# CS 2401
## Complexity Theory
# ASSIGNMENT # 1
## DUE DATE: Thursday, October 5, 2006

(1.) A language $L$ is *unary* if the underlying alphabet is $\{1\}$. (That is, every string in $L$ is of the form $1^i$ for some $i \geq 0$.) Prove that if a unary language is NP-complete, then P=NP. (This is exercise 11, Chapter 2 of the book. You may wish to consult the book to find a hint.)

(2.) (a) Show that 2SAT is in NL.

(b) Prove that 2SAT is NL-hard with respect to logspace reductions.

(3.) Prove that the PH has a complete language if and only if it collapses to some finite level $\Sigma_i^p$.

(4.) Let $S = \{S_1, S_2, \ldots, S_m\}$ be a collection of subsets of a finite set $U$. Let $|U| = n$. Then each $S_i$ will be represented by a bit string of length $n = |U|$, where the $j^{th}$ position will indicate whether or not the $j^{th}$ element of $U$ is in $S_i$. The VC-dimension of $S$, denoted by $VC(S)$, is the size of the largest set $X \subset U$ such that for every $X' \subseteq X$, there is an $i$ such that $S_i \cap X = X'$. (That is, $X$ is *shattered* by $S$.) Let $VCdim$ be the set of pairs $(S, k)$ such that the VC-dimension of $S$ is at least $k$.

(a) Prove that $VCdim$ is in $NP$.

(b) Explain why it is unlikely that $VCdim$ is NP-complete.

HINT: There is an algorithm for $VCdim$ that runs in *quasi-polynomial* time. That is, time $n^{O(\log n)}$, where $n$ is the total input size. The algorithm is based on a simple lemma which upper bounds the maximal size of the VC dimension of a set $S$, as a function of the size of $S$. State and prove this lemma, and show how it implies both an NP-algorithm, as well as a quasi-polynomial time algorithm for $VCdim$. Then explain why the existence of such an algorithm makes it unlikely that $VCdim$ is NP-complete.

(5.) This problem also concerns the VC-dimension of a set $S$, only now the set $S$ will be represented more succinctly. A boolean circuit $C$ succinctly represents collection $S$ if $S_i$ consists of exactly those elements $x \in U$ for which $C(i, x) = 1$. Let $|U| = n$. Then $C$ will have $\log m + \log n$ inputs, where the first $\log m$ inputs will be $i$ in binary notation, and the last $\log n$ inputs will be $x$ in binary notation. $C$ itself will be encoded by some string of length polynomial in the size of $C$. Define $VCdimSuccinct$ to be the set of all strings $< C, k >$ such that $C$ represents a collection $S$ such that the VC-dimension of $S$ is at least $k$.

(a) Show that $VCdimSuccinct$ is in $\Sigma_3^p$.

(b) Prove that $VCdimSuccinct$ is $\Sigma_3^p$ complete. (Hint: Reduce from $\Sigma_3$-3SAT.)