cs438/2404 Lecture 4

• HW 2: out! Due Oct 18
TODAY

First Order Logic

Quick Review – Syntax, semantics
LK completeness
Consequences of completeness

Lecture Notes: completeness p. 31-38
Underlying language $L$ specified by:

1. $\forall n \in \mathbb{N}$ a set of $n$-ary function symbols (i.e., $f, g, h, +, \cdot$)
   O-ary function symbols are called constants

2. $\forall n \in \mathbb{N}$ a set of $n$-ary predicate symbols (i.e., $P, Q, R, <, \leq$)

Plus:
- Variables: $x, y, z, \ldots a, b, c, \ldots$
- $\neg, \lor, \land, \exists, \forall$
- Parentheses: $\left(, \right)$
Terms over $\mathcal{L}$

1. Every variable is a term.
2. If $f$ is an $n$-ary function symbol, and $t_1, \ldots, t_n$ terms, then $f(t_1, \ldots, t_n)$ is a term.
Terms over $L$

1. Every variable is a term.
2. If $f$ is an $n$-ary function symbol, and $t_1, \ldots, t_n$ terms, then $ft_1, \ldots, t_n$ is a term.

Examples of terms ($0, s, f, +, \cdot$)

- $fossso$, $+x+fyz$, $\cdot+abssso$
- $f(ossso, 0)$, $x+f(y, z)$, $(a+b)*ssso$
First Order Formulas over $L$

1. $P(t_1, \ldots, t_n)$ is an atomic $L$-formula, where $P$ is an $n$-ary predicate in $L$, and $t_1, \ldots, t_n$ are terms over $L$.

2. If $A, B$ are $L$-formulas, so are $\neg A$, $(A \land B)$, $(A \lor B)$, $\forall x A$, $\exists x A$. 
Example: FO Formulas in $L_\alpha$

(3) Fermat's Last Theorem (actually Andrew Wiles' theorem)

$\forall n \geq 3 \ (\forall a,b,c \ a^n + b^n = c^n)$

**Problem:** How to say $a^n$?

(we'll see later how to do this!)

Hn $> 3$ (Va, b, c $a^n + b^n = c^n$)
An occurrence of $x$ in $A$ is bound if $x$ is in a subformula of $A$ of the form $\forall x.B$, or $\exists x.B$ (otherwise $x$ is free in $A$)

**Example** $\exists y \,(x = y + y)$

$P \land \forall x \,(\neg (x + 5x = x))$

A formula/term is closed if it contains no free variables

A closed formula is called a sentence
SEMANTICS OF FO LOGIC

An \( L \)-structure \( M \) (or model) consists of:

1. A nonempty set \( M \) called the **universe** (variables range over \( M \))

2. For every \( n \)-ary function symbol \( f \) in \( L \), an associated function \( f^M : M^n \rightarrow M \)

3. For each \( n \)-ary relation symbol \( P \) in \( L \), an associated relation \( P^M \subseteq M^n \)

* **Equality predicate** = is always true equality relation on \( M \).
Example

\[ L_A = \{0, s, +, \cdot \} \]

1. \[ \text{IN}: \text{standard model of } L_A \]

\[ M = \text{IN} \]

\[ 0 = 0 \in \text{IN} \]

\[ +, \cdot, S \text{ are usual plus, times, successor functions} \]

Jumping ahead a bit: Evaluation of a formula in \[ \text{IN} \]

\[ \forall x \forall z \forall z' \exists z^* (x = 1 + 3' \land (o = 3' \wedge z = x)) \rightarrow \exists z^* (s z + z^* = x) \]
Definition: Evaluation of terms/formulas on $M, s$

Let $M$ be an $L$-structure, $s$ an object assignment for $M$

Evaluation of terms over $M, s$

(a) $x^m_s \in s(x)$ for all variables $x$

(b) $(f t_1 \ldots t_n)^m_s = f^m(t_1^m s, \ldots, t_n^m s)$
Evaluation of formulas over $\mathcal{M}, \sigma$

