UNIVERSITY OF TORONTO
Faculty of Arts and Science

CSC438F/2404F Test 3

Duration — 2 hours

- **Make sure your examination booklet has ten pages, including this one.**
- **Write your answers in the spaces provided. Use the backs of pages for scratch work. BUDGET YOUR TIME.**

Family Name: ______________________ Given Names: ______________________

Student #: ______________________

1.	________	/ 10
2.	________	/ 10
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7.	________	/ 10
8.	________	/ 10
Total	________	/80

Question 1 [10 marks]
Let $\Gamma = \{\gamma_1, \gamma_2, \cdots\}$ be a set of axioms for a theory Σ over the language $\mathcal{L}_A$. Suppose that for all n there is a structure σ_n such that σ_n satisfies $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ but σ_n does not satisfy γ_{n+1}. Prove that Σ is not finitely axiomatizable. That is, show that there is no finite subset $S \subseteq \Sigma$ such that $\Sigma = \{\alpha \in \mathcal{L}_A \mid S \vdash \alpha\}$.

Question 2 [10 marks, 5 each]
Classify the following relations as (i) recursive, (ii) recursively enumerable but not recursive, (iii) not recursively enumerable. Briefly prove your answer. You may use any definition of r.e./recursive given in class or in the notes. For your reductions, you must use one of the languages given on the definition sheet.

(a) $A = \{x \mid range(\{x\}_1) = PRIME\}$, where PRIME is the set of all prime numbers.

(b) $A = \{x \mid \{x\}_1(x) \text{ halts within three steps}\}$

Question 3 [10 marks]
Suppose that $\mathcal{L}$ is a language which includes an infinite list $c_1, c_2, \cdots$ of constant symbols. Let Γ be a set of sentences, $\Gamma = \{c_i \neq c_j \mid i, j \in N,\ i < j\}$. Let A be a sentence such that $\Gamma \models A$. Prove that A has a model with a finite universe.

Question 4 [10 marks]
Suppose that Σ is a theory over a language with finitely many function and predicate symbols. Suppose that Σ is consistent but has no finite models. Suppose also that any two countable models of Σ are isomorphic. Prove that Σ is complete.

Question 5 [10 marks]
Prove that if a relation R is strongly representable in RA then R is recursive.

Question 6 [10 marks]
Recall that PA has as axioms $P1, \cdots P6$ and the induction scheme. Show that

$$PA \vdash \forall x \forall y (x \cdot (sy) = x \cdot y + y)$$

First give the explicit instance of the induction scheme that you need. Then show how that induction axiom is used to establish the result. You do not need to give an LK proof, but justify each step of your argument with one of the axioms (P1, ..., P6 or whatever is appropriate). You may take equality reasoning for granted.

Question 7 [10 marks]
State whether each of the following statements is true or false. Give a very brief (one sentence) explanation of your answer.

(a) TA is complete but not arithmetical.

(b) PA is arithmetical.

(c) PA is complete.

(d) $R_{PA}(n)$ is r.e.

(e) $R_{PA}(n)$ is recursive.

Question 8 [10 marks]
Recall $Truth(m)$ if and only if $m = \#A$ for some $A \in TA$. Sketch the proof that $Truth$ is not arithmetical. Hint: Recall that $sub(m, n) = \#A(s_n)$ where $\#A(x) = m$ and $d(n) = sub(n, n)$.