MAIN LEMMA Let $X_1 \ldots X_n \in \mathbb{R}$, $S_1 \ldots S_K$ a partition of $[n]$, $|S_i| = \frac{n}{K}$, s.t. $\forall i \in [K]$:

$$\mathbb{E}_{j \sim S_i} |X_j - \mu_i|^t = 2 \cdot t^{t/2}$$

Let w be a solution to degree $O(t)$ SDP for A.
(so v is a degree $O(t)$ pseudodistrib)

Then w satisfies

$$\sum_{i \in [K]} \left(\frac{|T \cap S_i|}{N} \right)^2 \geq 1 - \frac{2^{O(t)} t^{t/2} K^2}{\Delta^t}$$

where $|T \cap S_i| = \sum_{j \in S_i} w_j$

Lemma 1
[Fact 2]

Assume S satisfies $\sum_{i\in S} \frac{(\mu_S - X_i)^t}{N} \leq 2t^{t/2}$

$$\mathcal{A}: \begin{cases} w_i^2 - w_i = 0 & \forall i \in [n] \quad \leftarrow \text{vars in } \{0,1\} \\ \sum w_i = N & \leftarrow |T| = N \\ \frac{1}{N} \sum_{i\in[n]} w_i (X_i - \mu_T)^t \leq 2t^{t/2} & \leftarrow \text{says } T \subseteq X_1 \ldots X_n \text{ satisfies } t^{th} \text{ moment bound} \end{cases}$$

Then there is a degree $O(t)$ SoS derivation of $(*)$ from $\mathcal{A}$

$$(*): \underbrace{\left(\frac{\sum_{i\in S} w_i}{N}\right)^t}_{\left(\frac{|S\cap T|}{N}\right)^t = \alpha^t} \cdot (\mu_T - \mu_S)^t \leq 2^{O(t)}\, t^{t/2} \underbrace{\left(\frac{\sum_{i\in S} w_i}{N}\right)^{t-1}}_{\alpha^{t-1}}$$

<u>Interpretation</u> a solution to $\mathcal{A}$ satisfying $(*)$ also satisfies

$$\alpha\, |\mu_T - \mu_S| \leq c\sqrt{t}\, \alpha^{t-1/t}$$

$$\approx \quad |\mu_T - \mu_S| \leq c\sqrt{t}\, \alpha^{-1/t}$$

So intuitively lemma 1 is saying:

Let $S, S' \subseteq \mathbb{R}$ have $|S| = |S'| = N$.

Let X, X' be uniform sample from S, S'

Let $\mu = \mathbb{E}X$, $\mu' = \mathbb{E}X'$.

Suppose X, X' satisfy t^{th} moment bound:

$$\mathbb{E}|X-\mu|^t \leq 2 \cdot t^{t/2}$$
$$\mathbb{E}|X'-\mu'|^t \leq 2t^{t/2}$$

Then $$|\mu-\mu'| \leq 4\sqrt{t}\underbrace{\left(\frac{|S \cap S'|}{N}\right)}_{\alpha}^{-1/t}$$

$\Uparrow$ Says if we pick 2 groups of samples both of which satisfies t^{th} moment property, and that overlap a lot, then means are close

Prelims!

Holder's Inequality

Let $a_1 \ldots a_n$, $b_1 \ldots b_n$ nonneg,

p, q positive s.t. $\frac{1}{p} + \frac{1}{q} = 1$.

Then $\displaystyle\sum_{i=1}^{n} a_i \cdot b_i = \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}$

Triangle Inequality

$\|\tilde{v}\|_p = \left(\sum v_i^p\right)^{1/p}$

$$\|\tilde{v} + \tilde{w}\|_p \leq \|\tilde{v}\|_p + \|\tilde{w}\|_p$$

Cauchy Swartz is a special case where $p = q = 2$

<u>Claim</u> (pseudo-exp cauchy swartz)

Let $\tilde{E}$ be a degree d pseudoexp on variables $x_1 .. x_n$. Let $p(x)$, $q(x)$ be degree $\leq d/2$ polys. Then

$$\tilde{E}\, p(x) q(x) \leq \left(\tilde{E}\, p(x)^2\right)^{1/2} \left(\tilde{E}\, q(x)^2\right)^{1/2}$$

If $\tilde{E}$ has degree dt, t a power of 2

$$\tilde{E}\, p(x) \leq \left(\tilde{E}\, p(x)^t\right)^{1/t}$$

<u>Claim</u> (pseudo-exp Holders)

