(\mathcal{J}, \mathcal{H})$
[witnessing $\{\mathcal{J} = 0, \mathcal{H} \geq 0\} \Rightarrow P \geq c_0$]

is $(g_1, g_2, \ldots g_m, P_1, P_2, \ldots P_\ell)$ such that:

$$\sum_{i=1}^{m} g_i f_i + \sum_{\ell=1}^{s} p_\ell h_\ell = P - c_0$$

arbitrary poly's

juntas ie. $\prod_{i \in S} x_i \cdot \prod_{j \in T} (1 - x_j)$

$h_\ell \in \mathcal{H}$

$S \cap T = \phi$

Degree d:
max degree of $\{g_i f_i, P_\ell h_\ell\}$ is d

Lemma Let $\mathcal{H} = \{Ax \geq 0, P \geq 0\}$

Then $\min \{ E(P) \mid E \in \mathcal{E}_d(\mathcal{H}) \}$

equals

$\max \{ c_0 \mid$ There is a degree-d SA derivation of $P \geq c_0$ from $\mathcal{H}$, $\mathcal{J} = \{x_i^2 - x_i = 0\}$ $\}$

In words, recall that each $E \in \mathcal{E}_d(\mathcal{H})$ corresponds to a feasible solution of degree-d SA LP. The solution has some value $E(P)$.

The lemma says that there is a feasible solution of value c_0 iff SA can derive in degree d that $\mathcal{H}, \mathcal{J}$ implies $P \geq c_0$

So if there is no feasible solution of value c_0, then $\mathcal{H} \cup \{P \geq c_0\}$, $\mathcal{J} = \{x_i^2 - x_i\}$ has a degree d SA refutation.

Pf of this Lemma (derivational version of SA Lemma) is same/similar

Automatizability of SA

degree d SA LP has $n^{O(d)}$ linear constraints, so solvable in time $\mathrm{poly}(d)$

Equivalently degree-d SA refutations can be found in time $n^{O(d)}$

[just make a system of n^d linear equations + solve]

What about size automatizability?

UB: size $s \Rightarrow 2^{\sqrt{n \log s}}$

for $s = \mathrm{poly}(n)$ this is expl time!

Q: Prove or disprove: SA is poly automatizable (wrt size)

Uses of SA

1. Degree lower-bounded arborescences [Bateni, Charikar, Guruswami STOC '09]

2. Scheduling - [n unit size jobs, m machines, m constant, jobs have precedence constraints]

polylog(n) round SA gets $(1-\varepsilon)$ approx alg [Garg '17]

earlier $(\log n)^{\log\log n}$ [Levey, Rothvoss '16]

Max-Min allocation problem

n players, m items, utility additive p_{ij} $i \in [n]$ $j \in [m]$

Find allocation to maximize the min utility of the players

Technical core: Max Min Degree Arborescence

given directed G find subgraph H s.t. indegree ≤ 1, contains all sources of G, sinks of H are sinks of G, maximize min outdegree of vertices in H.

Other Proof Systems Related to Sherali-Adams:

1. Dynamic SA : LS_d

2. Nullstellensatz :
   like SA but only equalities

3. Polynomial Calculus
   like dynamic SA (LS_d)
   but only inequalities

4. Cutting Planes

LS_d Lovasz-Schrijver "Lift and Project" SA

Idea is to repeatedly apply SA_d algorithm (projecting down to original vars each time)

A proof is a sequence of lines

$P_1 \geq 0$
$P_2 \geq 0$
$\vdots$

where each $P_i \geq 0$ is
- an axiom
- a given inequality
- derived by an inference

Complexity Measures

Size : bit size of all coeff's (explicit sums of monomials)

Monomial-size

Length : # of inequalities

Degree : max degree

Height : longest path in proof dag

Notation $P_1 \geq 0 \ldots P_m \geq 0 \vdash^{H}_{D} P \geq 0$

SA / LS_d inferences / axioms

given $P_1 \geq 0 \ldots P_m \geq 0$ initial inequalities

axioms $x_i^2 - x_i \geq 0$

Inference Let Q_i be nnjuntas

[Non-neg linear combinations of juntas]

with

$$Q_0 + P_1 Q_1 + P_2 Q_2 + \ldots + P_m Q_m = P$$

can derive $P \geq 0$ ← linear ineq

Degree d SA: one-shot pf system so derivation dag is a star:

$P_1 \geq 0 \ \ldots \ P_m \geq 0 \quad x_i^2 - x_i = 0 \ \ldots$

P

LS_d : Can apply inference rule repeatedly so pf dag more complicated.