Let $A$ be an $\mathcal{L}$-formula. $\mathcal{M} \models A[\sigma]$ (if $\mathcal{M}$ satisfies $A$ under $\sigma$) iff

(a) $\mathcal{M} \models P_{t_1} \ldots P_{t_n}[\sigma]$ iff $\langle t_1^\mathcal{M}[\sigma], \ldots, t_n^\mathcal{M}[\sigma] \rangle \in P^\mathcal{M}$

(b) $\mathcal{M} \models (s = t)[\sigma]$ iff $s^\mathcal{M}[\sigma] = t^\mathcal{M}[\sigma]$

(c) $\mathcal{M} \models \neg A[\sigma]$ iff not $\mathcal{M} \models A[\sigma]$

(d) $\mathcal{M} \models (A \lor B)[\sigma]$ iff $\mathcal{M} \models A[\sigma]$ or $\mathcal{M} \models B[\sigma]$

(e) $\mathcal{M} \models (A \land B)[\sigma]$ iff $\mathcal{M} \models A[\sigma]$ and $\mathcal{M} \models B[\sigma]$

(f) $\mathcal{M} \models \forall x A[\sigma]$ iff $\forall m \in M \mathcal{M} \models A[\sigma](\sigma(x)^M)$

(g) $\mathcal{M} \models \exists x A[\sigma]$ iff $\exists m \in M \mathcal{M} \models A[\sigma](\sigma(x)^M)$
Example \( \mathcal{L} = \{ ; \}, \mathcal{R}, = \)

\[ M = ( \mathbb{N}, \leq, =, \mathcal{R} ) \]

\[ \mathcal{R}^M(m, n) \text{ if } m \leq n \]

Then

\[ M \models \forall x \exists y \mathcal{R}(x, y) \]

\[ M \not\models \exists y \forall x \mathcal{R}(x, y) \]

\[ \text{but } \exists y \forall x \mathcal{R}(x, y) \text{ is also satisfiable} \]
IMPORTANT DEFINITIONS

1. A is satisfiable iff there exists a model $M$ and an object assignment $g$ such that $M \models A[g]$

2. A set of formulas $\Phi$ is satisfiable iff $\exists M, g$ such that $M \models \Phi[g]$ [for all $A \in \Phi$]

3. $\Phi \models A$ (A is a logical consequence of $\Phi$) iff $\forall M, g$, if $M \models \Phi[g]$ then $M \models A[g]$

4. $\models A$ (A is valid) iff $\forall M, g$, $M \models A[g]$
Lines are again sequents

\[ A_1, \ldots, A_k \rightarrow B_1, \ldots, B_e \ \} \ S \]

where each \( A_i, B_j \) is a proper \( \mathcal{L} \)-formula

\[ A_s : A_1 \land A_2 \land \ldots \land A_k \Rightarrow B_1 \lor \ldots \lor B_e \]
FIRST ORDER SEQUENT CALCULUS LK

Lines are again sequents

\[ A_1, \ldots, A_k \rightarrow B_1, \ldots, B_e \]

where each \( A_i, B_j \) is a proper L-formula

RULES

OLD RULES OF PK

PLUS NEW RULES FOR \( \forall, \exists \)

like a large

AND

Large OR
New Logical Rules for $\forall, \exists$

$\forall$-left: \[ \frac{A(t), \Gamma \rightarrow \Delta}{\forall x A(x), \Gamma \rightarrow \Delta} \]

$\forall$-right: \[ \frac{\Gamma \rightarrow \Delta, A(b)}{\Gamma \rightarrow \Delta, \forall x A(x)} \]

$\exists$-left: \[ \frac{A(b), \Gamma \rightarrow \Delta}{\exists x A(x), \Gamma \rightarrow \Delta} \]

$\exists$-right: \[ \frac{\Gamma \rightarrow \Delta, A(t)}{\Gamma \rightarrow \Delta, \exists x A(x)} \]

