CS 438/2404
Computability and Logic
MIDTERM

Question 5:

The language of successor is $\mathcal{L}_s = [0, s, =]$. The theory of successor, $Th(s)$, is the set of all sentences over $\mathcal{L}_s$ that are logical consequences of the following set of axioms Ψ:

P1) $\forall x(sx \neq 0)$
P2) $\forall x \forall y(sx = sy \supset x = y)$
P3) $\forall x(x = 0 \vee \exists y(x = sy))$
S1) $\forall x(sx \neq x)$
S2) $\forall x(ssx \neq x)$
S3) $\forall x(sssx \neq x)$
$\vdots$

Prove that there is no finite set Γ of sentences in $Th(s)$ such that every sentence in $Th(s)$ is a logical consequence of Γ. Note that the sentences in Γ are not necessarily among the original set Ψ of axioms.

Assume that there is a finite set Γ of sentences in $Th(s)$ such that every sentence in $Th(s)$ is a logical consequence of Γ. Let $A = \Gamma \cap \Psi$; then $A \subset \Gamma$ is some finite set of sentences such that $A \models \Gamma$; by transitivity of logical consequence, $A \models Th(s)$.

Now define a model $\mathcal{M}$ that satisfies A but falsifies $Th(s)$. Let k be the largest integer such that the term with k consecutive 's'es ($ss \cdots sx$) is a term in A. The universe will be $\{a_0, a_1, a_2, \ldots, a_k, 0, 1, 2, 3, \ldots\}$ where the a_i are constants. Successor is interpreted as:

$$sx = \begin{cases} x+1 & x \in \mathbb{N} \\ a_{\ell+1 \bmod (k+1)} & x = a_\ell \end{cases}$$

Thus the successor function works as usual on the natural numbers. On the other elements of the universe, $sa_0 = a_1$, $sa_1 = a_2$, $\ldots$, $sa_k = a_0$. Axioms P1, P2, and P3 are satisfied. All the other axioms Si in A are also satisfied, but axiom S$k+1$ ($\forall x(sss \ldots sx \neq x)$ which has $k+1$ consecutive 's'es) is not satisfied, since $x = a_0$ falsifies it. Since this model satisfies A but does *not* satisfy all the axioms in Ψ, it cannot be that $A \models Th(s)$.