Can prove stuff that SA can prove in lower degree (due to cancellations)

Called "Lift and Project"

h = height = ~~to~~ # steps of lift/project

Now finding a degree d, ht h proof (or optimizing over H ounds of degree d SA Lift/project) can be done in time $n^{O(hd)}$

1. Apply SA_d to derive some new linear inequalities
2. Repeat on extended set of linear inequalities

Nullsatz (like SA but start with $\ell_i \geq 0$ $\ell_i \leq 0$ and don't have $1 \geq 0$)

Now we start with only polynomial equalities $[P_i \geq 0$ and $P_i \leq 0]$ plus $x_i^2 - x_i = 0$.

Inference rule simplifies to:

Start with $P_1 = 0, \ldots, P_m = 0, \; x_i^2 - x_i = 0$

Let

$$\sum P_i Q_i + \sum (x_i^2 - x_i) R_i = P$$

where now Q_i, R_i are arbitrary polynomials

Then can derive $P = 0$

Nullsatz for UNSAT CNF

2 possible conversions $x_1 \vee \bar{x}_2 \vee x_3$

① $x_1 + (1-x_2) + x_3 \geq 1$

②. $(1-x_1)(x_2)(x_3) = 0$ using $x_i^2 - x_i = 0$

Need to use ② for Nullsatz

Using this conversion, we have:

Theorem If $f = C_1 \wedge \ldots \wedge C_m$ (unsat kCNF) has a degree d Nullsatz refutation, then it has a degree d SA refutation.

Theorem There are unsat kCNFs that have small degree SA refutations but no small degree Nullsatz refutations [dont have non-neg juntas ≥ 0]

PC : polynomial Calculus

Same as Nullsatz but now proofs are *dynamic*

Degree d Nullsatz proofs can be found in time $n^{O(d)}$: Linear system of $n^{O(d)}$ eqns + vars

Degree d PC proofs can also be found in time $n^{O(d)}$ – proof is degree-d truncated multilinear version of grobner basis alg

Notes

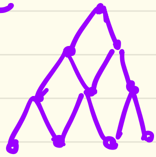
① Nullsatz, PC over arbitrary fields not just $\mathbb{R}$

② Soundness/completeness for arb. poly eq's over $\mathbb{F}$ follows by Hilbert's Nullstellensatz

Notes cont'd

③ Example where SA is stronger than Nullsatz (wrt degree)

graph pebbling. degree-d SA can p-simulate width-d Resolution.



④ Example where PC is stronger than Nullsatz (wrt degree):

Induction (pebbling on a line)

$x_1 \quad x_1 \to x_2 \quad x_2 \to x_3 \quad \ldots \quad x_{n-1} \to x_n \quad \neg x_n$

$x_n = 0 \quad 1 - x_1 = 0 \quad x_1(1-x_2) = 0 \quad x_2(1-x_3) = 0 \ldots \quad x_{n-1}(1-x_n) = 0$

Nullsatz : $\Theta(\log n)$ degree

PC : $\Theta(1)$ degree

Cutting Planes

again a proof system for tightening an LP.

Rules:

1. Nonneg linear combinations

2. Division with rounding

$$\sum a_i \cdot x_i \geq a_0$$, where $a_1 \ldots a_n$ divisible by k

$$\Rightarrow \sum \frac{a_i}{k} x_i \geq \left\lceil \frac{a_0}{k} \right\rceil$$

* Not known to be automatizable.