* $A, t$ are proper
* $b$ is a free variable not appearing in lower sequent of rule
Example of an LK proof

\[\begin{align*}
\text{Pa} &\Rightarrow \text{Pa} \\
\hline
\text{Pa, Qa} &\Rightarrow \text{Pa} \\
\hline
\text{Pa \land Qa} &\Rightarrow \text{Pa}
\end{align*}\]

\[\begin{align*}
\text{Qa} &\Rightarrow \text{Qa} \\
\hline
\text{Pa, Qa} &\Rightarrow \text{Qa} \\
\hline
\text{Pa \land Qa} &\Rightarrow \text{Qa}
\end{align*}\]

\[\begin{align*}
\text{Pa \land Qa} &\Rightarrow \exists x P_x \\
\hline
\exists x (P_x \land Q_x) &\Rightarrow \exists x P_x
\end{align*}\]

\[\begin{align*}
\text{Pa \land Qa} &\Rightarrow \exists x Q_x \\
\hline
\exists x (P_x \land Q_x) &\Rightarrow \exists x Q_x
\end{align*}\]

\[\exists x (P_x \land Q_x) \Rightarrow \exists x P_x \land \exists x Q_x\]
Theorem (LK Soundness)

Every sequent provable in LK is valid

Proof by induction on the number of sequents in proof.

Axiom $A \rightarrow A$ is valid

Induction step: use previous soundness lemma
Soundness (Proof): By induction on the number of segments in proof.

Example: \( \exists \text{Left} \)

Assume: \( A(c), P \Rightarrow \Delta \) has an \( LK \) proof and is valid.

Show: \( \exists \forall A(x), n \Rightarrow \Delta \) also valid.

By defn. \( A(c) \land \Pi \land \ldots \land \Pi_k \land \Delta \land \ldots \land \Delta_k \) is valid.

Let \( M \) be any structure, \( s \) any object assignment.

Show: \( M \models \neg \exists \forall A(x) \land \Pi \land \ldots \land \Pi_k \land \Delta \land \ldots \land \Delta_k [6] \) (4)

Case 1: \( M \models \Pi \land \ldots \land \Pi_k \land \Delta \land \ldots \land \Delta_k [6] \). Then (4) holds.

Case 2: Case 1 does not hold.
**Soundness (Proof):** By induction on the number of sequents in proof

**Example:** $\exists$ Left

**Assume:** $A(c), \Gamma \Rightarrow \Delta$ has an $LK$ proof and is valid

**Show:** $\exists x \ A(x), \Gamma \Rightarrow \Delta$ also valid

By defn $\overline{A(c) \lor \bar{\bar{\ldots}} \lor \bar{\bar{\ldots}} \lor \Delta_k}$ is valid

Let $M$ be any structure, $\sigma$ any object assignment.

**Show:** $M \models \exists x \ A(x), \Gamma \Rightarrow \Delta_k \ [G] \quad (4)$

**Case 1** $M \models \bar{\bar{\ldots}} \lor \Delta_k \ [G]$. Then (4) holds

**Case 2** Case 1 does not hold.

Since $b$ does not occur in $\Gamma$ or $\Delta$,

$M \models \bar{\bar{\ldots}} \lor \Delta_k \ [G (\overline{\overline{b}})]$ for all $n \in M$

Since $A(c), \Gamma \Rightarrow \Delta$ is valid, $M \models \overline{A(b) \ [G \overline{\overline{c}}]} \ \forall n \in M$

Thus $M \models \exists x \ A(x) \ [G]$; $\therefore \exists x \ A(x), \Gamma \Rightarrow \Delta$ is valid.
TODAY: gödel's completeness theorem

**Defn** An LK-\( \Phi \) proof is an LK-proof, but leaves are either axioms (A\( \rightarrow \)A) or of the form \( \rightarrow A \) for A\( \in \Phi \).

**Goal** Prove that if \( \Gamma \rightarrow \Delta \) is a logical consequence of \( \Phi \), then there is an LK-\( \Phi \) proof of \( \Gamma \rightarrow \Delta \) (called Derivational completeness).