Let p be a degree ℓ sos poly, $\tilde{E}$ a degree $O(t\ell)$ pseudoexp. Then

$$\tilde{E}\, p(x)^{t-2} \leq \left(\tilde{E}\, p(x)^t\right)^{t-2/t}$$

Notation Let $\mathcal{P}$ be a set of poly inequalities, q a poly ineq.

$$\mathcal{P} \vdash_d q$$

there is a degree d SOS deriv of q from $\mathcal{P}$

Facts (low degree SOS proofs)

(0) $x^2 = x \vdash_2 0 \leq x \leq 1$

(1) SOS triangle inequality. Let t be a power of 2.

$$\vdash_t (x+y)^t \leq 2^t (a^t + b^t)$$

(2) SOS Holder's inequality

[Unfortunately there are many versions of this]

Let $\mathcal{W} = \{w_i^2 - w_i = 0,\ i \in [n]\}$

Let $p_1(w) \ldots p_n(w)$ be degree $\leq \ell$ polys, & let t be a power of 2.

$$\mathcal{W} \vdash_{O(t\ell)} \Big(\sum_{i\in[n]} w_i \cdot p_i(w)\Big)^t \leq \Big(\sum_{i\in[n]} w_i\Big)^{t-1} \cdot \sum_{i\in[n]} p_i(w)^t$$

$$\mathcal{W} \vdash_{O(t\ell)} \Big(\qquad\qquad\Big) \leq \Big(\qquad\qquad\Big)^{t-1} \cdot \sum_{i\in[n]} w_i \cdot p_i(w)^t$$

Proof of Lemma 1

We want to show:

$$A \vdash_{O(t)} \left(\frac{\sum_{i\in S} w_i}{N}\right)^t (\mu_T - \mu_s)^t \leq 2^{O(t)} t^{\frac{t}{2}} \left(\frac{\sum_{i\in S} w_i}{N}\right)^{t-1}$$

① $$\left(\sum_{i\in S} w_i\right)^t (\mu_T - \mu_s)^t = \left(\underbrace{\sum_{i\in S} w_i \cdot [(\mu_T - X_i) - (\mu_s - X_i)]}_{(*)}\right)^t$$

② By SOS Holder's ineq (part 2)

$$A \vdash_{O(t)} (*) \leq \left(\sum_{i\in S} w_i\right)^{t-1} \cdot \sum_{i\in S} w_i \left[(\mu_T - X_i) - (\mu_s - X_i)\right]^t$$

③ using $w_i^2 - w_i = 0$

$$A \vdash_{O(t)} (*) \leq \left(\sum_{i\in S} w_i^2\right)^{t-1} \cdot \sum_{i\in S} w_i^2 \left[(\mu_T - X_i) - (\mu_s - X_i)\right]^t$$

$(a+b)^t \leq 2^t (a^t + b^t)$

④ By SOS triangle ineq,

$$\vdash_{O(t)} 2^t (a^t + b^t) - (a+b)^t$$

apply with $a = (\mu_T - X_i)$ $b = -(\mu_s - X_i)$

$$\vdash_{O(t)} w_i^2 \left[2^t (a^t + b^t) - (a+b)^t\right]$$ multiply by SOS

⑤ Add ③, ④ to get

$$A \vdash_{O(t)} (*) \leq \left(\sum_{i\in S} w_i^2\right)^{t-1} \cdot \sum_{i\in S} w_i^2 \cdot 2^t \left[(\mu_T - X_i)^t + (\mu_s - X_i)^t\right]$$

⑤ cont'd

$$\mathcal{A} \vdash_{O(t)} (*) \leq 2^t \left(\sum_{i \in S} w_i^2\right)^{t-1} \cdot \sum_{i \in S} w_i^2 (\mu_T - x_i)^t + w_i^2 (\mu_S - x_i)^t$$

⑥ Add ⑤ with

$$\underbrace{2^t \left(\sum_{i \in S} w_i^2\right)^{t-1} \sum_{i \notin S} w_i^2 (\mu_T - x_i)^t + w_i^2 (\mu_S - x_i)^t}_{\text{this is sos}}$$

to get

$$\mathcal{A} \vdash_{O(t)} (*) \leq 2^t \left(\sum_{i \in S} w_i^2\right)^{t-1} \sum_{i \in [n]} w_i^2 (\mu_T - x_i)^t + w_i^2 (\mu_S - x_i)^t$$

