Week 8

HW3 Due Today!
HW4 (Last one!) out
Week 7 Summary (2 weeks ago!)

1. We saw \( D = \{ x \mid \exists y \exists z \exists \gamma (x) \text{ does not accept } y \} \)

   is not r.e. by diagonalization

2. Using reductions we proved

   \( K, \text{ Halt are not recursive} \)
Using Reductions to show other (more natural) languages/functions are not computable/recursive/r.e.

High Level:

1. Say we know $L_1$ not recursive
   To show $L_2$ not recursive, design a TM $M_1$
   always halts & $L(M_1) = L_1$, assuming a
   TM $M_2$ that always halts & $L(M_2) = L_2$

2. Suppose $L_1$ not r.e.
   To show $L_2$ not r.e., construct $M_1$ st $L(M_1) = L_1$
   assuming a TM $M_2$ st $L(M_2) = L_2$
The Halting Problem is Not Recursive

\[ K^d = \{ x \mid \text{TM } \exists x \text{ halts on input } x \} \]

\[ \text{HALT}^d = \{ \langle x, y \rangle \mid \text{TM } \exists x \text{ halts on input } y \} \]

Theorem. HALT, K are both r.e., neither are recursive.
The Halting Problem is Not Recursive

\[ K = \{ x \mid \text{TM} \exists x \text{ halts on input } x \} \]

Theorem $K$ is not recursive

If $K$ recursive then $D$ also recursive

Theorem Halt Not recursive

If Halt recursive then $K$ recursive
Tips

(1.) Try obvious algorithms to see if you think language is recursive, re, or neither

(2.) To show $L$ not r.e., sometimes it helps to work with $\overline{L}$

(3.) Get reduction in correct direction. Many times constructed TM $M_1$ will ignore its own input
\[ L = \{ \{ x \} \mid \{ x \} \text{ accepts at least one input} \} \]
L = \{ x \mid \exists y^3(x) \text{ accepts at least one input}\}

- L is r.e. (Dovetailing)
- L is not recursive

L = K = \{ y \mid \exists y^3(y) \text{ halts}\}

Assume \( L_2 = L \) is recursive & let \( M_2 \) be TM \( L(M_2) = L \)
and \( M_2 \) always halts

\( M_1 \) on input \( y \):

- Construct encoding \( z \) of TM \( \exists y^3 \) where
- \( \{ \exists y^3 \} \) on input \( x \): Ignores \( x \) & runs \( \{ y^3 \} \) on \( y \)
- and accepts \( x \) if \( \exists y^3(y) \) halts

Run \( M_2 \) on \( z \) & accept \( y \) iff \( M_2(z) \) accepts

Claim \( L(M_1) = K \) & \( M_1 \) always halts

\( y \in K \Rightarrow \exists y^3(y) \text{ halts} \Rightarrow \exists y^3 \text{ accepts all inputs} \Rightarrow M_2(z) = 1 \Rightarrow M_1(y) = 1 \)

\( y \notin K \Rightarrow \exists y^3(y) \text{ doesn't} \Rightarrow \exists y^3 \text{ accepts no input} \Rightarrow M_2(z) \neq 1 \Rightarrow M_1(y) \neq 1 \)
Completeness

A set $A \subseteq \mathbb{N}$ is r.e.-complete if

1. $A$ is r.e.
2. $\forall B \subseteq \mathbb{N}$, if $B$ is r.e. then $B \leq_m A$.

So if $A$ is recursive then $B$ recursive.

$B$ reduces to $A$.

\[ N \xrightarrow{f} A \xrightarrow{g} N \]
Completeness

A set $A \subseteq \mathbb{N}$ is r.e.-complete if

1. $A$ is r.e.
2. $\forall B \subseteq \mathbb{N}$, if $B$ is r.e. then $B \leq_m A$

There exists a computable function $f: \mathbb{N} \Rightarrow \mathbb{N}$ such that

$\forall x \quad f(x) \in A \iff x \in B$
Hilbert's 10th Problem (1900)