**Defn** Let A\( (a_1...a_n) \) be a formula with free variables \( a_1...a_n \). Then \( \forall A \) is \( \forall x_1 \forall x_2...\forall x_n A(x_1...x_n) \) (called universal closure of A).
TODAY: LK COMPLETENESS

(MAIN LEMMA) completeness Lemma

If $\Gamma \Rightarrow \Delta$ is a logical consequence of a set of (possibly infinite) formulas $\forall \Phi$
then there exists a finite subset $\exists C_1, \ldots, C_n \subseteq \Phi$ such that

$$\forall C_1, \ldots, \forall C_n, \Gamma \Rightarrow \Delta$$

has a (cut-free) PK proof

* We will assume $=$ not in Language for now
Derivational Completeness Theorem

Let $\Phi$ be a set of sequents or formulas such that the sequent $\Gamma \rightarrow \Delta$ is a logical consequence of $\forall \Phi$. Then there is an $LK-\Phi$ proof of $\Gamma \rightarrow \Delta$.

Proof follows from Completeness Lemma (similar to derivational completeness of PK from completeness)
Proof of LK Completeness Lemma

High level idea (assume \( \emptyset \) is empty for now)

- As in PK completeness, we want to construct an LK proof in reverse.
- Start with \( \Gamma \Rightarrow \Delta \) at root, and apply rules in reverse (to break up a formula into one or 2 smaller ones)
- Tricky rules are \( \exists \text{right} \) and \( \Delta \text{left} \). When applying one of these in reverse, need to "guess" a term
New Logical Rules for $\forall, \exists$

$\forall$-left: $\frac{A(t), \Gamma \Rightarrow \Delta}{\forall x \ A(x), \Gamma \Rightarrow \Delta}$

$\forall$-right: $\frac{\Gamma \Rightarrow \Delta, A(b)}{\Gamma \Rightarrow \Delta, \forall x \ A(x)}$

$\exists$-left: $\frac{A(b), \Gamma \Rightarrow \Delta}{\exists x \ A(x), \Gamma \Rightarrow \Delta}$

$\exists$-right: $\frac{\Gamma \Rightarrow \Delta, A(t)}{\Gamma \Rightarrow \Delta, \exists x \ A(x)}$

* $A$, $t$ are proper
* $b$ is a free variable not appearing in lower sequent of rule
Proof of LK Completeness Lemma

High Level idea (assume $\emptyset$ is empty for now)

- As in PK completeness, we want to construct an LK proof in reverse.
- Start with $\Gamma \Rightarrow \Delta$ at root, and apply rules in reverse (to break up a formula into one or 2 smaller ones).
- Tricky rules are $\exists$ (right) + $\forall$ (left). When applying one of these in reverse, need to "guess" a term.
- Key is to systematically try all possible terms — without going down a rabbit hole.
Example of an LK proof

\[
\begin{align*}
Pa \rightarrow Pa \\
\hline
Pa, Qa \rightarrow Pa \\
\hline
Pa \land Qa \rightarrow Pa \\
\hline
Pa \land Qa \rightarrow \exists x P_x
\end{align*}
\]

\[
\begin{align*}
Qa \rightarrow Qa \\
\hline
Pa, Qa \rightarrow Qa \\
\hline
Pa \land Qa \rightarrow Qa \\
\hline
Pa \land Qa \rightarrow \exists x Q_x
\end{align*}
\]

\[
\exists x (P_x \land Q_x) \rightarrow \exists x P_x
\]

\[
\exists x (P_x \land Q_x) \rightarrow \exists x Q_x
\]

\[
\exists x (P_x \land Q_x) \rightarrow \exists x P_x \land \exists x Q_x
\]
Example of an LK proof

\( Pa \land Qa \rightarrow Pb \)

\( Pa \land Qa \rightarrow Pb \)

\( Pa \land Qa \rightarrow \exists x R_x \quad \exists x (R_x \land Q_x) \rightarrow \exists x Q_x \)

\( \exists x (R_x \land Q_x) \rightarrow \exists x R_x \land \exists x Q_x \)
Instead:

\[ P_a, Q_a \rightarrow P_b, \exists x P_x \]

\[ P_a \land Q_a \rightarrow P_b, \exists x P_x \]

\[ P_a \land Q_a \rightarrow \exists x P_x \]

\[ \exists x (P_x \land Q_x) \rightarrow \exists x P_x \]

\[ \exists x (P_x \land Q_x) \rightarrow \exists x Q_x \]

\[ \exists x (P_x \land Q_x) \rightarrow \exists x P_x \land \exists x Q_x \]
Instead