⑦ Add $(-w_i^2 + w_i)(\mu_T - x_i)^t$, $(-w_i^2 + w_i)(x_S - x_i)^t$ ← sos's to ⑥ :

$$\mathcal{A} \vdash_{O(t)} (*) \leq 2^t \left(\sum_{i \in S} w_i^2\right)^{t-1} \left[\underbrace{\sum_{i \in [n]} w_i (\mu_T - x_i)^t}_{} + \underbrace{\sum_{i \in [n]} w_i (\mu_S - x_i)^t}_{\leq N \cdot 2 \cdot t^{t/2}} \right]$$

$\mathcal{A}$ contains: $\dfrac{\sum_{i \in [n]} w_i (x_i - \mu_T)^t}{N} \leq 2 \cdot t^{t/2}$ by assumption

⑧ So $\mathcal{A} \vdash_{O(t)} (*) \leq 2^{O(t)} \cdot N \cdot t^{t/2} \cdot \left(\sum_{i \in S} w_i^2\right)^{t-1}$

⑨ Use $w_i^2 - w_i$ to get above but w_i^2 replaced by w_i

So we have

$$\mathcal{A} \vdash_{O(t)} \left(\sum_{i \in S} w_i \left[(\mu_T - x_i) - (\mu_S - x_i) \right] \right)^t \leq 2^{O(t)} \cdot \left(\sum_{i \in S} w_i \right)^{t-1} \cdot N \cdot t^{t/2}$$

rearranging:

$$\mathcal{A} \vdash \left(\sum_{i \in S} w_i (\mu_T - \mu_S) \right)^t \leq 2^{O(t)} \left(\sum_{i \in S} w_i \right)^{t-1} \cdot N \cdot t^{t/2}$$

$$\equiv \left(\sum_{i \in S} w_i \right)^t (\mu_T - \mu_S)^t \leq 2^{O(t)} \, t^{t/2} \cdot \left(\sum_{i \in S} w_i \right)^{t-1} \cdot N$$

$$\equiv \left(\frac{\sum_{i \in S} w_i}{N} \right)^t (\mu_T - \mu_S)^t \leq 2^{O(t)} \, t^{t/2} \left(\frac{\sum_{i \in S} w_i}{N} \right)^{t-1}$$

$$= \alpha^t (\mu_T - \mu_S)^t \leq 2^{O(t)} \, t^{t/2} \, \alpha^{t-1}$$

where $\alpha = \dfrac{|S \cap T|}{N}$

$\boxtimes$ (end lemma 1)

NOW: PROOF OF MAIN LEMMA

Recall $\tilde{E}$ is a solution pseudodist to degree $O(t)$ SOS SDP satisfying $\mathcal{A}$, and $X_1 .. X_N$, $S_1 .. S_k$ (real partition) satisfy moment bound $\mathbb{E}_{j \sim S_i} |X_j - \mu_i|^t \le 2 \cdot t^{t/2}$

WTS $\tilde{E}$ satisfies

$$\tilde{E}\left[\sum_{i \in [k]} \frac{|T \cap S_i|^2}{N}\right] \ge 1 - \frac{2^{O(t)}\, t^{t/2}\, k^2}{\Delta^t}$$

$$|T \cap S_j| = \sum_{i \in S_j} w_i$$

(1) By pseudo-exp Cauchy Swartz,

$$\tilde{E}\, |T \cap S_i|\,|T \cap S_j| \le \left(\tilde{E}\, |T \cap S_i|^t\, |T \cap S_j|^t\right)^{1/t}$$

(2) $|\mu_i - \mu_j| \ge \Delta^t$ (by assumpt they are well separated)

empirical mean of X_j $j \sim S_i$

By SOS triangle inequality,

$$\vdash_t (\mu_i - \mu)^t + (\mu_j - \mu)^t \ge 2^{-t}\left[(\mu_i - \mu) + (\mu_j - \mu)\right]^t$$
$$\ge 2^{-t} \Delta^t$$

where $\mu = \frac{1}{N} \sum_{i \in [n]} w_i X_i$

3 By 1 + 2 we have (take eqn 1 to power of t)

$$\tilde{\mathbb{E}}\, |T \cap S_i|^t |T \cap S_j|^t \leq$$

$$\tilde{\mathbb{E}} \Big[\underbrace{\frac{(\mu_i - \mu)^t + (\mu_j - \mu)^t}{2^{-t}\, \Delta^t}}_{\geq 1 \text{ by } 2} |T \cap S_i|^t \, |T \cap S_j|^t \Big]$$