A diophantine equation is of the form \( p(\bar{x}) = 0 \) where \( p \) is a polynomial over variables \( x_1, \ldots, x_n \) with integer coefficients

\[
\text{Ex: } 3x_1^5x_2^3 + (x_1 + 1)^8 - x_7^{10} = 0
\]

\( L_{\text{Dioph}} = \{ \langle p \rangle \mid p \text{ has a solution over } \mathbb{N}^3 \} \)

Theorem

\( L_{\text{Dioph}} \) is r.e.-complete
An Equivalent Characterization of RE Sets

Let \( f: \mathbb{N} \to \mathbb{N} \)

Then \( R_f \subseteq \mathbb{N} \times \mathbb{N} \)

is the set of all pairs \((x, y)\) such that \( f(x) = y \)

\*Theorem \quad f \text{ computable \ if and only if } R_f \text{ is r.e.} \)
An Equivalent Characterization of RE Sets

Let \( f : \mathbb{N} \rightarrow \mathbb{N} \)

Then \( R_f \subseteq \mathbb{N} \times \mathbb{N} \)

is the set of all pairs \((x, y)\) such that \( f(x) = y \)

**Theorem** \( f \) computable if and only if \( R_f \) is r.e.

**Proof** \( \Rightarrow \): Suppose \( f \) computable.

TM for \( R_f \) on input \((x, y)\):

Run TM computing \( f \) on \( x \).

If it halts and outputs \( y \) then accept \((x, y)\).

Otherwise reject \((x, y)\)
An Equivalent Characterization of RE Sets

Let $f : \mathbb{N} \rightarrow \mathbb{N}$

Then $R_f = \mathbb{N} \times \mathbb{N}$ is the set of all pairs $(x, y)$ such that $f(x) = y$

*Theorem* $f$ computable if and only if $R_f$ is r.e.

**Proof** $\Leftarrow$: Let $R_f$ be r.e. with TM $M$

On $x$: Enumerate all $\mathbb{N} : y_1, y_2, \ldots$

For $i = 1, 2, \ldots$

For all $j \leq i$: simulate $M$ on $(x, y_j)$ for $i$ steps

If simulation accepts $(x, y_j)$, halt and output $y_i$.
A second characterization of RE sets

*Theorem* A relation $A \subseteq \mathbb{N}^k$ is r.e. if and only if there is a recursive relation $R \subseteq \mathbb{N}^{k+1}$ such that

$$\exists \bar{x} \in A \iff \exists y R(\bar{x}, y) \quad \forall \bar{x} \in \mathbb{N}^n$$

Note we defined $A$ to be r.e. iff there is a TM $M$ such that $\forall \bar{x} \in \mathbb{N}^n \ (M(\langle \bar{x} \rangle) \text{ accepts } \iff \exists \bar{x} \in A)$.
A Second Characterization of RE Sets

*Theorem* A relation $A \subseteq \mathbb{N}^k$ is r.e. if and only if there is a recursive relation $R \subseteq \mathbb{N}^{k+1}$ such that

$$\exists x \in A \iff \exists y R(x, y) \quad \forall x \in \mathbb{N}^n$$

**Proof sketch**

$\Rightarrow$: Let $A$ be r.e., $L(M) = A$

$R(x, y)$: view $y$ as encoding of an $m \times m$ tableaux for some $m \in \mathbb{N}$

$(x, y) \in R \iff M(x)$ halts in $m$ steps and accepts and $y$ is the $m \times m$ tableaux of $M(x)$
A second characterization of RE sets

*Theorem*  A relation $A \subseteq \mathbb{N}^k$ is r.e. if and only if there is a recursive relation $R \subseteq \mathbb{N}^{k+1}$ such that

$$\exists \overline{x} \in A \iff \exists y \ R(\overline{x}, y) \quad \forall \overline{x} \in \mathbb{N}^n$$

**Proof sketch**

$\Leftarrow$ Let $R \subseteq \mathbb{N}^{k+1}$ be recursive relation such that

$$\exists \overline{x} \in A \iff \exists y \ R(\overline{x}, y), \quad \text{and let } L(M) = R$$

on input $\overline{x}$:

For $i = 1, 2, \ldots$

For $j = 1$ to $i$

Run $M$ on $(\overline{x}, y_j)$

halt & accept if $M(\overline{x}, y_j)$ accepts
Review of Definitions

$\mathcal{L}_A = \{ 0, 1, +, \cdot, j \}$ Language of arithmetic

$\Phi_0 = \text{all } \mathcal{L}_A - \text{sentences}$

$TA = \{ A \in \Phi_0 \mid \text{IN } \models A \}$ True Arithmetic

A theory $\Sigma$ is a set of sentences (over $\mathcal{L}_A$) closed under logical consequence.