Try again

\[ \begin{align*}
P_a, Q_a & \rightarrow P_b, P_f a, \exists x P_x \\
\therefore P_a, Q_a & \rightarrow P_b, \exists x P_x
\end{align*} \]

\[ \begin{align*}
P_a \land Q_a & \rightarrow P_b, \exists x P_x \\
P_a \land Q_a & \rightarrow \exists x P_x \\
\exists x (P_x \land Q_x) & \rightarrow \exists x P_x
\end{align*} \]

\[ \begin{align*}
P_a \land Q_a & \rightarrow \exists x Q_x \\
\exists x (P_x \land Q_x) & \rightarrow \exists x Q_x \\
\exists x (P_x \land Q_x) & \rightarrow \exists x P_x \land \exists x Q_x
\end{align*} \]
Instead

\[ Pa, Qa \rightarrow Pb, Pfa, Pf_b, \exists x P_x \]

Try again

\[ Pa, Qa \rightarrow Pb, \exists x P_x \]

Try again

\[ Pa^\land Qa \rightarrow Pb, \exists x P_x \]

\[ Pa^\land Qa \rightarrow \exists x P_x \]

\[ Pa^\land Qa \rightarrow \exists x Q_x \]

\[ \exists x (P_x \land Q_x) \rightarrow \exists x P_x \]

\[ \exists x (P_x \land Q_x) \rightarrow \exists x Q_x \]

\[ \exists x (P_x \land Q_x) \rightarrow \exists x P_x \land \exists x Q_x \]
Instead

There are infinitely many choices! Need a systematic way to try all

\[
\begin{align*}
\text{Pa, Qa} & \rightarrow \text{Pb, Pfa, Pb, } \exists x \text{Px} \\
\text{Pa, Qa} & \rightarrow \text{Pb, } \exists x \text{Px} \\
\text{Pa} \land \text{Qa} & \rightarrow \text{Pb, } \exists x \text{Px} \\
\text{Pa} \land \text{Qa} & \rightarrow \exists x \text{Px} \\
\exists(x \land Qx) & \rightarrow \exists x \text{Px} \quad \exists(x \land Qx) & \rightarrow \exists x \text{Qx} \\
\exists(x \land Qx) & \rightarrow \exists x \text{Px} \land \exists x \text{Qx}
\end{align*}
\]
Instead

There are infinitely many choices! Need a systematic way to try all and for all frontier sequents in current proof!

\[ \text{Try again} \]

\[ \{ \text{ParQa} \rightarrow P_{\text{F}}xP_{\text{x}} \}
   \]

\[ \text{Pa, Qa} \rightarrow P_{\text{b}}, P_{\text{f}}, P_{\text{f}}, P_{\text{f}}, \exists \text{x} \text{P}_x \]

\[ \text{Pa, Qa} \rightarrow P_{\text{b}}, \exists \text{x} \text{P}_x \]

\[ \text{Pa, Qa} \rightarrow P_{\text{b}}, \exists \text{x} \text{P}_x \]

\[ \text{Pa, Qa} \rightarrow \exists \text{x} \text{P}_x \]

\[ \exists \text{x}(P_{\text{x}} \land Q_{\text{x}}) \rightarrow \exists \text{x} \text{P}_x \]

\[ \exists \text{x}(P_{\text{x}} \land Q_{\text{x}}) \rightarrow \exists \text{x} \text{P}_x \]

\[ \exists \text{x}(P_{\text{x}} \land Q_{\text{x}}) \rightarrow \exists \text{x} \text{P}_x \land \exists \text{x} \text{Q}_x \]
Completeness: Proof Search Algorithm

Enumeration of formulas + terms:

Since the number of underlying symbols of \( L \) is finite, there is an enumeration of pairs \( \langle A_i, t_i \rangle, \langle A_2, t_2 \rangle, \langle A_3, t_3 \rangle, \ldots \) such that every term and every formula in \( L \) occur infinitely often in the enumeration.
More details of enumeration (L finite)

