LP as a Proof System

Sound, complete proof system for linear inequalities over $\mathbb{R}$

LP: $\max \; c^T x$

s.t. $\left.Ax \le b\right\}$ linear constraints $(*)$

Decision version:

Is there a value of x satisfying $(*)$?

Farkas' Lemma

(Soundness + completeness of Decision version)

A set $\{Ax - b \ge 0\}$ of linear inequalities is unsat over $\mathbb{R}$ iff

$\exists y \ge 0$ s.t. $y^T A = 0$, $y^T b = -1$

Duality (Implicational completeness)

Consider any nonneg y^T s.t. $y^T A = c^T$

Then for any feasible solution x to $\{Ax \le b\}$ we have

$$c^T x \le y^T A x \le y^T b$$

so $y \ge 0$ witnesses the upper bound $b^T y$.

How tight is such an upper bound?

Duality

(P) Primal

$$\max \ c^T x$$
$$\text{s.t.} \quad Ax \leq b$$
$$x \geq 0$$

(D) Dual:

$$\min \ b^T y$$
$$\text{s.t.} \quad A^T y \geq c$$
$$y \geq 0$$

Duality Theorem (Implied by Farkas' Lemma)

Exactly one of the following holds

(i) Neither (P) nor (D) has a feasible solution

(ii) (P) has solns with arbitrarily large values & (D) is unsat

(iii) (P) unsat & (D) has arb. large solutions

(iv) Both (P) & (D) have optimal solns x^* & y^*. Then $c^T x^* = b^T y^*$

[So there is a solution to dual that witnesses tight bound]

So an LP "refutation" of $\{Ax \le b,\ x \ge 0\}$ is a nonneg linear combination of these inequalities that equals $\bot$

An LP "derivation" of $\{Ax \le b,\ x \ge 0\} \Longrightarrow c^T x \le c_0$ is a nonneg y^* s.t.

$$(y^*)^T b = c_0$$

$$\left[\text{since } c^T x \le (y^*)^T A x \le (y^*)^T b = c_0\right.$$

Soundness / Completeness

Farkas' Lemma / Duality Thm

Polynomial Automatizability of LP

Satisfiability of
linear inequalities:

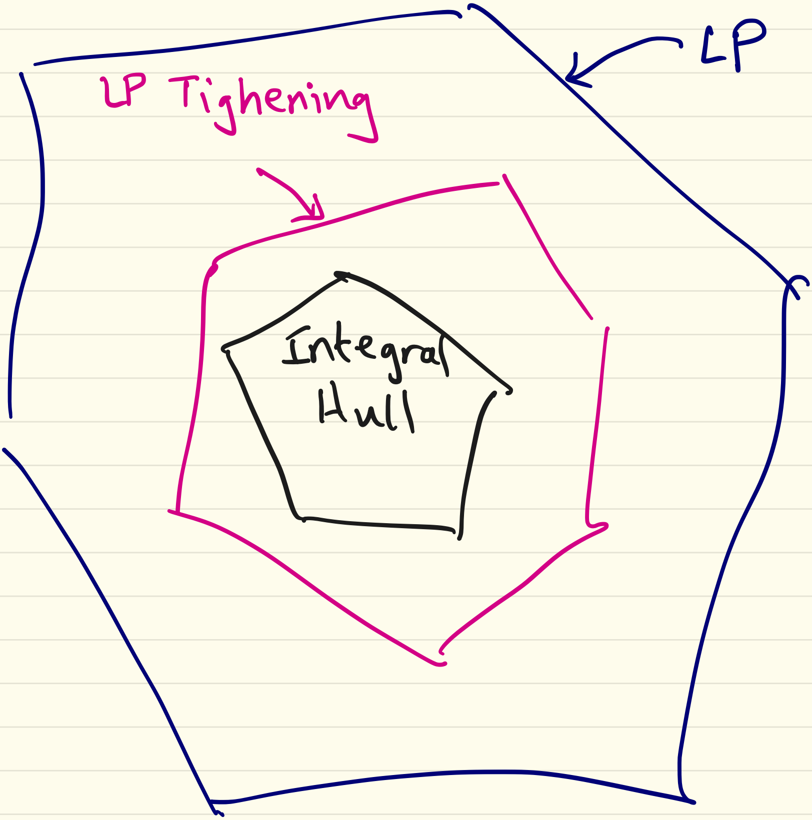
- in NP
- By Farkas' Lemma, in coNP

$\therefore$ in $NP \cap coNP$

Ellipsoid Algorithm [Khachiyan '79]