- We can specify a theory by a subset of sentences that logically implies all sentences in $\Sigma$

$\Sigma$ is consistent iff $\Phi_0 \not\models \Sigma$ (iff $\forall A \in \Phi_0$, either $A$ or $\neg A$ not in $\Sigma$)

$\Sigma$ is complete iff $\Sigma$ is consistent and $\forall A$ either $A$ or $\neg A$ is in $\Sigma$
$\Sigma$ is **sound** iff $\Sigma \subseteq TA$

Let $M$ be a model/structure over $\mathcal{L}_A$

$Th(M) = \{ A \in \Phi_\sigma \mid M \models A \}$

$Th(M)$ is **complete** (for all structures $M$)

**Note**: $TA = Th(IN)$ is complete, consistent, & sound

$VALID = \{ A \in \Phi_\sigma \mid \models A \}$ ← smallest theory
Let $\Sigma$ be a theory

$\Sigma$ is axiomatic if there exists a set $\Gamma \subseteq \Sigma$ such that

1. $\Gamma$ is recursive
2. $\Sigma = \{ A \in \Phi_0 | \Gamma \vdash A \}$

Theorem $\Sigma$ is axiomatizable iff $\Sigma$ is r.e.

(P. 76 of Notes)
Let $\Sigma$ be a theory

$\Sigma$ is axiomatizable if there exists a set $\Gamma \subseteq \Sigma$

such that

1. $\Gamma$ is recursive
2. $\Sigma = \{ \text{A} \in \mathcal{B} \mid \Gamma \vdash A \}$

**Theorem** $\Sigma$ is axiomatizable iff $\Sigma$ is r.e.

**Proof** $\Rightarrow$. Suppose $\Sigma$ is axiomatizable, $\Gamma$ recursive

Define $R(x,y) = \text{true}$ iff $y$ encodes a $\Gamma$-LK proof of (the formula encoded by $y$) $x$

$R$ is recursive, so by previous *Theorem*, $\Sigma$ is r.e.
Let \( \Sigma \) be a theory

\( \Sigma \) is axiomatic if there exists a set \( \Gamma \subseteq \Sigma \) such that

1. \( \Gamma \) is recursive
2. \( \Sigma = \{ A \in \Sigma : \Gamma \vdash \neg A \} \)

**Theorem** \( \Sigma \) is axiomatizable iff \( \Sigma \) is r.e.

**Proof** \( \Rightarrow \). Suppose \( \Sigma \) is axiomatizable, \( \Gamma \) recursive

Define \( R(x, y) = \text{true} \) iff \( y \) encodes a \( \Gamma \)-LK proof of \((\text{the formula encoded by} y) \times \)

\( \Gamma \) is recursive, so by previous *Theorem, \( \Sigma \) is r.e.

\( \Leftarrow \) By *Theorem, \( \Sigma = \text{range of total computable function } f \)

\( \therefore \Sigma = \{ f(0), f(1), f(2), \ldots \} \)
Incompleteness - Introduction

1. TA is not r.e. (so by previous theorem, not axiomatizable)

First Incompleteness Theorem Every sound axiomatizable theory is incomplete
\[ \Phi_0 : \]

all \( L_A \) sentences

\[ \exists \text{ sound and axiomatizable} \Rightarrow \exists A, \forall A, \exists \]
Incompleteness - Introduction

1. TA is not r.e. (so by previous theorem, not axiomatizable)
   **First Incompleteness Theorem**: Every sound axiomatizable theory is incomplete

2. Define PA - Peano arithmetic
   Sound, axiomatizable
   So by Tarski's Thm, PA is incomplete

3. Gödel's second Incompleteness Thm:
   A specific sentence asserting "PA is consistent" is not a theorem of PA
First Incompleteness Theorem