Enumerate all L-formulas $A_1, A_2, \ldots$
Enumerate L-terms $t_1, \ldots$

such that every formula/term occurs infinitely often

Enumerate all pairs to have same property

\begin{array}{cccc}
A_1 & A_2 & A_3 & A_4 \\
\hline \\
t_1 \\
t_2 \\
t_3 \\
\vdots \\
\end{array}
Completeness: Proof Search Algorithm

- Initially $\Pi$ is the sequent $\Gamma \Rightarrow \Delta$
- At each stage, modify $\Pi$ by adding some $A_i \in \phi$ to antecedent of all sequents in $\Pi$, and adding onto the "frontier" or "active" sequents in $\Pi$
- Active sequent: a leaf sequent in $\Pi$, not a weakening of $A \Rightarrow A$
- At stage $k$: we will use the $k^{th}$ pair $<A_k, \Gamma_k>$ in the enumeration
Completeness: Proof Search Algorithm

Stage \( \kappa \): \( \langle A, t \rangle_\kappa \)

1. If \( A_\kappa \in \overline{\Delta} \), replace \( \Gamma' \Rightarrow \Delta \) in \( \Pi \) by \( \Gamma', A_\kappa \Rightarrow \Delta' \)

2. If \( A_\kappa \) atomic, skip this step. Otherwise for all leaf sequents containing \( A_\kappa \), break up outermost connective of \( A_\kappa \) using the appropriate logical rule, and \( t_\kappa \) if necessary.
Completeness: Proof Search Algorithm

Stage $k$:

1. If $A_k \notin \Phi$, replace $\Gamma \Rightarrow \Delta$ in $\Pi$ by $\Gamma', A_k \Rightarrow \Delta'$
2. If $A_k$ atomic, skip this step. Otherwise, for all leaf sequents containing $A_k$, break up outermost connective of $A_k$ using the appropriate logical rule, and $t_k$ if necessary.

Examples:
- $A_k = \exists x B(x)$
- $\Gamma, \exists x B(x) \Rightarrow \Delta$
- $\Gamma, \exists x B(x) \Rightarrow \Delta$
- $\Gamma \Rightarrow \Delta, \exists x B(x), B(t_k)$
- $\Gamma \Rightarrow \Delta, \exists x B(x)$
Stage $k$:

1. If $A_k \in \bar{\Phi}$, replace $\Gamma \Rightarrow \Delta$ in $\Pi$ by $\Gamma', A_k \Rightarrow \Delta'$

2. If $A_k$ atomic, skip this step. Otherwise, for all leaf sequents containing $A_k$, break up outermost connective of $A_k$ using the appropriate logical rule, and $t_k$ if necessary.

Examples:
- $A_k = \forall x \ B(x)$

\[ \frac{\Gamma \Rightarrow \Delta, \ B(c)}{\Gamma \Rightarrow \Delta, \ \forall x \ B(x)} \]

\[ \frac{B(t_k), \ \forall x \ B(x), \ \Gamma \Rightarrow \Delta}{\Gamma, \ \forall x B(x) \Rightarrow \Delta} \]

C a new variable

Keep both here

Exit when no more active sequents
Proof of correctness

We want to show:

- If Algorithm halts, \( T \) is an LK-\( \Phi \) proof of \( \forall \psi_1, \ldots, \forall \psi_n \psi \Rightarrow \Delta \)  

- If Algorithm never halts, then  
  \( \forall \Phi \, \exists \, \Gamma \Rightarrow \Delta \)
Proof of correctness

We want to show: If Algorithm never halts, then $\forall \Phi \in \Gamma \rightarrow \Delta$

Suppose Algorithm doesn’t halt and let $T$ be the (typically infinite) tree that results

Leaf “sequents” of $T$ look like $\Gamma', C_1, C_2, \ldots \rightarrow \Delta'$

infinite sequence containing all of $\Phi$ each infinitely often

Find a bad path $\beta$ in the tree:

If $T$ finite, $\exists$ some active leaf node containing only atomic formulas. Choose $\beta$ to be path from root to this leaf
Proof of correctness

We want to show: If Algorithm never halts, then $\forall \Phi \in \mathcal{P} \rightarrow \Delta$

Find a bad path $\beta$ in the tree:

If $T$ finite, $\exists$ some active leaf node containing only atomic formulas. Choose $\beta$ to be path from root to this leaf.