$LP \in P$

LP as relaxation of Integer Program



SA : A proof system for integer programming obtained by successive LP tightenings

CP : Another pf system for LP

Sherali Adams (SA)

Some (equivalent) views

- LP tightening
- Pseudo distributions
- As a proof system

SA as LP tightening (degree d)

- Add new variables to represent all degree $\leq d$ terms
- This "lifts" polytope from n dimensions to $n^{O(d)}$ dimensions.
- Projection back to $x_1 .. x_n$ preserves all 0/1 solutions (+ removes some fractional ones)

SA degree d tightening

Original LP: [ignore max $c^T x$]

$$\underset{1\geq 0}{Ax \geq b}, \quad 0 \leq x \leq 1$$

add new variables $y_S \quad \forall S \subseteq [n], \ |S| \leq d$

Impose constraints $\underbrace{\prod_{i\in S} x_i \cdot \prod_{i \in T} (1-x_i)}_{\substack{S\cap T = \phi \ |S\cup T| \leq d \\ \text{"junta"}}} \cdot (a^T x - b) \geq 0 \quad \forall \text{rows } a \in A$

New constraints:

lifted SA_d constraints (*)

$$y_\phi = 1$$

$$y_{\{i\}} = x_i$$

$$0 \leq y_S \leq 1$$

$$\sum_{T' \subseteq T} (-1)^{|T'|} \left(\sum_{i=1}^{n} a_i y_{S\cup T' \cup \{i\}} - b y_{S \cup T'} \right) \geq 0 \quad \forall \text{rows } a \in A$$

above constraints translated to linear inequalities using new y vars plus multilinearization ($x_i^2 = x_i$)

since we have $1 \geq 0$ as an initial constraint this gives $\forall S,T$ $|S \cup T| \leq d$, $S \cap T = \phi$

$$\prod_{i \in S} x_i \prod_{j \in T} (1 - x_j) \geq 0$$

translates to

$$\sum_{T' \subseteq T} (-1)^{|T'|} \left(Y_{S \cup T'} \right) \geq 0$$

Example: $S = 1,2$ $T = 3,4,5$ $x_1 x_2 (1-x_3)(1-x_4)(1-x_5)$

Multiplying out:

$$x_1 x_2 - x_1 x_2 x_3 - x_1 x_2 x_4 - x_1 x_2 x_5 + x_1 x_2 x_3 x_4 + x_1 x_2 x_3 x_5$$
$$+ x_1 x_2 x_4 x_5 - x_1 x_2 x_3 x_4 x_5$$

$$Y_{12} - Y_{123} - Y_{124} - Y_{125} + Y_{1234} + Y_{1235} + Y_{1245} - Y_{12345} \geq 0$$

Degree-d pseudoexpectations for SA

Let $\mathcal{H} = \{Ax - b \geq 0,\ x \geq 0,\ 1 \geq 0\}$

$\mathcal{E}_d(\mathcal{H})$ is a set of linear functionals

$$E : \mathbb{R}[x_1 \ldots x_n]_d \Rightarrow \mathbb{R} \quad \text{s.t.}$$

$\forall E \in \mathcal{E}_d(\mathcal{H})$:

(1) $E(1) = 1$

(2) $E(Q) \geq 0$ $\quad \forall$ nn-junta Q with degree$(Q) \leq d$

(3) $E(PQ) \geq 0$ for $P \in \mathcal{H}$, and nn-junta Q with degree$(PQ) \leq d$

nn-junta : $\alpha \cdot \prod_{i \in S} x_i \cdot \prod_{j \in T} (1 - x_j)$

$S \cup T = \phi$

non-neg coeff

Each $E \in \mathcal{E}_d(\mathcal{H})$ is called a degree-d "pseudo-distrib" for $\mathcal{H}$

Feasible Solutions to degree-d SA polytope are exactly degree-d "pseudo-distributions"

Recall $E \in \mathcal{E}_d(\mathcal{H})$ is a linear functional s.t.:

$$(*) \begin{cases} \bullet\ E[1] = 1 \\ \bullet\ E(Q) \geq 0 \text{ for any junta } Q \text{ of degree} \leq d \\ \bullet\ E(PQ) \geq 0 \text{ for } P \in S \text{ and junta } Q \\ \qquad \text{with } \deg(PQ) \leq d \end{cases}$$

A feasible soln α gives a value $\alpha(y_S)$ to every variable Y_S (every degree $\leq d$ junta) such that all linear constraints are satisfied.