4 Recall Lemma 1:

$$\mathcal{A} \vdash_{O(t)} \left(\frac{|S_i \cap T|}{N}\right)^t (\mu_i - \mu_T)^t \leq 2^{O(t)} t^{t/2} \left(\frac{|S_i \cap T|}{N}\right)^{t-1}$$

so

$$\tilde{\mathbb{E}}\, |T \cap S_i|^t |T \cap S_j|^t \leq$$

$$2^{O(t)} t^{t/2} \Delta^{-t} N \left[\tilde{\mathbb{E}}\, |T \cap S_i|^t |T \cap S_j|^{t-1} + \tilde{\mathbb{E}}\, |T \cap S_i|^{t-1} |T \cap S_j|^t \right]$$

4 Since $\mathcal{A} \vdash_t |T \cap S_i| \leq N$, $\mathcal{A} \vdash_t |T \cap S_j| \leq N$

$$\text{LHS} \leq 2^{O(t)} t^{t/2} \Delta^{-t} N^2\, \tilde{\mathbb{E}}\, |T \cap S_i|^{t-1} |T \cap S_j|^{t-1}$$

⑤ By pseudo-exp cauchy Swartz

$$\tilde{E}\,|T\cap S_i|^{t-1}\,|T\cap S_j|^{t-1} \le$$

$$\left(\tilde{E}\,|T\cap S_i|^{t}\,|T\cap S_j|^{t}\right)^{1/2}\left(\tilde{E}\,|T\cap S_i|^{t-2}\,|T\cap S_j|^{t-2}\right)^{1/2}$$

combining with ④ & rearranging gives

$$\text{LHS} \le \frac{2^{O(t)}\,t^t\,N^4}{\Delta^{2t}}\,\underbrace{\tilde{E}\,|T\cap S_i|^{t-2}\,|T\cap S_j|^{t-2}}_{\le\left(\tilde{E}\,|T\cap S_i|^{t}\,|T\cap S_j|^{t}\right)^{\frac{t-2}{t}}\ \text{by pseudo-exp Holders}}$$

⑥ so we have

$$\underbrace{\tilde{E}\,|T\cap S_i|^{t}\,|T\cap S_j|^{t}}_{A} \le \frac{2^{O(t)}\,t^t\,N^4}{\Delta^{2t}}\left(\tilde{E}\,|T\cap S_i|^{t}\,|T\cap S_j|^{t}\right)^{\frac{t-2}{t}}$$

$$A \le \text{bla}\cdot A^{1-2/t}$$

$$A^{2/t} \le \text{bla}$$

$$A^{t} \le \text{bla}^{1/2}$$

$$\tilde{E}\,|T\cap S_i|^{t}\,|T\cap S_j|^{t} \le \frac{2^{O(t)}\,t^{t/2}\,N^2}{\Delta^{t}}$$

⑦ Lastly

$$\tilde{E} \sum_{i,j \in [k]} |T \cap S_i| \, |T \cap S_j|$$

$$= \tilde{E} \Big(\sum_{i \in [n]} w_i \Big)^2 = N^2$$

So

$$\tilde{E} \Big[\sum_{i \in [k]} \Big(\frac{|T \cap S_i|}{N} \Big)^2 \Big]$$

$$= \frac{1}{N^2} \tilde{E} \Big[\sum_{i \in k} |T \cap S_i|^2 \Big]$$

$$\left(\begin{aligned} & a_1^2 + \dots + a_k^2 \\ & = (a_1 + \dots + a_k)^2 \\ & \quad - \sum_{i \neq j} a_i a_j \end{aligned} \right)$$

$$= \frac{1}{N^2} \Big[\tilde{E} \sum_{i,j \in [k]} |T \cap S_i| \, |T \cap S_j| - \sum_{i \neq j} |T \cap S_i| \, |T \cap S_j| \Big]$$

$$= \frac{1}{N^2} \Big[N^2 - \frac{k^2 \, 2^{O(t)} \, t^{t/2} N^2}{\Delta^t} \Big]$$

$$= 1 - \frac{k^2 \, 2^{O(t)} \, t^{t/2}}{\Delta^t}$$

$\square$