Welcome to CSC 438/2404!

Instructor: Toniann Pitassi (Toni)
TA: Noah Fleming
Brief Bio

I received bachelors and masters degrees from Pennsylvania State University and then received a PhD from the University of Toronto in 1992. After that, I spent 2 years as a postdoc at UCSD, and then 2 years as an assistant professor (in mathematics with a joint appointment in computer science) at the University of Pittsburgh. For the next four years, I was a faculty member of the Computer Science Department at the University of Arizona. In the fall of 2001, I moved back to Toronto, where I am currently a professor in the Computer Science Department, with a joint appointment in Mathematics.

The above picture was taken in London in front of Bertrand Russell's flat. If you click on the picture to see an enlarged version, and then go to the upper right quadrant, the blue sign mentioning this landmark will be legible.
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CSC 438F/2404F: Computability and Logic  
Fall, 2019

ANNOUNCEMENTS: (Students, please check for announcements every week.)

    Posted on Aug 24: The first class is Monday Sept 9, 2019.

COURSE TIMES, CONTACT INFO

    Instructor: Toniann Pitassi, email: toni@cs
    Office Hours: Monday 5:15-6pm, Sandford Fleming 2305A
    Lectures: Monday 3-5 BA 1200
    Tutorial: Friday 12-1 BA 1200

    Tutor: Noah Fleming, noahfleming@cs
    Noah's Office Hours: to be announced soon

• Course Information Sheet

HOMEWORK ASSIGNMENTS:

• Homework 1, Coming Soon

GRADES AND MARKING:

• Coming Soon

COURSE NOTES:

• Propositional Calculus
• Predicate Calculus
• Completeness
• Herbrand, Equality, Compactness
Exclusions: MAT 309H1, PHL348H1
Prerequisites (ugrads): (CSC363H1/CSC463H1)/CSC365H1/CSC373H1/CSC375H1/MAT247H1
Lectures: Monday 3-5, BA 1200
Tutorial: Friday 12-1, BA 1200
Instructor: Toniann Pitassi, toni@cs.toronto.edu
Office hours: Monday 5:10-6, SF2305A
Tutor: Noah Fleming, SF 4306, noahfleming@cs.toronto.edu

Course Notes: Postscript files for course notes and all course handouts will be available on the web page.

Topics:

Marking Scheme:
Class attendance/participation (2% of final grade)
4 assignments (each worth 12% of final grade)
First Term test (25% of final grade)
Second Term Test (25% of final grade)

Due Dates:
First Term Test: Monday Oct 21, 3-5pm BA 1200
Second Term Test: Thursday Dec 5, 3-5pm BA 1200
Assignment 1 due date: Friday Sept 27 12pm, before tutorial
Assignment 2 due date: Friday Oct 18 12pm, before tutorial
Assignment 3 due date: Friday Nov 1 12pm, before tutorial
Assignment 4 due date: Friday Nov 29 12pm, before tutorial

Assignments are due at the beginning of class, since solutions will be discussed during the beginning of class/tutorial.

The work you submit must be your own. You may discuss problems with each other; however, you should prepare written solutions alone. Copying assignments is a serious academic offence and will be dealt with accordingly.
Supplementary References:

H.B. Enderton, A Mathematical Introduction to Logic (undergrad)
G Boolos and R.C. Jeffrey, Computability and Logic (undergrad)
E. Mendelson, Introduction to Mathematical Logic, 3rd edition (undergrad/ grad)
J.N. Crossley and others, What is Mathematical Logic? (informal, readable)
A.J.Kfoury, R.Moll, and M. Arbib, A Programming Approach to Computability (undergrad)
M.Davis, R. Sigal, and E. Weyuker, Computability, Complexity, and Languages: Fundamentals of Theoretical Computer Science (undergrad/grad)
Important

→ All lectures and tutorials are mandatory.
Sometimes Friday 12-1 will be a lecture, other times a tutorial

→ All assignments due at start of lecture/tutorial
Late assignments not accepted

→ You may discuss your solutions with other students in the current course.
Discussing with anyone outside course or consulting web is prohibited
- Work hard on understanding lecture notes, work hard on assignments.
- Start early -- cannot cram/solve in a couple of days.
- Come to office hrs!
- Writeups must be completed independently.
Course Intro

Foundations of mathematics involves the axiomatic method - write down axioms (basic truths) and prove theorems from axioms from purely logical reasoning.
Example 1  Euclidean geometry (300 BC, "Elements")