The corresponding pseudodistribution E_α is obtained in the obvious way. For any degree-d polynomial, we make it multilinear, and then use $\alpha(Y_S)$ values.

Ex. $f = -x_1 x_2 x_4 + x_7 - 3 x_8 x_1$

$$E_\alpha[f] = -\alpha(y_{124}) + \alpha(y_7) - 3\, y_{18}$$

The degree-d SA linear constraints (in y_S vars) convert to properties (*) when we replace y_S by corresponding monomials

For each feasible point α of degree-d SA LP, the functional E_α defines a "pseudodistribution" in the sense that for every set S of $\leq d$ original variables, E_α gives a probability distrib. to all $\{0,1\}$ assignments to S

<u>and</u> $\forall S', S$ s.t. $S' \subseteq S$, and $|S| \leq d$, the marginal distrib. on S' [w.r.t. distribution over S) is equal to the distrib on $\{0,1\}$ ass's to S'

Ex Let $S = \{x_1\ x_2\}$. $Pr(x_1 = 1 \wedge x_2 = 1) = \alpha(y_{12})$

$Pr(x_1 = 1\ x_2 = 0)$: $x_1(1 - x_2) = x_1 - x_1 x_2$

So $pr(x_1 = 1\ x_2 = 0) = \alpha(y_1) - \alpha(y_{12})$

this is positive since we have

$x_1(1 - x_2) \geq 0 \Rightarrow x_1 - x_1 x_2 \geq 0 \Rightarrow \alpha(y_1) - \alpha(y_{12}) \geq 0$

Similarly $Pr(x_1 = 0\ x_2 = 1)$: $(1 - x_1)(x_2) = x_2 - x_1 x_2$

So $Pr(x_1 = 0\ x_2 = 1) = \alpha(y_2) - \alpha(y_{12}) \geq 0$

and $Pr(x_1 = 0\ x_2 = 0)$: $(1 - x_1)(1 - x_2) = 1 - x_1 - x_2 - x_1 x_2 \geq 0$

so $Pr(x_1 = 0\ x_2 = 0) = \alpha(1) - \alpha(x_1) - \alpha(x_2) - \alpha(y_{12}) \geq 0$

<u>Consistency</u>: $Pr(x_1 = 1) = \underbrace{Pr(x_1 = 1 \wedge x_2 = 1)}_{y_{12}} + \underbrace{Pr(x_1 = 1 \wedge x_2 = 0)}_{y_1 - y_{12}}$

y_1

$= y_1$

So from any assignment to y_S vars satisfying the degree-d SA constraints we can define a pseudodistrib over all subsets S of variables, $|S| \leq d$ s.t. they are consistent

SA as a Refutation System

Let $\mathcal{I}$ be a set of polynomial equalities
(includes $x_i^2 - x_i = 0$)

Let $\mathcal{H}$ be a set of poly inequalities $[Ax - b \geq 0]$
(includes $1 \geq 0$)

A degree-d SA derivation of -1 from $(\mathcal{I}, \mathcal{H})$

[witnessing no feasible solution]

is $(g_1, g_2, \ldots g_m, p_1, p_2, \ldots p_s)$ such that:

$$\sum_{i=1}^{m} g_i f_i + \sum_{\ell=1}^{s} p_\ell h_\ell = -1$$

g_i: arbitrary poly's

$h_\ell \in \mathcal{H}$

p_ℓ: juntas ie. $\prod_{i \in S} x_i \cdot \prod_{j \in T} (1 - x_j)$

$S \cap T = \phi$

Degree d:

max degree of $\{g_i f_i, p_\ell h_\ell\}$ is d

* Note $\mathcal{I}$ includes $x_i^2 - x_i = 0$

SA as a Derivation System

Let $\mathcal{J}$ be a set of polynomial equalities
(includes $x_i^2 - x_i = 0$)

