Week 11

- Test II: Thursday 3-5 pm
- Extra office hrs posted
- See course webpage for practice problems
Today:

1. A specific sentence "I am not provable" = \( g \) such that neither \( g \) nor \( \neg g \) are provable in PA (assuming PA is consistent)

2. Consistency of PA, \( \text{Con}(PA) \) is not provable in PA (assuming PA is consistent)
• Let $\Gamma_{PA}$ be the set of axioms of PA

• Let $\text{Proof}(x,y)$: true if and only if $y$ codes a $\text{LK-}\Gamma_{PA}$ proof of the sentence coded by $x$

• Recall $d(n) = \#A(s_n)$ where $\#A(x) = n$

  ($so n codes the formula A(x),$ and $d(n)$ codes $A(s_n)$ )
Let \( \Gamma_{PA} \) be the set of axioms of PA

Let \( \text{Proof}(x, y) \) be true if and only if \( y \) codes a LK-\( \Gamma_{PA} \) proof of the sentence coded by \( x \)

Recall \( d(n) = \#A(s_n) \) where \( \#A(x) = n \)

\( (\text{so } n \text{ codes the formula } A(x), \text{ and } d(n) \text{ codes } A(s_n) ) \)

Let \( s(x) \) be the r.e. relation: \( \exists y \text{ Proof}(d(x), y) \)
Let $\Gamma_{PA}$ be the set of axioms of PA

Let $\text{Proof}(x, y)$: true if and only if $y$ codes a $\text{LK-}\Gamma_{PA}$ proof of the sentence coded by $x$

Recall $d(n) = \#A(s_n)$ where $\#A(x) = n$

( so $n$ codes the formula $A(x)$, and $d(n)$ codes $A(s_n)$ )

Let $S(x)$ be the r.e. relation: $\exists y \text{Proof}(d(x), y)$

By RA representation Theorem, let $A(x)$ be a $\exists \Delta_0$ formula that represents $S(x)$ in RA (so hence in PA)

Then $\forall n \in \mathbb{N} \exists y \text{Proof}(d(n), y) \iff \text{PA} \vdash A(s_n)$ (⋆)
Recall \( d(n) = \#A(s_n) \) where \( \#A(x) = n \)

(\( \) so \( n \) codes the formula \( A(x) \), and \( d(n) \) codes \( A(s_n) \) \( ) \)

Let \( S(x) \) be the r.e. relation: \( \exists y \) Proof\((d(x), y)\)

By RA representation Theorem, let \( A(x) \) be a \( \exists \Delta_0 \)

formula that represents \( S(x) \) in RA (\( d \) hence in PA)

Let \( e = \# \neg A(x) \), so \( d(e) = \# \neg A(s_e) \)

Let \( q \models \neg A(s_e) \)

\( \) says that “I am not provable”

since \( \neg A(s_e) \) says the formula encoded
by \( d(e) \) -- which is \( q \) -- is not provable in PA
• Let \( S(x) \) be the r.e. relation: \( \exists y \) Proof\((d(x), y)\)

• By RA representation Theorem, let \( A(x) \) be a \( \exists \Delta_0 \)
  formula that represents \( S(x) \) in RA (so hence in PA)

• Let \( e = \# \neg A(x) \), so \( d(e) = \# \neg A(Se) \)

• Let \( g \equiv \neg A(Se) \)

**Theorem** \( \text{PA consistent} \implies \text{PA} \vdash g \)

**PF** suppose \( \text{PA} \vdash g \)

Then sentence number \( d(e) \) is provable, so \( \exists y \) Proof\((d(e), y)\)
holds

Thus \( \text{PA} \vdash A(Se) \) by left-to-right direction of \((*)\)

Thus \( \text{PA} \vdash g \) and \( \text{PA} \vdash \neg g \) so \( \text{PA} \) not consistent
• Let \( s(x) \) be the r.e. relation: \( \exists y \text{ Proof}(d(x),y) \)

• By RA representation Theorem, let \( A(x) \) be a \( \exists \Delta_0 \)
  formula that represents \( s(x) \) in RA (so hence in PA)

• Let \( e = \# \neg A(x) \), so \( d(e) = \# \neg A(Se) \)

• Let \( g \equiv \neg A(Se) \)

**Theorem** PA consistent \( \Rightarrow \) PA \( \vdash \neg g \)

**PF** suppose \( \text{PA} \vdash \neg g \) i.e. \( \text{PA} \) proves \( A(Se) \)

Then \( \exists y \text{ Proof}(d(e),y) \) by rt-to-left direction \( g \) (\( \ast \))

So \( \text{PA} \) proves \( \neg A(Se) \)

So \( \text{PA} \vdash g \) and \( \text{PA} \vdash \neg g \), so \( \text{PA not consistent} \)
Formulating consistency in PA

Let $B(x,y)$ be a $\exists \Delta_0$ formula that represents $\text{Proof}(x,y)$ in RA (and thus also in PA).