We define a predicate \( \text{Truth} \in \mathbb{N} \)

\[
\text{Truth} = \{ m \mid m \text{ encodes a sentence } \langle m \rangle \in \Phi_0 \text{ that is in TA} \}
\]

We will show that \( \text{Truth} \) is not r.e.
**FIRST INCOMPLETENESS THEOREM**

We define a predicate $\text{Truth} = \mathbb{N}$

$\text{Truth} = \exists \ m \mid m \text{ encodes a sentence } \langle m \rangle \in \bar{\Phi}$

that is in $TA$.

We will show that $\text{Truth}$ is not r.e.

**Defn** A predicate is arithmetical if it can be represented by a formula over $\mathcal{L}_A$.

**We'll show:**

1. Every r.e. predicate/language is arithmetical
2. $\text{Truth}$ is not arithmetical

$\therefore$ $\text{Truth}$ is not r.e.
Since *Truth* is not r.e.,
there is no r.e. TM that accepts exactly the sentences in TA

\[ \therefore TA \text{ is not axiomatizable} \]

\[ \therefore \text{Any sound, axiomatizable theory } \Sigma \text{ is incomplete} \]

(There is a sentence \( A \in \Phi \), such that neither \( A \) or \( \neg A \) are in \( \Xi \).)
**FIRST INCOMPLETENESS THEOREM**

We define a predicate \( \text{Truth} \in \mathbb{N} \)

\[
\text{Truth} = \{ m \mid m \text{ encodes a sentence } \langle m \rangle \in \Phi \}
\]
that is in TA.

We will show that \( \text{Truth} \) is not r.e.:

**Defn**
A predicate is arithmetical if it can be represented by a formula over \( \mathcal{L}_A \)

**Show**

1. Every r.e. predicate/language is arithmetical
2. \( \text{Truth} \) is not arithmetical

\[\therefore \text{Truth} \text{ is not r.e.}\]

*Exists-Delta Theorem* PP 68-71
*Tarski Theorem* PP 73-74
1. Every R.E. predicate is arithmetical

**Definition** Let \( s_0 = 0, s_1 = s_0, s_2 = s_0 s_0, \) etc.

Let \( R(x_1, \ldots, x_n) \) be an \( n \)-ary relation \( R \subseteq \mathbb{N}^n \). Let \( A(x_1, \ldots, x_n) \) be an \( \mathcal{L}_A \) formula, with free variables \( x_1, \ldots, x_n \).

\( A(x) \) **represents** \( R \) iff \( \forall \bar{a} \in \mathbb{N}^n \ R(\bar{a}) \iff \mathbb{N} \models A(s_{a_1}, s_{a_2}, \ldots, s_{a_n}) \)

**Example** \( R \subseteq \mathbb{N} \) \( R = \{ a \in \mathbb{N} \mid a \text{ is even} \} \)

\( A : \exists y \ (y + y = x) \)

3 \( \in R \) and \( \mathbb{N} \models A(s_{s_0}) = \exists y \ (y + y = s_{s_0}) \)

4 \( \in R \) and \( \mathbb{N} \models A(s_{s_{s_0}}) = \exists y \ (y + y = s_{s_{s_{s_0}}}) \quad y = s_{s_0} \)
1. Every R.e. predicate is arithmetical

Definition. Let \( s_0 = 0, s_1 = s_0, s_2 = s_{s_0}, \text{ etc.} \)

Let \( R(x_1, \ldots, x_n) \) be an \( n \)-ary relation \( R \subseteq \mathbb{N}^n \)

Let \( A(x_1, \ldots, x_n) \) be an \( \mathcal{L}_A \) formula, with free variables \( x_1, \ldots, x_n \)

\( A(x) \) represents \( R \) iff \( \forall \bar{a} \in \mathbb{N}^n \ R(\bar{a}) \iff \mathbb{N} = A(s_{a_1}, s_{a_2}, \ldots, s_{a_n}) \)

\( R \) is arithmetical iff there is a formula \( A \in \mathcal{L}_A \) that represents \( R \)