Axiomatic system where all theorems are derivable from a small number of simple axioms/postulates

Postulate 5

If sum of $\angle d + \angle f$ is $< 180$ then the 2 lines (blue + yellow) eventually meet (on same side as $\angle a$, $\angle b$ angles)

The School of Athens, Rafael
Example 2 - Group Theory (Cayley, 1854)

axiom 1: \( \forall x y z \ [x \cdot (y \cdot z) = (x \cdot y) \cdot z] \) (associativity)

axiom 2: \( \exists u \)

\[ \forall x \ [x \cdot u = u \cdot x = u] \ \land \ \forall x \exists y \ [x \cdot y = y \cdot x = u] \]

there exists an identity element and every element has an inverse

A group is a model for the axioms

\((G, \cdot)\) a function from \(G \times G \to G\)

a set
Examples of groups

1. \( g = \mathbb{Z} \) (the integers) \( \cdot = \) addition
Examples of groups

1. $g = \mathbb{Z}$ (the integers) $\cdot = \text{addition}$

2. Rubik's cube group

<table>
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<tr>
<th>Basic 90°</th>
<th>180°</th>
<th>-90°</th>
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<tr>
<td>$F$ turns the front clockwise</td>
<td>$F^2$ turns the front clockwise twice</td>
<td>$F'$ turns the front counter-clockwise</td>
</tr>
<tr>
<td>$B$ turns the back clockwise</td>
<td>$B^2$ turns the back clockwise twice</td>
<td>$B'$ turns the back counter-clockwise</td>
</tr>
<tr>
<td>$U$ turns the top clockwise</td>
<td>$U^2$ turns the top clockwise twice</td>
<td>$U'$ turns the top counter-clockwise</td>
</tr>
<tr>
<td>$D$ turns the bottom clockwise</td>
<td>$D^2$ turns the bottom clockwise twice</td>
<td>$D'$ turns the bottom counter-clockwise</td>
</tr>
<tr>
<td>$L$ turns the left face clockwise</td>
<td>$L^2$ turns the left face clockwise twice</td>
<td>$L'$ turns the left face counter-clockwise</td>
</tr>
<tr>
<td>$R$ turns the right face clockwise</td>
<td>$R^2$ turns the right face clockwise twice</td>
<td>$R'$ turns the right face counter-clockwise</td>
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$g = \text{all possible moves}$

$\cdot = \text{composition of moves}$
Course Outline

We will study First Order Logic (Predicate Logic).

I. Start with simpler Propositional Logic (no quantifiers)
   • Language of propositional logic ("syntax")
   • Meaning ("semantics")
   • Two proof systems for prop. logic: Resolution, and PK
   • We will prove Soundness + Completeness for both
II. FIRST ORDER (PREDICATE) LOGIC

- Language ("syntax")
- Meaning ("semantics")
- Proof system LK (extends PK)

  Soundness

  ** Completeness

  Major corollaries of Completeness
COURSE OUTLINE (cont'd)

III. Computability

IV. Axiomatizable Theories

Incompleteness Theorems
Interplay/connections between computability & logic
**PROPOSITIONAL LOGIC**

**Vocabulary:** $P_1, P_2, Q, \ldots$, propositional variables

$\neg, \lor, \land, (, )$

**Examples:**

$((P \lor Q) \lor R)$

$(\neg P \lor \neg Q)$
Inductive Definition of a Propositional Formula

1. Atoms/propositional variables: \( P_1, P_2, \ldots \) are formulas.
2. If \( A \) is a formula, then so is \( \neg A \).
3. If \( A, B \) are formulas, so is \( (A \land B) \).
4. \( \neg \neg A \) is equivalent to \( A \) (Double Negation Law).
\((A \equiv B)\) is shorthand for \((\neg A \lor B)\)
\((A \leftrightarrow B)\) is shorthand for \((\neg A \lor B) \land (\neg B \lor A)\)

A **subformula** of a formula is any substring of \(A\) which itself is a formula.