Let $\mathcal{H}$ be a set of poly inequalities $[Ax - b \geq 0]$
(includes $1 \geq 0$)

A degree-d SA derivation of f from $(\mathcal{J}, \mathcal{H})$
[witnessing $\{\mathcal{J} = 0, \mathcal{H} \geq 0\} \Rightarrow f$]

is $(g_1, g_2, \ldots g_m, p_1, p_2, \ldots p_s)$ such that:

$$\sum_{i=1}^{m} g_i f_i + \sum_{\ell=1}^{s} p_\ell h_\ell = f$$

g_i: arbitrary poly's

p_ℓ: juntas ie. $\prod_{i \in S} x_i \cdot \prod_{j \in T} (1 - x_j)$, $S \cap T = \phi$

$h_\ell \in \mathcal{H}$

Degree d:
max degree of $\{g_i f_i, p_\ell h_\ell\}$ is d

Lemma Let $\{Ax - b \geq 0, 1 \geq 0, x \geq 0\} = \mathcal{H}$

Then the degree-d SA LP has NO feasible solution iff there is a degree-d SA refutation of $\mathcal{H}$

Pf

(1) degree-d SA LP has a feas. soln $\Rightarrow$ there is NO degree-d SA refutation

plug in feas. soln into alleged SA ref. The LHS will evaluate to something ≥ 0

(2) degree-d SA LP has no feas soln $\Rightarrow$ $\exists$ a degree-d SA refutation

By Farkas Lemma, $\exists$ nonneg linear comb of inequalities (*) that is -1

They convert to a degree-d SA refutation — need $x_i^2 - x_i$ to multilinearize

Example – INDependent Set

$G = (V, E)$
$|V| = n$

$$\max \sum x_i$$
$$\text{s.t. } x_i + x_j \leq 1 \quad \forall (i,j) \in E$$
$$0 \leq x_i \leq 1$$

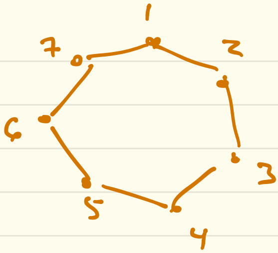
Degree-2 SA LP:

Each original constraint $\Rightarrow$ 2n new constraints

$$x_k [0 \leq x_i \leq 1] \Rightarrow 0 \leq y_{i,k} \leq x_k$$
$$(1-x_k)[0 \leq x_i \leq 1] \Rightarrow 0 \leq x_i - y_{ik} \leq 1 - x_k$$
$$x_k [x_i + x_j \leq 1] \Rightarrow y_{ik} + y_{jk} \leq x_k$$
$$(1-x_k)[x_i + x_j \leq 1] \Rightarrow x_i - y_{ik} + x_j - y_{jk} \leq 1 - x_k$$

$$[y_{ii} = x_i, \ y_{i,j} = y_{ji}]$$

Odd Cycle Example



original constraints:

$$(*)\left\{\begin{array}{l}\left.\begin{array}{l} x_1+x_2\le 1\\ x_2+x_3\le 1\\ x_3+x_4\le 1\\ x_4+x_5\le 1\\ x_5+x_6\le 1\\ x_6+x_7\le 1\end{array}\right\}(**)\\ x_7+x_1\le 1\end{array}\right.$$

(*): No feasible solution

We will consider the $d=2$ SA LP.

It's opt will be 3 $\left(= \dfrac{|cycle| - 1}{2} \right)$

instead of 3.5 (opt for original LP)

$k=2$

Example - Independent Set

SA

$$\begin{cases} (1) & 0 \le y_{ik} \le x_k & \forall k \\ (2) & 0 \le x_i - y_{ik} \le 1 - x_k & \forall k \\ (3) & y_{ik} + y_{jk} \le x_k & \forall \text{ edges } (i,j), \forall k \\ (4) & x_i - y_{ik} + x_j - y_{jk} \le 1 - x_k & \forall \text{ edges } (i,j), \forall k \end{cases}$$

also have

Derive:

(a) $y_{12} \le 0$

(b) $x_2 - y_{12} + x_3 - y_{13} \le 1 - x_1$

(c) $y_{1,3} + y_{1,4} \le x_1$

(d) $x_4 - y_{14} + x_5 - y_{15} \le 1 - x_1$

(e) $y_{1,5} + y_{1,6} \le x_1$

(f) $x_6 - y_{16} + x_7 - y_{17} \le 1 - x_1$

(g) $y_{17} \le 0$

By:

(3) $k=1$ $(i,j)=(1,2)$

(4) with $k=1$ $(i,j)=(2,3)$

(3) with $k=1$ $(i,j)=(3,4)$

(4) with $k=1$, $(i,j)=(4,5)$

(3) with $k=1$, $(i,j)=(5,6)$

(4) with $k=1$, $(i,j)=(6,7)$

(3) $k=1$ $(i,j)=(1,7)$

sum up: $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \le 3$

Example - Independent Set

Now Let's add $\sum x_i \geq 3.5$ to LP

By Farka's Lemma, $\exists$ nonneg linear comb that equals 1:

$$
\begin{array}{lll}
(a) & y_{12} \leq 0 & x_1(x_1+x_2 \leq 1) \\
(b) & x_2 - y_{12} + x_3 - y_{13} \leq 1-x_1 & (1-x_1)(x_2+x_3 \leq 1) \\
(c) & y_{13} + y_{14} \leq x_1 & x_1(x_3+x_4 \leq 1) \\
(d) & x_4 - y_{14} + x_5 - y_{15} \leq 1-x_1 & (1-x_1)(x_4+x_5 \leq 1) \\
(e) & y_{15} + y_{16} \leq x_1 & x_1(x_5+x_6 \leq 1) \\
(f) & x_6 - y_{16} + x_7 - y_{17} \leq 1-x_1 & (1-x_1)(x_6+x_7 \leq 1) \\
(g) & y_{17} \leq 0 & x_1(x_1+x_7 \leq 1) \\
(h) & x_1+x_2+x_3+x_4+x_5+x_6+x_7 \geq 3.5 &
\end{array}
$$

$$
\left.
\begin{array}{l}
x_1(x_1+x_2-1) + (1-x_1)(x_2+x_3-1) + x_1(x_3+x_4-1) + \\
1-x_1(x_4+x_5-1) + x_1(x_5+x_6-1) + (1-x_1)(x_6+x_7-1) \\
+ x_1(x_1+x_7-1) \quad + \quad 2(x^2-x_1) \\
+ (4-x_1-x_2-x_3-x_4-x_5-x_6-x_7) \\
\qquad = 1 \leq 0
\end{array}
\right\}
\begin{array}{l}
\text{SA} \\
\text{refutation} \\
(d=2)
\end{array}
$$

Notes

Thus degree $d=2$ SA LP derives all odd cycle constraints.

Unfortunately, there are easy examples of ind. set where $d=2$ SA performs very badly:

Ex the complete graph.

Have: $\forall (i,j)$ $\quad x_i + x_j \leq 1$

Cannot derive in low degree

$$\sum_{i=1}^{n} x_i \leq 1$$

SA degree d tightening [Alternative & equivalent way]

Original LP: $\max \quad c^T x$

s.t. $\quad Ax \geq b \quad , \quad 0 \leq x \leq 1$

$1 \geq 0$

add new variables $J_{S,T} \quad \forall S,T \subseteq [n], \quad S \cap T = \phi$

$|S| + |T| \leq d$

$J_{S,T}$ represents the junta $\prod_{i \in S} x_i \prod_{j \in T} (1-x_j)$

New constraints:

$$J_{\phi,\phi} = 1$$

$$J_{\{i\},\phi} = x_i \qquad J_{\phi,\{i\}} = (1-x_i)$$

$$Y_{S,T}\left(\sum a_i x_i\right) \geq Y_{S,T} \cdot b$$

converted to a linear constraint using

$$Y_{S \cup i, T} = x_i \cdot Y_{S,T}$$

$$Y_{S, T \cup i} = (1-x_i) \cdot Y_{S,T}$$

Automatizability of SA

degree d SA LP has $n^{O(d)}$ linear constraints, so solvable in time poly(d)

Equivalently degree-d SA refutations can be found in time $n^{O(d)}$

[just make a system of n^d linear equations + solve]

What about size automatizability?

UB: size $s \Rightarrow 2^{\sqrt{n \log s}}$

for $s = \text{poly}(n)$ this is expl time!

Q: Prove or disprove: SA is poly automatizable (wrt size)