CS 2401

Complexity Theory

ASSIGNMENT # 4

DUE DATE: Thursday, December 7, 2006

1. Consider the following problem about computing a relation. Associate the following communication problem with any boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$. Player A gets an input x such that $f(x) = 0$, and player B gets an input y such that $f(y) = 1$. They have to communicate in order to find a bit position i, $1 \leq i \leq n$ such that $x_i \neq y_i$. Prove that the communication complexity of this problem is exactly the minumum depth of any circuit that computes f (where the maximum fanin of each gate is 2).

2. Now consider the case where f is monotone. In this case, again Player A gets in input x such that $f(x) = 0$, and Player B gets an input y such that $f(y) = 1$, but now they want to find a position i such that $x_i = 0$ and $y_i = 1$. prove that the communication complexity of this problem is exactly the minimum depth of any monotone circuit that computes f.

3. Using the characterization from question (1), give a circuit of the smallest depth that you can for computing the majority function. That is, describe a communication protocol for the majority function. (Majority takes as input n bits and outputs 1 if and only if at least $\lceil n/2 \rceil$ of the bits are 1.)

4. Using the characterization from question (2), now give a monotone circuit of the smallest depth that you can for computing the majority function.

5. Let f be a boolean function on n variables, computed by a boolean formula over AND, OR and $\neg$. That is, f is computed by a directed, acyclic circuit F of fanin-in 2 (and fan-out 1). The size of F is the total number of internal nodes/connectives and the depth is the maximum height of the circuit. Prove that for any such F, there exists another *balanced* boolean formula F' computing the same function as f and such that the depth of F' is $O(\log(size(F)))$. (Hint: First prove the "1/3 - 2/3 trick" : For any boolean formula G of size S, there exists a subformula G' of G such that $size(G)/3 \leq size(G') \leq 2size(G)/3$.)