**Unique Readability Thm** says the grammar for generating formulas is not ambiguous.
Semantics

A truth assignment $T: \{\text{atoms}\} \rightarrow T, F$

Extending $T$ to every formula:

1. $(\neg A)^T = T$ iff $A^T = F$
2. $(A \land B)^T = T$ iff $A^T = T \land B^T = T$
3. $(A \lor B)^T = T$ iff either $A^T = T$ or $B^T = T$

Example
Definitions

\( \tau \) satisfies \( A \) iff \( A^\tau = T \)

\( \tau \) satisfies a set \( \Phi \) of formulas iff
\( \tau \) satisfies \( A \) for all \( A \in \Phi \)

\( \Phi \) is satisfiable iff \( \exists \tau \) that satisfies \( \Phi \)
otherwise \( \Phi \) is unsatisfiable

\( \Phi \vdash A \) (A is a logical consequence of \( \Phi \)) iff
\( \forall \tau [ \tau \text{ satisfies } \Phi \Rightarrow \tau \text{ satisfies } A ] \)

\( \vdash A \) (A is valid or A is a tautology) iff
\( \forall \tau [ \tau \text{ satisfies } A ] \)
Examples

1. \((A \land B) \models (A \lor B)\)

2. \(\models (A \lor \neg A)\)
Some easy facts (check them)

1. If $\Phi \vdash A$ and $\Phi \cup \{A\} \vdash B$ then $\Phi \vdash B$

2. $\Phi \vdash A$ iff $\Phi \cup \{\neg A\}$ is unsatisfiable

3. $A$ is a tautology iff $\neg A$ is unsatisfiable
Equivalence

A and B are equivalent (written $A \iff B$) iff $A \models B$ and $B \models A$

Examples

1. $(A \land B) \iff (B \land A)$
2. $(\neg A \lor B) \iff (\neg B \lor A)$
Resolution: Proof System for Prop Logic

- Resolution is basis for most automated theorem provers.
- Proves that formulas are unsatisfiable (recall $F$ is a tautology iff $\neg F$ is valid).
- Formulas have to be in a special form: CNF

$$(x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_2 \lor x_4) \land (\bar{x}_4) \land (x_1 \lor x_3) \land (x_1)$$
Converting a formula to CNF

- Obvious method (deMorgan) could result in an exponential blowup in size

  Example: \((x_1 \land x_2) \lor (x_3 \land x_4) \lor (x_5 \land x_6) \lor \ldots\)

- Better method: SAT THEOREM
  There is an efficient method to transform any propositional formula \(F\) into a CNF formula \(g\) such that \(F\) is satisfiable iff \(g\) is satisfiable.
SAT THEOREM: proof by example

F: \((Q \land R) \lor \neg Q\)

\[ P_B \]

\[ P_A \]

\[ Q = 1 \]

\[ R = 1 \]

\[ \left( \neg P_B \lor \phi \right) \left( \neg P_B \lor R \right) \left( \neg Q \lor R \lor P_B \right) \]

\[ q: P_B \leftrightarrow (Q \land R) \quad P_A \leftrightarrow P_B \lor \neg Q \lor P_A \]
RESOLUTION

Start with CNF formula $F = C_1 \land C_2 \land \ldots \land C_m$

view $F$ as a set of clauses $\{C_1, C_2, \ldots, C_m\}$

Resolution Rule:

$$(A \lor x), (B \lor \neg x) \Rightarrow (A \lor B)$$

A Resolution Refutation of $F$ is a sequence
of clauses $D_1, D_2, \ldots, D_q$ such that:

each $D_i$ is either a clause from $F$, or follows
from 2 previous clauses by Resolution rule,
and final clause $D_q = \emptyset$ (the empty clause)
Resolution Refutation

\[ F = (a \lor b \lor c) (a \lor \bar{c}) (\bar{b}) (\bar{a} \lor d) (\bar{a} \lor b) \]
Resolution Soundness

Fact: If \( C_1, C_2 \) derive \( C_3 \) by Resolution rule, then \( C_1, C_2 \models C_3 \)

From above Fact we can prove:

Resolution Soundness Theorem

If a CNF formula \( F \) has a RES refutation, then \( F \) is unsatisfiable
**Resolution Completeness THM**

Every unsatisfiable CNF formula $F$ has a Resolution Refutation.

**Proof idea**

We describe a canonical procedure for obtaining a RES refutation for $F$.

The procedure exhaustively tries all truth ass's — via a decision tree.

Then we show that any such decision tree can be viewed as a RES refutation.
Decision Trees

\[ F = (a \vee b \vee c) (a \vee \overline{c}) (\overline{b}) (\overline{a} \land d) (\overline{a} \land \overline{b}) \]
Resolution Refutation

\[ F = (avbvc)(av\overline{c})(\overline{b})(\overline{a}vd)(\overline{a}vb) \]