Exists-\( \Delta \)-Theorem. Every R.e. relation is arithmetical. In fact every R.e. relation is represented by a \( \exists \Delta \) \( \mathcal{L}_A \)-formula.
\[ t_1 \leq t_2 \] stands for \( \exists z (t_1 + z = t_2) \)

\[ \exists x \leq t \ A \] stands for \( \exists x (x \leq t \land A) \)

\[ \forall x \leq t \ A \] stands for \( \forall x (x \leq t \supset A) \)

**Definition** A formula is a \( \Delta_0 \)-formula if it has the form \( \forall x_1 \leq t_1 \exists x_2 \leq t_2 \forall x_3 \leq t_3 \ldots \exists x_k \leq t_k A(x_1 \ldots x_k \bar{y}) \)

**Definition** A relation \( R(\bar{x}) \) is a \( \Delta_0 \)-relation iff some \( \Delta_0 \)-formula represents it.
Example: \( \text{Prime} = \{ x \in \mathbb{N} \mid x \text{ is prime} \} \) is a \( \Delta_0 \) -relation, represented by the following \( \Delta_0 \) -formula:

\[
(x = 2 \lor 1 = 2) \land \forall z \leq x \forall y \exists a (x - 2 = y \land y < 0) \land x > 0 \land \forall z \leq 2 \forall x \leq y \forall a \leq z \forall x \leq y
\]
**$\exists \Delta_0$ Formulas**

$t_1 \leq t_2$ stands for $\exists w (t_1 + w = t_2)$

$\exists z \leq t A$ stands for $\exists z (z \leq t \land A)$

$\forall z \leq t A$ stands for $\forall z (z \leq t \supset A)$

$\{ \text{Bounded} \}$

**Definition** A formula is a $\Delta_0$-formula if it has

the form $\forall z_1 \leq t, \exists z_2 \leq t_2 \forall z_3 \leq t_3 \ldots \exists z_k \leq t_k A(z_1, z_2, z_3, \ldots, z_k, t)$

**Definition** $\exists \Delta_0$ formula has the form $\exists \eta B_{\Delta_0}$

**Definition** A relation $R(\bar{x})$ is a $\Delta_0$-relation iff

some $\Delta_0$-formula represents it

**Definition** $R(\bar{x})$ is a $\exists \Delta_0$-relation iff some $\exists \Delta_0$-formula represents it
Every $\Delta_0$ relation is recursive.

Every $\exists \Delta_0$ relation is r.e.

$\exists \Delta_0$ (Exists-Delta) Theorem: Every r.e. relation is represented by a $\exists \Delta_0$ formula.
**Main Lemma**

Let $f : \mathbb{N}^n \rightarrow \mathbb{N}$ be a total computable function.

Let $R_f = \{(x, y) \in \mathbb{N}^{n+1} \mid f(x) = y\}$

Then $R_f$ is a $\exists \Delta_0$-relation.
Main Lemma  \( \text{Let } f : \mathbb{N}^n \rightarrow \mathbb{N} \text{ be total, computable.} \) Then \( R_f = \{ (x, y) \mid f(x) = y \} \) is an \( \exists \Delta_0 \) relation.

### Proof of \( \exists \Delta_0 \) Theorem from Main Lemma

Let \( R(x) \) be an r.e. relation. Then \( R(x) = \exists y \exists z (x, y, z) \) where \( S \) is recursive. Since \( S \) is recursive, \( f_S(x, y) = 1 \) if \( (x, y) \in S \). This is total computable. By the main lemma, \( R_{fs} \) is represented by a \( \exists \Delta_0 \) relation. So \( R(x) = \exists y \exists z R_{fs} \) is represented by a \( \exists \Delta_0 \) relation.
Proof of Main Lemma (see pp 70-71)

Main idea: is a way of representing sequences of numbers by numbers using $\Omega_2$ formulas.