Then for every sentence $C$:

$$ PA \vdash C \iff PA \vdash \exists y \ B(\#C, y) $$

stands for $B(S^*_k, y)$

Then $PA \vdash A(S_n) = \exists y \ B(S_{dn}(n), y)$

[recall $A(x)$ represents $\exists y \ B(d(x), y)$]
Formulating consistency in PA

Let $B(x,y)$ be a $\exists \Delta_0$ formula that represents $\text{Proof}(x,y)$ in RA (and thus also in PA)

Then for every sentence $C$

$$\text{PA} \vdash C \iff \text{PA} \vdash \exists y B(\#C, y)$$

stands for $B(S_{\#}, y)$

Then $\text{PA} \vdash A(s_n) = \exists y B(s_{dn^e}, y)$

[recall $A(x)$ represents $\exists y B(d(x), y)$]

Define $\text{con}(\text{PA}) \overset{d}{=} \neg \exists y B(\#0 \neq 0, y)$
Theorem: If PA is consistent, then \( PA \not\vdash \text{con}(PA) \)

Proof:

Main Lemma: \( PA \vdash (\text{con}(PA) \Rightarrow q) \)

[recall \( q \triangleq \neg \varphi(e) \), \( e = \# \# \varphi(x) \) says \\
"\# \varphi(x) is not provable"
]

If \( PA \vdash \text{con}(PA) \) by main lemma, \( PA \vdash q \)

But by previous theorem

\( PA \) consistent \( \Rightarrow \) \( PA \not\vdash q \)

\( \therefore \) \( PA \) consistent \( \Rightarrow \) \( PA \not\vdash \text{con}(PA) \)
It is left to prove:

**Main Lemma**: $\text{PA} \vdash \text{con}(\text{PA}) \Rightarrow g$

(recall $g \equiv \neg A(s_e)$, $e = \# \neg A(x)$ says

"$I$ am not provable"

Need to formalize proof of Gödel's Incompleteness Thm in PA. Main step is to formalize in PA that every true $\exists \sigma$ sentence is provable in RA.
Review for Test II

1. Completeness of LK
   derivational completeness

2. Computability: diagonalization
   recursive / r.e.
   recursive, re not rec, not re

$D \xleftarrow{\text{Halp}}$
3. Incompleteness.

- Define: consistent, sound, axiomatizable

- A relation $R(x)$ is represented by a (EA) formula $A(x)$

$$\forall a \in N \ R(a) \iff TA \models A(S_a)$$

- A relation $R(x)$ is represented in $\Sigma$ by $N(x)$

$$\forall a \in N \ R(a) \iff \Sigma \vdash \neg A(S_a)$$
Theorem

every $\exists x$ is represented by a $\exists \Delta_0$ formula

every $\exists \Delta_0$ formula is r.e.

Corollaries

Cor 2: TA is not axiomatizable

(Tarski's thm: TA is not arithmetical so not r.e.)

Cor 3: every sound axiom theory is incomplete
PA, RA

RA represent. Thm

every r.e. relation is represented in RA by \exists \forall formula.

Strong Rep Thm

every recursive reln. is strongly represented in RA by \exists \forall formula

Strongly:

\[ R(\bar{a}) \Rightarrow RA \vdash A(S_{\bar{a}}) \]
\[ \neg R(\bar{a}) \Rightarrow RA \vdash \neg A(S_{\bar{a}}) \]
Corollary every consistent extension of RA is undecidable.

Today

1. $PA \not\vdash g$, $PA \vdash \neg g$

So $PA$ is incomplete (assuming $PA$ is consistent).

$\Rightarrow$ 2. $PA \not\vdash \text{cons}(PA)$
5 questions (~65 pts)

20 - 25 computability

rest one q on PA prob I someone

rest (other half)

incompleteness / defs

PA axioms, equal axioms