If $T$ infinite by König's Lemma, $\exists$ an infinite path. Let $\beta$ be this path.
Proof of correctness

Properties of \( \beta \)

1. \( \beta \) is a path starting at root
2. All sequents in \( \beta \) were once active
3. For all sequents in \( \beta \), no formula occurs on both the left and right side of sequent
4. All atomic formulas \( \mathsf{A} \in \Phi \) in root sequent of \( \beta \) on LEFT, and thus occur on LEFT of all sequents in \( \beta \)

By (3)+(4), we know that no atomic \( \mathsf{A} \in \Phi \) occurs on the Right of any sequent in \( \beta \)
Proof of correctness (cont'd)

We will construct a "term" model $M$, $\Phi$ object assignment $G$ from $\beta$ such that $M \models \Phi \cup G$ but $M \not\models \Gamma \rightarrow \Delta$ (and thus our algorithm fails to halt and produce a proof only when $\Gamma \rightarrow \Delta$ is not a logical consequence of $\Phi$.)
Proof of Correctness (cont'd)

We will construct a "term" model $\mathcal{M}$, an object assignment $G$ from $\beta$ such that $\mathcal{M} \models \Phi (G)$ but $\mathcal{M} \not\models \Gamma \Rightarrow \Delta$

Universe $M$: all $L$-terms $t$ (containing only free vars)

$G$: map variable $a$ to itself

\[
\begin{align*}
\varphi^m (r_1, \ldots, r_k) & \overset{d}{=} f r_1 \ldots r_k \\
p^m (r_1, \ldots, r_k) & \overset{d}{=} \text{true if and only if } Pr_1^{r_k}
\end{align*}
\]

is on the LEFT of some sequent in $\beta$
Claim: For every formula $A,$

$M, \varepsilon$ satisfies $A$ iff $A$ is on the LEFT of some sequent in $\beta$, and

$M, \varepsilon$ falsifies $A$ iff $A$ is on the RIGHT of some sequent in $\beta$
Claim: For every formula $A$,
- $M, \sigma$ satisfies $A$ iff $A$ is on the LEFT of some sequent in $\beta$, and
- $M, \sigma$ falsifies $A$ iff $A$ is on the RIGHT of some sequent in $\beta$

Proof (induction on $A$)

A atomic: $A$ cannot occur on LEFT of some sequent in $\beta$ and on RIGHT of some sequent in $\beta$ (since $A$ persists up $\beta$)
Proof of correctness (cont’d)

Claim: For every formula $A$,

- $M, \sigma$ satisfies $A$ iff $A$ is on the LEFT of some sequent in $\beta$, and
- $M, \sigma$ falsifies $A$ iff $A$ is on the RIGHT of some sequent in $\beta$

Proof (induction on $A$)

Induction Step  Example $A = \exists x B(x)$ on RIGHT

High level: 'If $A$ occurs in some sequent in $\beta$, then $A$ persists upward until it becomes the active formula (at stage $K$, $A_K = A$) then use inductive hypothesis.'
Proof of correctness (cont'd)

Claim: For every formula $A$,

1. $M, e$ satisfies $A$ iff $A$ is on the LEFT of some sequent in $\beta$, and
2. $M, e$ falsifies $A$ iff $A$ is on the RIGHT of some sequent in $\beta$.

Proof (induction on $A$)

**Induction Step** $A = \exists x B(x)$ on RIGHT

By Ind hyp, $M, e$ falsify $B(t_j)$.

Since $\exists x B(x)$ persists, we have $\forall t B(t)$ on RIGHT of some sequent in $\beta$.

Thus $M, e$ falsify $B(t)$ for all terms $t$.