Note: Prime power decomposition not useful here since we only have $s, t, e$

(i.e. represent $(a_1, a_2, a_3, a_4)$ by $2^a \cdot 3^b \cdot 5^c \cdot 7^d$)

Definition $\beta$-function

$\beta(c, d, i) = \text{rm}(c, d(i+1)+1)$

where $\text{rm}(x, y) = x \text{mod} y$
Proof of Main Lemma (see pp 70-71)

**Definition** $\beta$-function

$$\beta(c, d, i) = \text{rm}(c, d(i+1) + 1) \quad \text{where} \quad \text{rm}(x, y) = x \mod y$$

**Lemma 0.** $\forall n, r_0, r_1, \ldots, r_n \exists c, d$ such that

$$\beta(c, d, i) = r_i \quad \forall i, 0 \leq i \leq n$$

So the pair $(c, d)$ represents the sequence $c, r_0, r_1, \ldots, r_n$ using $\beta$
Proof of Main Lemma (see pp 70-71)

**Definition** $\beta$ - function

$$\beta(c, d, i) = \text{rm}(c, d(i+1)+1)$$

where $\text{rm}(x, y) = x \mod y$

**Lemma 0**

$$\forall \eta, r_0, r_1, \ldots, r_n \exists c, d \text{ such that }$$

$$\beta(c, d, i) = r_i \quad \forall i, 0 \leq i \leq n$$

**ERT (Chinese Remainder Theorem)**

Let $r_0, \ldots, r_n, m_0, \ldots, m_n$ be such that

$$0 \leq r_i \leq m_i \quad \forall i, 0 \leq i \leq n \quad \text{and} \quad \gcd(m_i, m_j) = 1 \quad \forall i, j$$

Then $\exists r$ such that $\text{rm}(r, m_i) = r_i \quad \forall i, 0 \leq i \leq n$
Proof of Main Lemma (see pp 70-71)

Lemma 0 \forall n, r_0, r_1, \ldots, r_n \exists c, d \text{ such that } 
\beta(c, d, i) = r_i \quad \forall i, 0 \leq i \leq n \quad \beta(c, d, i) = r_m(c, d(i+1)+1) 
\text{ where } \text{mod } y

Chinese Remainder Theorem

Let \( r_0, \ldots, r_n, m_0, \ldots, m_n \) be such that 
0 \leq r_i \leq m_i \text{ and } \text{gcd}(m_i, m_j) = 1. \text{ Then } \exists r \text{ such that } \text{mod } y \text{ and } \beta(r, c, i) = r \quad \forall i

Proof of Lemma 0

Let \( d = (n! + r_0 + \ldots + r_n + 1)! \)
Let \( m_i = d(i+1)+1 \)
Claim \( \forall i, j \text{ such that } \text{gcd}(m_i, m_j) = 1 \text{ (see notes) } \)
By CRT \( \exists r = c \text{ such that } \beta(c, d, i) = \text{mod } y \text{ and } \beta(c, d, i) = r_i \quad \forall i \in [n] \)
Proof of Main Lemma (see pp. 10-71)

**Lemma 0** \( \forall n, r_0, r_1, \ldots, r_n \exists c, d \text{ such that} \)
\[ \beta(c, d, i) = r_i \quad \forall i, 0 \leq i \leq n \]

**Lemma 1** \( R_p \) is a \( \Delta_0 \) relation

\[ Pf: \quad y = \beta(c, d, i) \iff \left[ \exists q \leq c (c = q(d(i+1) + y) \land y < d(i+1) + 1) \right] \]

**Lemma 2** If \( R(x, y) \) is a \( \exists \Delta_0 \) relation, \( R_p \) is a \( \exists \Delta_0 \) relation

then \( S(x) = \exists y \left(R_p(x, y) \land R(x, y)\right) \) is a \( \exists \Delta_0 \) relation
Proof of Main Lemma (see pp 70-71)

Let \( f: \mathbb{N} \rightarrow \mathbb{N} \) be unary, total computable function, and let \( M_f \)
be TM computing \( f \)

\( R(x, y) \) will be a \( \exists \delta_0 \) relation saying:

- \( \exists m, c, d \) such that
  - \( m, c, d \) describe the tableaux given by \( \tau, \ldots, \tau_m, \ldots, \tau_{m^2} \)
  - \( \tau, \ldots, \tau_m \) encode start config of \( M_f \) on \( x \)
  - \( \tau_{m+1}, \ldots, \tau_{m^2} \) encode last config, containing \( y \) in first cells then \( B \), and \( state \ is \ q_2 \)
  - For all other configs, state is not \( q_2 \)
  - all \( 2 \times 3 \) local cells are consistent with transition function of \( M_f \)
Recap: We wanted to prove

\[ \exists \Delta_0 \text{ (Exists-Delta) Theorem } \quad \text{every r.e. relation is represented by a } \exists \Delta_0 \text{ formula} \]

which followed by **Main Lemma**:  

\[ f \text{ total, computable } \Rightarrow R_f \text{ is a } \exists \Delta_0 \text{ relation} \]
We define a predicate \( \text{Truth} = \mathbb{N} \)
\[
\text{Truth} = \{ m \mid m \text{ encodes a sentence } <m> \in \Phi \text{ that is in } TA \}\]

We will show that \( \text{Truth} \) is not r.e.

**Defn** A predicate is arithmetical if it can be represented by a formula over \( \mathcal{L}_A \)

1. Every r.e. predicate/language is arithmetical
2. \( \text{Truth} \) is not arithmetical

\[ \therefore \text{Truth is not r.e.} \]
Tarski Theorem

Define the predicate \( \text{Truth} \subseteq \mathbb{N} \)

\[
\text{Truth} = \{ m \mid m \text{ encodes a sentence } \langle m \rangle \in \text{TA} \}
\]

Then \( \text{Truth} \) is not arithmetical.
Define the predicate $\text{Truth} = \mathbb{N}$

$$\text{Truth} = \{ m \mid m \text{ encodes a sentence } \langle m \rangle \in TA \}$$

Then $\text{Truth}$ is not arithmetical.

High Level Idea:
Formulate a sentence "I am false" which is self-contradictory.
Let \( \text{sub}(m,n) = \begin{cases} 0 & \text{if } m \text{ is not a legal encoding of a formula} \\ \text{otherwise say } m \text{ encodes the formula } A(x) \text{ with free variable } x. \\ \text{Then } \text{sub}(m,n) = m' \text{ where } m' \text{ encodes } A(s_n) \end{cases} \)

Let \( d(n) = \text{sub}(n,n) \)
\[
\begin{cases} 
\quad d(n) = 0 & \text{if } n \text{ not a legal encoding.} \\
\text{otherwise say } n \text{ encodes } A(x), \text{ then } d(n) = n' \text{ where } n' \text{ encodes } A(s_n) 
\end{cases}
\]

clearly \( \text{sub, d} \) are both computable
Proof of Tarski's Thm

Suppose that Truth is arithmetical. Then define $R(x) = \neg \text{Truth}(d(x))$.

Since $d$, Truth both arithmetical, so is $R$

Let $\overline{R(x)}$ represent $R(x)$, and let $e$ be the encoding of $R(x)$.

Let $d(e) = e'$ so $e'$ encodes $R(\overline{e})$ encodes "I am false".

Then $\overline{R(\overline{e})} \in TA \iff \neg \text{Truth}(d(e))$

since $\overline{R}$ represents $R$ by defn of truth

TA contains exactly one of $A, \overline{A}$

\[ \iff \neg \overline{R(\overline{e})} \in TA \]

\[ \iff \overline{R(\overline{e})} \notin TA \]

\[ \iff \overline{R(\overline{e})} \notin TA \]

\[ \iff \text{this is a contradiction}. \therefore \text{Truth is not arithmetical} \]
FIRST INCOMPLETENESS THEOREM

FINALLY WE HAVE PROVEN:

1. Every r.e. predicate/language is arithmetical
2. Truth is not arithmetical

∴ Truth is not r.e.

Truth not r.e. ⇒ TA not axiomatizable

∴ Any sound, axiomatizable theory is incomplete
$\Phi_0$:

all $L_A$ sentences

\[ \Gamma \text{ sound and axiomatizable } \Rightarrow \exists A, \forall A \in \Gamma \]
2nd Incompleteness Theorem

- We will define PA (Peano Arithmetic), an axiomatizable sound theory.
- Most of number theory provable in PA
- We will see that PA cannot prove its own consistency (2nd Incompleteness Thm)